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PERTURBATIONS OF NONLINEAR EIGENVALUE PROBLEMS

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Abstract. We consider perturbations of nonlinear eigenvalue problems driven by a nonhomogeneous differential operator plus an indefinite potential. We consider both sublinear and superlinear perturbations and we determine how the set of positive solutions changes as the real parameter λ varies. We also show that there exists a minimal positive solution $\overline{u}_\lambda$ and determine the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$. Special attention is given to the particular case of the p-Laplacian.

1. **Introduction.** The aim of this paper is to study the following nonlinear nonhomogeneous parametric Robin problem

$$\left\{ \begin{array}{l} -\mathrm{div}\, a(Du(z)) + \xi(z)u(z)^{p-1} = \lambda u(z)^{p-1} + f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega,\ u > 0,\ \lambda \in \mathbb{R},\ 1 < p < \infty. \end{array} \right\} \qquad (P_\lambda)$$

In this problem, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial\Omega$. The map $a : \mathbb{R}^N \to \mathbb{R}^N$ involved in the differential operator, is continuous, strictly monotone (hence maximal monotone, too) and satisfies some other regularity and growth conditions listed in hypotheses $H(a)$ below (see Section 2). These extra-conditions on

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$a(\cdot)$ are not restrictive and so our framework incorporates many differential operators of interest such as the p-Laplacian and the (p,q)-Laplacian (that is, the sum of a p-Laplacian and a q-Laplacian). The potential function $\xi \in L^\infty(\Omega)$ is indefinite (that is, sign changing). In the reaction (right-hand side of the equation), we have a parametric term $u \mapsto \lambda u^{p-1}$ and a perturbation $f(z,x)$ which is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z,x)$ is measurable and for almost all $z \in \Omega$ the mapping $x \mapsto f(z,x)$ is continuous).

We consider two distinct cases. In the first one, $f(z,\cdot)$ is $(p-1)$-sublinear near $+\infty$, while in the second we assume that $f(z,\cdot)$ is $(p-1)$-superlinear. In the boundary condition, $\frac{\partial u}{\partial n_a}$ denotes the conormal derivative of u, defined by extension of the map

$$L^1(\overline{\Omega}) \ni u \mapsto (a(Du), n)_{\mathbb{R}^N},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$. The boundary coefficient $\beta(\cdot)$ is non-negative and the case $\beta \equiv 0$ is also included and corresponds to the Neumann problem.

We look for positive solutions of (P_λ) and we want to determine how the set of positive solutions changes as the parameter λ moves on the real line $\mathbb{R}$. More precisely, we show that there is a critical parameter value $\lambda^* \in \mathbb{R}$ such that for $\lambda < \lambda^*$ problem (P_λ) has

- at least one positive smooth solutions, when $f(z,\cdot)$ is $(p-1)$-sublinear;
- at least two positive smooth solutions, when $f(z,\cdot)$ is $(p-1)$-superlinear.

For $\lambda \geq \lambda^*$, problem (P_λ) has no positive solutions.

In the particular case of the p-Laplace differential operator (that is, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$), problem (P_λ) can be viewed as a perturbation of the classical eigenvalue problem for the p-Laplacian. In this particular case, we can identify λ^* as the principal eigenvalue $\hat{\lambda}_1$ of the differential operator $u \mapsto -\Delta_p u + \xi(z)|u|^{p-2}u$ with the Robin boundary condition. This was already observed by these authors for the semilinear problem (that is, $p = 2$ hence $a(y) = y$ for all $y \in \mathbb{R}^N$), see Papageorgiou, Rădulescu & Repovš [21]. Also, for both cases (sublinear and superlinear), we establish the existence of a smallest positive solution $\overline{u}_\lambda$ and determine the monotonicity and continuity properties of the map $\lambda \mapsto \overline{u}_\lambda$. Finally in the sublinear case we address the question of uniqueness of the solution.

Nonlinear, nonhomogeneous parametric Robin problems were also studied by Autuori & Pucci [2], Colasuonno, Pucci & Varga [5], Fragnelli, Mugnai & Papageorgiou [7], Papageorgiou, Rădulescu & Repovš [22, 23], and Perera, Pucci & Varga [24].

Our approach is variational, using results from the critical point theory and also truncation, perturbation and comparison techniques.

2. **Mathematical background and hypotheses.**

Let X be a Banach space and X^* its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X,\mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that

$\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + ||u_n||)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,

admits a strongly convergent subsequence."

In what follows, we denote by K_φ the critical set of φ, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Also, if $c \in \mathbb{R}$, then

$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

Using the notion of the C-condition, we have the following minimax theorem, known in the literature as the "mountain pass theorem".

Theorem 2.1. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $||u_1 - u_0|| > \rho > 0$,*

$$\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : ||u - u_0|| = \rho\} = m_\rho$$

and $c = \inf\limits_{\gamma \in \Gamma} \max\limits_{0 \le t \le 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \ge m_\rho$ and c is a critical value of φ (that is, we can find $\hat{u} \in X$ such that $\varphi'(\hat{u}) = 0$ and $\varphi(\hat{u}) = c$).

In the analysis of problem (P_λ), we will use the Sobolev space $W^{1,p}(\Omega)$, the Banach space $C^1(\overline{\Omega})$ and the "boundary" Lebesgue spaces $L^r(\partial\Omega), 1 \le r \le \infty$. We denote by $|| \cdot ||$ the norm of $W^{1,p}(\Omega)$ defined by

$$||u|| = \left(||u||_p^p + ||Du||_p^p\right)^{1/p} \text{ for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone $C_+ = \{u \in C^1(\overline{\Omega}) : u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Evidently, $\operatorname{int} C_+$ contains the open set D_+ defined by

$$D_+ = \{u \in C_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact, D_+ is the interior of C_+ when $C^1(\overline{\Omega})$ is furnished with the $C(\overline{\Omega})$-norm topology.

On $\partial\Omega$ we consider the $(N-1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. Using this measure on $\partial\Omega$, we can define in the usual way the "boundary" Lebesgue spaces $L^r(\partial\Omega)$. From the theory of Sobolev spaces, we know that there exists a unique continuous linear map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the "trace map", such that

$$\gamma_0(u) = u|_{\partial\Omega} \text{ for all } u \in W^{1,p}(\Omega) \cap C(\overline{\Omega}).$$

So, the trace map extends the notion of boundary value to all Sobolev functions. The map $\gamma_0(\cdot)$ is compact into $L^r(\partial\Omega)$ for all $r \in [1, (N-1)p/(N-p))$ when $p < N$, and into $L^r(\partial\Omega)$ for all $1 \le r < \infty$ when $p \ge \mathbb{N}$. Also, we have

$$\operatorname{im} \gamma_0 = W^{\frac{1}{p'},p}(\partial\Omega) \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \text{ and } \ker \gamma_0 = W_0^{1,p}(\Omega).$$

In what follows, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0(\cdot)$. All restrictions of Sobolev functions on $\partial\Omega$, are understood in the sense of traces.

Let $\vartheta \in C^1(0, \infty)$ and assume that it satisfies the following growth conditions:

$$0 < \hat{c} \le \frac{\vartheta'(t)t}{\vartheta(t)} \le c_0 \text{ and } c_1 t^{p-1} \le \vartheta(t) \le c_2(t^{\tau-1} + t^{p-1}) \text{ for all } t > 0, \tag{1}$$

with $0 < c_1, c_2$ and $1 \le \tau < p$.

The hypotheses on the map $a(\cdot)$ involved in the differential operator of (P_λ), are as follows:

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0,\infty)$, $t \mapsto a_0(t)t$ is strictly increasing on $(0,\infty), a_0(t)t \to 0^+$ as $t \to 0^+$ and $\lim_{t\to 0^+} \frac{a_0'(t)t}{a_0(t)} > -1$;

(ii) $|\nabla a(y)| \leq c_3 \frac{\vartheta(|y|)}{|y|}$ for all $y \in \mathbb{R}^N \backslash \{0\}$, and for some $c_3 > 0$;

(iii) $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{\vartheta(|y|)}{|y|}|\xi|^2$ for all $y \in \mathbb{R}^N\backslash\{0\}$, $\xi \in \mathbb{R}^N$;

(iv) if $G_0(t) = \int_0^t a_0(s)sds$ then there exists $q \in (1,p]$ such that $t \mapsto G_0(t^{1/q})$ is convex, $\lim_{t\to 0^+} \frac{qG_0(t)}{t^q} = \tilde{c} > 0$ and $0 \leq pG_0(t) - a_0(t)t^2$ for all $t \geq 0$.

Remark 1. Hypotheses $H(a)(i)(ii)(iii)$ permit the use of the nonlinear regularity theory of Lieberman [13] and of the nonlinear maximum principle of Pucci & Serrin [25]. Hypothesis $H(a)(iv)$ serves the needs of our problem. It is a mild condition which is satisfied in all cases of interest (see the examples below). These hypotheses imply that $G_0(\cdot)$ is strictly increasing and strictly convex. We set $G(y) = G_0(|y|)$ for all $y \in \mathbb{R}^N$. We have

$$\nabla G(y) = G_0'(|y|)\frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N, \nabla G(0) = 0.$$

So, $G_0(\cdot)$ is the primitive of $a(\cdot)$ and $G(\cdot)$ is convex, $G(0) = 0$. Therefore

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{2}$$

The next lemma summarizes the main properties of the map $a(\cdot)$. It is an easy consequence of hypotheses $H(a)(i)$, (ii), (iii).

Lemma 2.2. *If hypotheses $H(a)(i)(ii)(iii)$ hold, then*

(a) $y \mapsto a(y)$ is continuous and strictly monotone (thus, maximal monotone, too);

(b) $|a(y)| \leq c_4 \left(1 + |y|^{p-1}\right)$ for all $y \in \mathbb{R}^N$, and for some $c_4 > 0$;

(c) $(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$ for all $y \in \mathbb{R}^N$.

Using this lemma and relation (2), we obtain the following growth properties for the primitive $G(\cdot)$

Corollary 1. *If hypotheses $H(a)(i)$, (ii), (iii) hold, then $\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1 + |y|^p)$ for all $y \in \mathbb{R}^N$, and for some $c_5 > 0$.*

Examples. The following maps $a(\cdot)$ satisfy hypotheses $H(a)$ (see also Papageorgiou & Rădulescu [18, 19]):

(a) $a(y) = |y|^{p-2}y,\ 1 < p < \infty$.
This map corresponds to the p-Laplace differential operator defined by

$$\Delta_p u = \text{div } \left[|Du|^{p-2}Du\right] \text{ for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = |y|^{p-2}y + |y|^{q-2}y,\ 1 < q < p < \infty$.
This map corresponds to the (p,q)-Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \text{ for all } u \in W^{1,p}(\Omega).$$

Such operators arise in problems of mathematical physics (see Cherfils & Ilyasov [4]). A survey of some recent results on such equations with several relevant references, can be found in Marano & Mosconi [14].

(c) $a(y) = [1 + |y|^2]^{\frac{p-2}{2}}y,\ 1 < p < \infty$.
This map corresponds to the generalized p-mean curvature differential operator defined by

$$\text{div } [(1 + |Du|^2)^{\frac{p-2}{2}}Du] \text{ for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) : |y|^{p-2}y\left(1+\frac{1}{1+|y|^p}\right), \ 1<p<\infty$

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$\langle A(u), h\rangle = \int_\Omega (a(Du), Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W^{1,p}(\Omega).$$

The next proposition establishes the properties of $A(\cdot)$ and is a special case of a more general result of Gasinski & Papageorgiou [9] (see also Gasinski & Papageorgiou [10, Problem 2.192]).

Proposition 1. *If hypotheses $H(a)$ hold, then $A(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, monotone (thus maximal monotone, too) and of type $(S)_+$, that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$.*

We will also need the following strong comparison principle due to Papageorgiou, Rădulescu & Repovš [21].

Proposition 2. *If hypotheses $H(a)$ hold, $k \in L^\infty(\Omega)$ with $k(z) \ge 0$ for almost all $z \in \Omega$, $h_1, h_2 \in L^\infty(\Omega)$*

$$0 < \tilde{\gamma} \le h_2(z) - h_1(z) \textit{ for almost all } z \in \Omega$$

and $u, v \in C^{1,\alpha}(\overline{\Omega})$ with $\alpha \in (0,1], u \le v$ and

$$-\operatorname{div} a(Du) + k(z)|u|^{p-2}u = h_1(z) \textit{ for almost all } z \in \Omega,$$
$$-\operatorname{div} a(Dv) + k(z)|v|^{p-2}v = h_2(z) \textit{ for almost all } z \in \Omega,$$

then $v - u \in \operatorname{int} C_+$.

We introduce the hypotheses on the potential function $\xi(\cdot)$ and the boundary coefficient $\beta(\cdot)$

$H(\xi) : \xi \in L^\infty(\Omega)$;

$H(\beta) : \beta \in C^{0,\alpha}(\partial\Omega)$ with $\alpha \in (0,1)$ and $\beta(z) \ge 0$ for all $z \in \partial\Omega$.

Remark 2. The case $\beta \equiv 0$ corresponds to the Neumann problem.

Let $\mu : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\mu(u) = \int_\Omega pG(Du)dz + \int_\Omega \xi(z)|u|^p dz + \int_{\partial\Omega} \beta(z)|u|^p d\sigma \text{ for all } u \in W^{1,p}(\Omega).$$

Consider a Carathéodory function $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_0(z,x)| \le a_0(z)(1+|x|^{r-1}) \text{ for almost all } z \in \Omega, \text{ and for all } x \in \mathbb{R},$$

with $a_0 \in L^\infty(\Omega)$ and $1 \le r \le p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \le p \end{cases}$ (the critical Sobolev exponent).

We set $F_0(z,x) = \int_0^x f_0(z,s)ds$ and consider the C^1-functional $\varphi_0 : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_0(v) = \frac{1}{p}\mu(u) - \int_\Omega F_0(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

From Papageorgiou & Rădulescu [20] we have the following proposition. The result is essentially an outgrowth of the nonlinear regularity theory of Lieberman [13].

Proposition 3. *Assume that hypotheses $H(a)$ hold and $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$-minimizer of $\varphi_0(\cdot)$, that is, there exists $\rho_1 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}) \text{ with } ||h||_{C^1(\overline{\Omega})} \leq \rho_1.$$

Then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ and u_0 is a local $W^{1,p}(\Omega)$-minimizer of φ_0, that is, there exists $\rho_2 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } ||h|| \leq \rho_2.$$

We will also use some facts about the spectrum of the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -\Delta_r u(z) + \xi(z)|u(z)|^{r-2}u(z) = \hat{\lambda}|u(z)|^{r-2}u(z) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_r} + \beta(z)|u|^{r-2}u = 0 \text{ on } \partial\Omega, 1 < r < \infty. \end{array} \right\} \tag{3}$$

In this case, the conormal derivative $\frac{\partial u}{\partial n_r}$ is defined by

$$\frac{\partial u}{\partial n_r} = |Du|^{r-2}\frac{\partial u}{\partial n} \text{ for all } u \in W^{1,r}(\Omega).$$

As before, $n(\cdot)$ denotes the outward unit normal on $\partial\Omega$. We say that $\hat{\lambda} \in \mathbb{R}$ is an "eigenvalue", if problem (3) admits a nontrivial solution $\hat{u} \in W^{1,r}(\Omega)$, known as an "eigenfunction" corresponding to the eigenvalue $\hat{\lambda}$. The nonlinear regularity theory of Lieberman [13] (see also Gasinski & Papageorgiou [8, pp. 737-738]), implies that $\hat{u} \in C^1(\overline{\Omega})$. From Fragnelli, Mugnai & Papageorgiou [7] (see also Mugnai & Papageorgiou [16] and Papageorgiou & Rădulescu [17], where special cases of (3) are discussed), we have the following property.

Proposition 4. *If hypotheses $H(\xi), H(\beta)$ hold, then problem (3) admits a smallest eigenvalue $\hat{\lambda}_1 = \hat{\lambda}_1(r, \xi, \beta) \in \mathbb{R}$ such that*

- *(a) $\hat{\lambda}_1$ is isolated (that is, if $\hat{\sigma}(r)$ denotes the spectrum of (3), then we can find $\epsilon > 0$ such that $(\hat{\lambda}_1, \hat{\lambda}_1 + \epsilon) \cap \hat{\sigma}(r) = \emptyset$);*
- *(b) $\hat{\lambda}_1$ is simple (that is, if $\hat{u}, \hat{v} \in C^1(\overline{\Omega})$ are eigenfunctions corresponding to $\hat{\lambda}_1$, then $\hat{u} = \hat{\xi}\hat{v}$ for some $\hat{\xi} \in \mathbb{R}\backslash\{0\}$);*
- *(c) we have*

$$\hat{\lambda}_1 = \inf\left\{\frac{\mu_r(u)}{||u||_r^r} : u \in W^{1,r}(\Omega), u \neq 0\right\}, \tag{4}$$

with

$$\mu_r(u) = \|Du\|_r^r + \int_\Omega \xi(z)|u|^r dz + \int_{\partial\Omega} \beta(z)|u|^r d\sigma.$$

In (4), the infimum is realized on the corresponding one-dimensional eigenspace. The above properties imply that the elements of this eigenspace have fixed sign. We denote by $\hat{u}_1 = \hat{u}_1(r, \xi, \beta)$ the positive, L^r-normalized (that is, $||\hat{u}_1||_r = 1$) eigenfunction corresponding to $\hat{\lambda}_1 = \hat{\lambda}(r, \xi, \beta)$. The nonlinear Hopf lemma (see Pucci & Serrin [25, pp. 111, 120] and Gasinski & Papageorgiou [8, p. 738]) implies that $\hat{u}_1 \in D_+$. Moreover, if $\hat{u}$ is an eigenfunction corresponding to an eigenvalue $\hat{\lambda} \neq \hat{\lambda}_1$, then $\hat{u}$ is nodal (that is, sign-changing).

For every $x \in \mathbb{R}$, let $x^\pm = \max\{\pm x, 0\}$. Then given $u \in W^{1,p}(\Omega)$, we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega),\ |u| = u^+ + u^-,\ u = u^+ - u^-.$$

Given a measurable function $k : \Omega \times \mathbb{R} \to \mathbb{R}$ (for example, a Carathéodory function), we denote by $N_k(\cdot)$ the Nemytskii map corresponding to $k(\cdot,\cdot)$, that is,

$$N_k(u)(\cdot) = k(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega).$$

If $v, u \in W^{1,p}(\Omega)$ and $v \le u$, then we set

$$[v,u] = \{y \in W^{1,p}(\Omega) : v(z) \le y(z) \le u(z) \text{ for almost all } z \in \Omega\}$$
$$[u) = \{y \in W^{1,p}(\Omega) : u(z) \le y(z) \text{ for almost all } z \in \Omega\}.$$

3. **$(p-1)$-sublinear perturbation.** In this section, we examine the case where the perturbation $f(z,x)$ in problem (P_λ) is $(p-1)$-sublinear near $+\infty$. More precisely, the hypotheses on $f(z,x)$ are the following:

$H(f)_1$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z,x)| \le a_\rho(z) \text{ for almost all } z \in \Omega, \text{ and for all } 0 \le x \le \rho;$$

(ii) $\lim_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for almost all $z \in \Omega$;

(iii) with $q \in (1,p]$ as in hypothesis $H(a)(iv)$ we have

$$\lim_{x\to 0^+} \frac{f(z,x)}{x^{q-1}} = +\infty \text{ uniformly for almost all } z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$, then function

$$x \mapsto f(z,x) + \hat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0,\rho]$.

Remark 3. Since we are looking for positive solutions and all the above hypotheses concern the positive semi-axis $\mathbb{R}_+ = [0,+\infty)$, we may assume without any loss of generality that $f(z,x) = 0$ for almost all $z \in \Omega$, and for all $x \le 0$. Hypothesis $H(f)_1(ii)$ implies that $f(z,\cdot)$ is $(p-1)$-superlinear near $+\infty$. Hypothesis $H(f)_1(iii)$ implies that $f(z,\cdot)$ is $(q-1)$-superlinear near 0^+ (that is, $f(z,\cdot)$ exhibits a q-concave term near 0^+). Hypothesis $H(f)_1(iv)$ is satisfied if for example $f(z,\cdot)$ is differentiable and for every $\rho > 0$, there exists $\eta_\rho > 0$ such that $f'_x(z,x)x \ge -\eta_\rho x^{p-1}$ for almost all $z \in \Omega$, and for all $0 \le x \le \rho$. We stress that no global sign condition is imposed on $f(z,\cdot)$.

Example. The following function satisfies hypotheses $H(f)_1$. For the sake of simplicity we drop the z-dependence:

$$f(z) = \begin{cases} 0 & \text{if } x < 0 \\ x^{\tau-1} - 2x^{q-1} & \text{if } 0 \le x \le 1 \\ x^{r-1} - 2x^{s-1} & \text{if } 1 < x \end{cases}$$

with $\tau < q \le p$ and $1 < s < r < p$. Note that $f(\cdot)$ changes sign.

Let

$$\mathcal{L} = \{\lambda \in \mathbb{R} : \text{ problem } (P_\lambda) \text{ has a positive solution}\},$$

$$S_\lambda = \text{the set of all positive solutions of problem } (P_\lambda).$$

Proposition 5. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold, then $\mathcal{L} \neq \emptyset$ and $S_\lambda \subseteq D_+$.*

Proof. Let $\eta > ||\xi||_\infty$ and consider the following Carathéodory function

$$e_\lambda(z,x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (\lambda+\eta)\, x^{p-1} + f(z,x) & \text{if } 0 < x, \end{cases} \tag{5}$$

for all $\lambda \in \mathbb{R}$.

We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and consider the C^1-functional $\varphi_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u||_p^p - \int_\Omega E_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Let $F(z,x) = \int_0^x f(z,s)ds$. Hypotheses $H(f)_1(i), (ii)$ imply that given $\epsilon > 0$, we can find $c_6 = c_6(\epsilon) > 0$ such that

$$F(z,x) \leq \frac{\epsilon}{p}x^p + c_6 \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{6}$$

Using (5), (6), Corollary 1 and hypothesis $H(\beta)$, we have

$$\varphi_\lambda(u) \geq \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}\int_\Omega [\xi(z) + \eta - (\lambda + \xi)]\, |u|^p dz - c_7 \text{ for some } c_7 > 0.$$

Choosing $\lambda \in \mathbb{R}$ such that $\lambda + \epsilon < \eta - ||\xi||_\infty$, we can write

$$\begin{aligned} & \varphi_\lambda(u) \geq \frac{c_1}{p(p-1)}||Du||_p^p + c_8||u||_p^p - c_7 \text{ for some } c_8 > 0, \\ \Rightarrow \quad & \varphi_\lambda(\cdot) \text{ is coercive.} \end{aligned}$$

By the Sobolev embedding theorem and the compactness of the trace map we deduce that $\varphi(\cdot)$ is sequentially weak lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\varphi_\lambda(u_\lambda) = \inf\left\{\varphi_\lambda(u) : u \in W^{1,p}(\Omega)\right\}. \tag{7}$$

Hypothesis $H(a)(iv)$ implies that given $\tilde{c}_0 > \tilde{c}$, we can find $\delta \in (0,1)$ such that

$$G(y) \leq \frac{\tilde{c_0}}{q}|y|^q \text{ for all } |y| \leq \delta. \tag{8}$$

Hypothesis $H(f)(iii)$ implies that given any $\vartheta > 0$, by choosing $\delta > 0$ even smaller if necessary, we can also have

$$F(z,x) \geq \frac{\vartheta}{q}x^q \text{ for almost all } z \in \Omega, \text{ and for all } 0 \leq x \leq \delta. \tag{9}$$

Let $\hat{\lambda}_1 = \hat{\lambda}_1(q,\xi_0,\beta_0)$ and $\hat{u}_1 = \hat{u}_1(q,\xi_0,\beta_0) \in D_+$ with $\xi_0 = \frac{1}{\tilde{c}_0}\xi$, $\beta_0 = \frac{1}{\tilde{c}_0}\beta$. We choose small $t \in (0,1)$ such that

$$0 \leq t\hat{u}_1(z) \leq \delta \text{ and } |D(t\hat{u}_1)(z)| \leq \delta \text{ for all } z \in \overline{\Omega}. \tag{10}$$

Using (5), (8), (9), (10), we have

$$\begin{aligned} \varphi_\lambda(t\hat{u}_1) \quad & \leq \frac{\tilde{c}_0 t^q}{q}||D\hat{u}_1||_q^q + \frac{1}{p}\int_\Omega \xi(z)|t\hat{u}_1|^p dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)|t\hat{u}_1|^p d\sigma \\ & \quad -\frac{\lambda t^p}{p}||\hat{u}_1||_p^p - \frac{\vartheta t^q}{q} \text{ (recall that } ||\hat{u}_1||_q = 1) \\ & \leq \frac{\tilde{c}_0 t^q}{q}\left(||D\hat{u}_1||_q^q + \int_\Omega \xi_0|\hat{u}_1|^q dz + \int_{\partial\Omega}\beta_0(z)|\hat{u}_1|^q d\sigma\right) - \frac{\vartheta}{q}t^q \\ & \qquad \text{(since } 0 < \delta < 1 \text{ and } q \leq p) \\ & \leq \frac{t^q}{q}\left(\tilde{c}_0\hat{\lambda}_1 - \vartheta\right). \end{aligned}$$

But $\vartheta > 0$ is arbitrary. So, choosing $\vartheta > \tilde{c}_0\hat{\lambda}_1$, we see that

$$\begin{aligned} & \varphi_\lambda(t\hat{u}_1) < 0, \\ \Rightarrow \quad & \varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \text{ (see (7))}, \\ \Rightarrow \quad & u_\lambda \neq 0. \end{aligned}$$

From (7) we have for all $h \in W^{1,p}(\Omega)$

$$\begin{aligned} & \varphi'_\lambda(u_\lambda) = 0 \\ \Rightarrow \quad & \langle A(u_\lambda), h\rangle + \int_\Omega (\xi+\eta)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta_\lambda |u_\lambda|^{p-2} u_\lambda h d\sigma \\ & = \int_\Omega e_\lambda(z, u_\lambda) h dz. \end{aligned} \tag{11}$$

In (11) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \frac{c_1}{p-1}||Du_\lambda^-||_p^p + \tfrac{1}{p}\textstyle\int_\Omega [\xi(z)+\eta](u_\lambda^-)^p dz \leq 0 \\ & \text{(see Lemma 2.2, hypothesis } H(\beta) \text{ and (5))} \\ \Rightarrow \quad & u_\lambda \geq 0,\ u_\lambda \neq 0 \text{ (recall that } \eta > ||\xi||_\infty). \end{aligned}$$

It follows from (5) and (11) that

$$\left.\begin{aligned} & -\operatorname{div} a(Du_\lambda(z)) + \xi(z)u_\lambda(z)^{p-1} = \lambda u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) \text{ for almost all } z \in \Omega \\ & \frac{\partial u}{\partial n_a} + \beta(z) u_\lambda^{p-1} = 0 \text{ on } \partial\Omega \text{ (see Papageorgiou \& Rădulescu [17])}. \end{aligned}\right. \tag{12}$$

From (12) and Papageorgiou & Rădulescu [20], we have $u_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [13] implies that $u_\lambda \in C_+\backslash\{0\}$.

Let $\rho = ||u_\lambda||_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)(iv)$. Then from (35) we have

$$\begin{aligned} & \operatorname{div} a(Du_\lambda(z)) \leq \left(||\xi||_\infty + \hat{\xi}_\rho\right) u_\lambda(z)^{p-1} \text{ for almost all } z \in \Omega, \\ \Rightarrow \quad & u_\lambda \in D_+ \text{ (see Pucci \& Serrin [25, pp. 111, 120])}. \end{aligned}$$

Therefore we conclude that $\lambda \in \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$ and also $S_\lambda \subseteq D_+$. □

Next, we show that $\mathcal{L}$ is a half-line.

Proposition 6. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold, $\lambda \in \mathcal{L}$ and $\vartheta < \lambda$, then $\vartheta \in \mathcal{L}$.*

Proof. By hypothesis, $\lambda \in \mathcal{L}$. So, we can find $u_\lambda \in S_\lambda \subseteq D_+$. With $\eta > ||\xi||_\infty$ as before, we introduce the following truncation-perturbation of the reaction in problem (P_λ):

$$k_\vartheta(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ (\vartheta+\eta)\, x^{p-1} + f(z,x) & \text{if } 0 \leq x \leq u_\lambda(z) \\ (\vartheta+\eta)\, u_\lambda(z)^{p-1} + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x. \end{cases} \tag{13}$$

This is a Carathéodory function. We set $K_\vartheta(z,x) = \int_0^x k_\vartheta(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_\vartheta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\varphi}_\vartheta(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u||_p^p - \int_\Omega K_\vartheta(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Clearly, $\hat{\varphi}_\vartheta(\cdot)$ is coercive (see (13)) and sequentially weakly lower semicontinuous. So, we can find $u_\vartheta \in W^{1,p}(\Omega)$ such that

$$\hat{\varphi}_\vartheta(u_\vartheta) = \inf\left\{\hat{\varphi}_\vartheta(u) : u \in W^{1,p}(\Omega)\right\}. \tag{14}$$

As in the proof of Proposition 5, using hypotheses $H(a)(iv)$ and $H(f)(iii)$, we show that $\hat{\varphi}_\vartheta(u_\vartheta) < 0 = \hat{\varphi}_\vartheta(0)$, hence $u_\vartheta \neq 0$. From (14) we have

$$\begin{aligned} &\hat{\varphi}'_\vartheta(u_\varphi) = 0, \\ \Rightarrow\quad &\langle A(u_\vartheta), h\rangle + \int_\Omega (\xi(z)+\eta)\,|u_\vartheta|^{p-2}u_\vartheta h dz + \int_{\partial\Omega}\beta(z)|u_\vartheta|^{p-2}u_\vartheta h d\sigma \\ &= \int_\Omega k_\vartheta(z,u_\vartheta)hdz \text{ for all } h \in W^{1,p}(\Omega) \end{aligned} \tag{15}$$

In (15) we first choose $h = -u_\vartheta^- \in W^{1,p}(\Omega)$. Then using Lemma 2 and (13) we obtain

$$\begin{aligned} &\frac{c_1}{p-1}||Du_\vartheta^-||_p^p + \int_\Omega [\xi(z)+\eta]\,(u_\vartheta^-)^p dz \leq 0 \text{ (see hypothesis } H(\beta)), \\ \Rightarrow\quad & u_\vartheta \geq 0,\ u_\vartheta \neq 0. \end{aligned}$$

Next, in (15) we choose $h = (u_\vartheta - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\langle A(u_\vartheta), (u_\vartheta - u_\lambda)^+\rangle + \int_\Omega (\xi(z)+\eta)\,u_\vartheta^{p-1}(u_\vartheta - u_\lambda)^+ dz \\ &+ \int_{\partial\Omega}\beta(z)u_\vartheta^{p-1}(u_\vartheta - u_\lambda)^+ d\sigma \\ = &\int_\Omega ((\vartheta+\eta)\,u_\lambda^{p-1} + f(z,u_\lambda))(u_\vartheta - u_\lambda)^+ dz \text{ (see (13))} \\ \leq &\int_\Omega ((\lambda+\eta)\,u_\lambda^{p-1} + f(z,u_\lambda)(u_\vartheta - u_\lambda)^+ dz \text{ (recall that } \vartheta < \lambda) \\ = &\langle A(u_\lambda), (u_\vartheta - u_\lambda)^+)\rangle + \int_\Omega (\xi(z)+\eta)\,u_\lambda^{p-1}(u_\vartheta - u_\lambda)^+ dz \\ &+ \int_{\partial\Omega}\beta(z)u_\lambda^{p-1}(u_\vartheta - u_\lambda)^+ d\sigma \\ &\text{(recall that } u_\lambda \in S_\lambda) \\ \Rightarrow\quad & u_\vartheta \leq u_\lambda. \end{aligned}$$

We have proved that

$$u_\vartheta \in [0, u_\lambda],\ u_\vartheta \neq 0. \tag{16}$$

It follows from (13), (15), (16) that $\vartheta \in \mathcal{L}$ and $u_\vartheta \in S_\vartheta \subseteq D_+$. □

Let $\lambda^* = \sup \mathcal{L}$.

Proposition 7. *If hypotheses $h(a), H(\xi), H(\beta), H(f)_1$ hold, then $\lambda^* < +\infty$.*

Proof. Hypotheses $H(\xi), H(f)_1$ imply that for large enough $\tilde{\lambda} > 0$ we have

$$(\tilde{\lambda} - \xi(z))x^{p-1} + f(z,x) \geq x^{p-1} \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{17}$$

Let $\lambda > \tilde{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then by Proposition 5 we can find $u \in S_\lambda \subseteq D_+$. We set

$$m = \min_{\overline{\Omega}} u > 0 \quad (\text{since } u \in D_+). \tag{18}$$

For $\delta > 0$ we set $m_\delta = m + \delta > 0$. Also, let $\rho = ||u||_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)_1(iv)$. We can always take $\hat{\xi}_\rho > \max\{\lambda, ||\xi||_\infty\}$.

We have that for almost all $z \in \Omega$ the function $x \mapsto (\lambda + \hat{\xi}_\rho)x^{p-1} + f(z,x)$ is nondecreasing on $[0,\rho]$. We have

$$\begin{aligned}
& -\mathrm{div}\, a(Dm_\delta) + (\xi(z) + \hat{\xi}_\rho)m_\delta^{p-1} \\
\leq\ & (\xi(z) + \hat{\xi}_\rho)m^{p-1} + \gamma(\delta) \text{ with } \gamma(\delta) \to 0^+ \text{ as } \delta \to 0^+ \\
\leq\ & (\tilde{\lambda} + \hat{\xi}_\rho)m^{p-1} + f(z,m) + \gamma(\delta) \text{ (see (17))} \\
=\ & (\lambda + \hat{\xi}_\rho)m^{p-1} + f(z,m) - (\lambda - \tilde{\lambda})m^{p-1} + \gamma(\delta) \\
\leq\ & (\lambda + \hat{\xi}_\rho)m^{p-1} + f(z,m) \text{ for small enough } \delta > 0 \text{ (so that } \gamma(\delta) < (\lambda - \tilde{\lambda})m^{p-1}) \\
\leq\ & (\lambda + \hat{\xi}_\rho)u^{p-1} + f(z,u) \text{ (see (18))} \\
=\ & -\mathrm{div}\, a(Du) + (\xi(z) + \hat{\xi}_\rho)u^{p-1}.
\end{aligned}$$

Let

$$\begin{aligned}
h_1(z) &= (\xi(z) + \hat{\xi}_\rho)m^{p-1} + \gamma(\delta) \\
h_2(z) &= (\lambda + \hat{\xi}_\rho)u^{p-1} + f(z,u).
\end{aligned}$$

Evidently, $h_1, h_2 \in L^\infty(\Omega)$ and for $\delta > 0$ small we have

$$0 < \tilde{\gamma} \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega.$$

So, by Proposition 2 for small enough $\delta > 0$ we have

$$u - m_\delta \in \mathrm{int}\, C_+,$$

a contradiction to (18). Therefore $\lambda \notin \mathcal{L}$ and so $\lambda^* \leq \tilde{\lambda} < +\infty$. $\square$

Fix $\lambda < \lambda^*$. Then by Proposition 6 we have $\lambda \in \mathcal{L}$. We will show that $S_\lambda \subseteq D_+$ admits a smallest element. Let $r \in (p, p^*)$. On account of hypotheses $H(f)_1$ we can find $c_9 > 0$ and $c_{10} = c_{10}(\lambda) > 0$ both large enough such that

$$\lambda x^{p-1} + f(z,x) \geq c_9 x^{q-1} - c_{10}x^{r-1} \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \quad (19)$$

Motivated by this one-sided growth condition on the reaction of problem (P_λ), we consider the following auxiliary nonlinear nonhomogeneous Robin problem

$$\left\{ \begin{array}{l} -\mathrm{div}\, a(Du(z)) + |\xi(z)|u^{p-1} = c_9 u(z)^{q-1} - c_{10}u(z)^{r-1} \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, \ u > 0. \end{array} \right\} \quad (20)$$

Proposition 8. *If hypotheses $H(a), H(\xi), H(\beta)$ hold and $c_9, c_{10} > 0$ are both large enough, then problem (20) admits a unique solution $u_*^\lambda \in D_+$*

Proof. Let $\Psi : W^{1,p}(\Omega) \to \mathbb{R}$ be the C^1-functional defined by

$$\begin{aligned}
\Psi(u) = \int_\Omega G(Du)dz + \frac{1}{p}\int_\Omega |\xi(z)||u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma + \frac{1}{p}||u^-||_p^p \\
+ \frac{c_{10}}{r}||u^+||_r^r - \frac{c_9}{q}||u^+||_q^q \text{ for all } u \in W^{1,p}(\Omega).
\end{aligned}$$

By Corollary 1 and the fact that $q \leq p < r$, by taking $c_{10} > 0$ large enough (see (19)), we see that $\Psi_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_*^\lambda \in W^{1,p}(\Omega)$ such that

$$\Psi(u_*^\lambda) = \inf\left\{\Psi(u) : u \in W^{1,p}(\Omega)\right\}. \quad (21)$$

On account of hypothesis $H(f)(iii)$, we can choose $c_9 > 0$ large enough so that

$$\begin{aligned}
& \Psi(u_*^\lambda) < 0 = \Psi(0) \text{ (recall that } q \leq p < r) \\
\Rightarrow\ & u_*^\lambda \neq 0.
\end{aligned}$$

From (21) we have

$$
\begin{aligned}
& \Psi'(u_*^\lambda) = 0, \\
\Rightarrow\ & \langle A(u_*^\lambda), h\rangle + \int_\Omega |\xi(z)||u_*^\lambda|^{p-2}u_*^\lambda h dz + \int_{\partial\Omega} \beta(z)|u_*^\lambda|^{p-2}u_*^\lambda h d\sigma \\
& - \int_\Omega ((u_*^\lambda)^-)^{p-1} h dz \\
=\ & c_9 \int_\Omega ((u_*^\lambda)^+)^{q-1} h dz - c_{10} \int_\Omega ((u_*^\lambda)^+)^{r-1} h dz \text{ for all } h \in W^{1,p}(\Omega).
\end{aligned}
\tag{22}
$$

In (22) we choose $h = -(u_*^\lambda)^- \in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
& \frac{c_1}{p-1}||D(u_*^\lambda)^-||_p^p + \int_\Omega (|\xi(z)|+1)((u_*^\lambda)^-)^p dz \leq 0 \\
& \text{(see Lemma 2.2 and hypothesis } H(\beta)) \\
\Rightarrow\ & u_*^\lambda \geq 0,\ u_*^\lambda \neq 0.
\end{aligned}
$$

It follows from (22) that u_*^λ is a positive solution of (20). The nonlinear regularity theory implies that $u_*^\lambda \in C_+ \backslash \{0\}$. Moreover, we have

$$
\operatorname{div} a(Du_*^\lambda(z)) \leq \left(||\xi||_\infty + c_{10}||u_*^\lambda||_\infty^{r-p}\right) u_*^\lambda(z) \text{ for almost all } z \in \Omega.
$$

Next, we show that this positive solution of (20) is unique. For this purpose we introduce the functional $l : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$
l(u) = \begin{cases} \displaystyle\int_\Omega G(Du^{\frac{1}{q}})dz + \frac{1}{p}\int_\Omega |\xi(z)|u^{\frac{p}{q}}dz + \frac{1}{p}\int_{\partial\Omega}\beta(z)u^{\frac{p}{q}}d\sigma \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \geq 0,\ u^{\frac{1}{q}} \in W^{1,p}(\Omega) \\ +\infty \qquad\qquad\qquad\qquad\qquad\quad \text{otherwise.} \end{cases}
$$

Here, $q \leq p$ is as in hypothesis $H(a)(iv)$. Let $\operatorname{dom} l = \{u \in L^1(\Omega) : l(u) < +\infty\}$ (the effective domain of $l(\cdot)$). Let $u_1, u_2 \in \operatorname{dom} l$ and consider $u = [(1-t)u_1 + tu_2]^{\frac{1}{q}}$ with $t \in [0, 1]$. From Lemma 1 of Diaz & Saa [6], we have

$$
\begin{aligned}
& |Du(z)| \leq [(1-t)|Du_1(z)^{\frac{1}{q}}|^q + t|Du_2(z)^{\frac{1}{q}}|^q]^{\frac{1}{q}} \text{ for almost all } z \in \Omega \\
\Rightarrow\ & G_0(|Du(z)|) \leq G_0([(1-t)|Du_1(z)^{\frac{1}{q}}|^p + t|Du_2(z)^{\frac{1}{q}}|^q]^{\frac{1}{q}}) \\
& \quad \text{(since } G_0(\cdot) \text{ is increasing)} \\
& \leq (1-t)G_0(|Du_1(z)^{\frac{1}{q}}|) + tG_0(|Du_2(z)^{\frac{1}{q}}|) \text{ for almost all } z \in \Omega \\
& \quad \text{(see hypotheses } H(a)(iv)) \\
\Rightarrow\ & G(Du(z)) \leq (1-t)G(Du_1(z)^{\frac{1}{q}}) + tG(Du_2(z)^{\frac{1}{q}}) \text{ for almost all } z \in \Omega, \\
\Rightarrow\ & \operatorname{dom} l \ni u \mapsto \textstyle\int_\Omega G(Du^{\frac{1}{q}})dz \text{ is convex.}
\end{aligned}
$$

Since $q \leq p$ and $\beta \geq 0$ (see hypotheses $H(\beta)$), we deduce that the mapping

$$
\operatorname{dom} l \ni u \mapsto \int_\Omega |\xi(z)|u^{\frac{p}{q}}dz + \int_{\partial\Omega}\beta(z)u^{\frac{p}{q}}d\sigma \text{ is convex.}
$$

Therefore, we conclude that $l(\cdot)$ is convex and by Fatou's lemma it is also lower semicontinuous.

Assume that v_*^λ is another positive solution for problem (20). Again, we can show that $v_*^\lambda \in D_+$. Let $h \in C^1(\overline{\Omega})$. For $|t| < 1$ small enough we have

$$
(u_*^\lambda)^q + th \in \operatorname{dom} l \text{ and } (v_*^\lambda)^q + th \in \operatorname{dom} l.
$$

It is easily seen that $l(\cdot)$ is Gâteaux differentiable at $(u_*^\lambda)^q$ and at $(v_*^\lambda)^q$ in the direction h. Using the chain rule and the nonlinear Green identity (see Gasinski &

Papageorgiou [8, p. 210]), we obtain

$$l'((u_*^\lambda)^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Du_*^\lambda) + |\xi(z)|(u_*^\lambda)^{p-1}}{(u_*^\lambda)^{q-1}} hdz$$
$$l'((v_*^\lambda)^q)(h) = \frac{1}{q}\int_\Omega \frac{-\operatorname{div} a(Dv_*^\lambda) + |\xi(z)|(v_*^\lambda)^{p-1}}{(v_*^\lambda)^{q-1}} hdz.$$

The convexity of $l(\cdot)$ implies the monotonicity of $l'(\cdot)$. Therefore

$$\begin{aligned} 0 \quad &\leq \int_\Omega \left[\frac{-\operatorname{div} a(Du_*^\lambda) + |\xi(z)|(u_*^\lambda)^{p-1}|}{(u_*^\lambda)^{q-1}} - \frac{-\operatorname{div} a(Dv_*^\lambda) + |\xi(z)|(v_*^\lambda)^{p-1}|}{(v_*^\lambda)^{q-1}}\right] \\ &\quad ((u_*^\lambda)^q - (v_*^\lambda)^q)dz \\ &= \int_\Omega c_{10}[(u_*^\lambda)^{r-q} - (v_*^\lambda)^{r-q}](u_*^\lambda)^q - (v_*^\lambda)^q)dz \text{ (see (20))} \\ \Rightarrow \quad & u_*^\lambda = v_*^\lambda \text{ (recall that } q \leq p < r). \end{aligned}$$

So, the positive solution $u_*^\lambda \in D_+$ of problem (20) is unique. □

Proposition 9. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold and $\lambda < \lambda^*$, then $u_*^\lambda \leq u$ for all $u \in S_\lambda$.*

Proof. From Proposition 6 we know that $\lambda \in \mathcal{L}$. Let $u \in S_\lambda \subseteq D_+$ (see Proposition 5). Again we fix $\eta > ||\xi||_\infty$ and consider the Carathéodory function $\vartheta : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$\vartheta(z,x) = \begin{cases} 0 & \text{if } x < 0 \\ c_9 x^{q-1} - c_{10}x^{r-1} + \eta x^{p-1} & \text{if } 0 \leq x \leq u(z) \\ k_9 u(z)^{q-1} - c_{10}u(z)^{r-1} + \eta u(z)^{p-1} & \text{if } u(z) < x. \end{cases} \tag{23}$$

We set $\Theta(z,x) = \int_0^x \vartheta(z,s)ds$ and consider the C^1-functional $\zeta : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\zeta(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u||_p^p - \int_\Omega \Theta(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

As before, $\zeta(\cdot)$ is coercive and sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_*^\lambda \in W^{1,p}(\Omega)$ such that

$$\zeta(\tilde{u}_*^\lambda) = \inf\{\zeta(u) : u \in W^{1,p}(\Omega)\}. \tag{24}$$

Since $q \leq p < r$, for $c_9, c_{10} > 0$ large enough as in Proposition 7, we have

$$\begin{aligned} &\zeta(\tilde{u}_*^\lambda) < 0 = \zeta(0), \\ \Rightarrow \quad &\tilde{u}_*^\lambda \neq 0. \end{aligned}$$

From (24) we have

$$\begin{aligned} &\zeta'(\tilde{u}_*^\lambda) = 0, \\ \Rightarrow \quad &\langle A(\tilde{u}_*^\lambda), h\rangle + \int_\Omega [\xi(z)+\eta]|\tilde{u}_*^\lambda|^{p-2}\tilde{u}_*^\lambda hdz + \int_{\partial\Omega} \beta(z)|\tilde{u}_*^\lambda|^{p-2}\tilde{u}_*^\lambda hd\sigma \\ &= \int_\Omega \vartheta(z,\tilde{u}_*^\lambda)hdz \text{ for all } h \in W^{1,p}(\Omega). \end{aligned} \tag{25}$$

Let $h = -(\tilde{u}_*^\lambda)^- \in W^{1,p}(\Omega)$ in (25). Then

$$\begin{aligned} &\frac{c_1}{p-1}||D(\tilde{u}_*^\lambda)^-||_p^p + \int_\Omega [\xi(z)+\eta]((\tilde{u}_*^\lambda)^-)^p dz \leq 0 \\ &\quad \text{(see Lemma 2.2, hypothesis } H(\beta), \text{ and (23))} \\ \Rightarrow \quad &\tilde{u}_*^\lambda \geq 0,\ \tilde{u}_*^\lambda \neq 0. \end{aligned}$$

Also, let $h = (\tilde{u}_*^\lambda - u)^+ \in W^{1,p}(\Omega)$ in (25). Then

$$\begin{aligned}
&\langle A(\tilde{u}_*^\lambda), (\tilde{u}_*^\lambda - u)^+\rangle + \int_\Omega [\xi(z) + \eta](\tilde{u}_*^\lambda)^{p-1}(\tilde{u}_*^\lambda - u)^+ dz \\
&+ \int_{\partial\Omega} \beta(z)(\tilde{u}_*^\lambda)^{p-1}(\tilde{u}_*^\lambda - u)^+ d\sigma \\
=& \int_\Omega (c_9 u^{q-1} - c_{10} u^{r-1} + \eta u^{p-1})(\tilde{u}_*^\lambda - u)^+ dz \text{ (see (23))} \\
\leq& \int_\Omega [(\lambda + \eta)u^{p-1} + f(z,u)](\tilde{u}_*^\lambda - u)^+ dz \text{ (see (19))} \\
=&\langle A(u), (\tilde{u}_*^\lambda - u)^+\rangle + \int_\Omega [\xi(z)+\eta]u^{p-1}(\tilde{u}_*^\lambda - u)dz + \int_{\partial\Omega} \beta(z)u^{p-1}(\tilde{u}_*^\lambda - u)^+ d\sigma \\
&\text{(since } u \in S_\lambda) \\
\Rightarrow& \tilde{u}_*^\lambda \leq u.
\end{aligned}$$

So, we have proved that

$$\tilde{u}_*^\lambda \in [0, u], \ \ \tilde{u}_*^\lambda \neq 0. \tag{26}$$

Then from (23), (25), (26) we infer that

$$\begin{aligned}
& \tilde{u}_*^\lambda \text{ is a positive solution of (20)}, \\
\Rightarrow \quad & \tilde{u}_*^\lambda = u_*^\lambda \in D_+ \text{ (see Proposition 8)}, \\
\Rightarrow \quad & u_*^\lambda \leq u \text{ for all } u \in S_\lambda \text{ (see (26))}.
\end{aligned}$$

The proof is complete. ☐

From Papageorgiou, Rădulescu & Repovš [21] (proof of Proposition 4), we know that the set S_λ is downward directed (that is, if $u_1, u_2 \in S_\lambda$, then we can find $u \in S_\lambda$ such that $u \leq u_1, u \leq u_2$).

Proposition 10. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold and $\lambda < \lambda^*$, then S_λ admits a smallest element $\overline{u}_\lambda \in D_+$, that is,*

$$\overline{u}_\lambda \leq u \textit{ for all } u \in S_\lambda.$$

Proof. According to Lemma 3.10 of Hu & Papageorgiou [11, p. 178], we can find $\{u_n\}_{n\geq 1} \subseteq S_\lambda$ such that

$$\inf S_\lambda = \inf_{n\geq 1} u_n.$$

Moreover, since S_λ is downward directed, we can choose $\{u_n\}_{n\geq 1} \subseteq S_\lambda$ to be decreasing. We have

$$\begin{aligned}
&\langle A(u_n), h\rangle + \int_\Omega \xi(z)u_n^{p-1} h dz + \int_{\partial\Omega} \beta(z)u_n^{p-1} h d\sigma = \lambda \int_\Omega u_n^{p-1} h dz + \\
&\int_\Omega f(z, u_n) h dz \text{ for all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N}
\end{aligned} \tag{27}$$

$$0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \tag{28}$$

From (27), (28), we infer that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} \overline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to \overline{u}_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{29}$$

In (27) we choose $h = u_n - \overline{u}_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (29). Then

$$\begin{aligned}
& \lim_{n\to\infty} \langle A(u_n), u_n - \overline{u}_\lambda\rangle = 0, \\
\Rightarrow \quad & u_n \to \overline{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}.
\end{aligned} \tag{30}$$

In (27) we pass to the limit as $n \to \infty$ and use (30). Then

$$\begin{aligned}&\langle A(\overline{u}_\lambda), h\rangle + \int_\Omega \xi(z)\overline{u}_\lambda^{p-1} h dz + \int_{\partial\Omega} \beta(z)\overline{u}_\lambda^{p-1} h d\sigma = \lambda \int_\Omega \overline{u}_\lambda^{p-1} h dz \\ &+ \int_\Omega f(z, \overline{u}_\lambda) h dz \text{ for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{31}$$

Moreover, from Proposition 8 we have

$$\begin{aligned}& u_*^\lambda \leq u_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow\quad & u_*^\lambda \leq \overline{u}_\lambda, \text{hence } \overline{u}_\lambda \neq 0, \\ \Rightarrow\quad & \overline{u}_\lambda \in S_\lambda \text{ (see (31)) and } \overline{u}_\lambda = \inf S_\lambda.\end{aligned}$$

The proof is now complete. □

In the next proposition we establish the monotonicity and continuity properties of the map $\mathcal{L} \ni \lambda \mapsto u_\lambda \in C^1(\overline{\Omega})$.

Proposition 11. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold, then the map $\lambda \mapsto \overline{u}_\lambda$ from $\mathcal{L}$ into $C^1(\overline{\Omega})$ is:*

(a) strictly increasing in the sense that

$$\vartheta < \lambda \Rightarrow \overline{u}_\lambda - \overline{u}_\vartheta \in \operatorname{int} C_+;$$

(b) left continuous, that is, if $\lambda_n \to \lambda^-$ with $\lambda \in \mathcal{L}$, then $\bar{u}_{\lambda_n} \to \bar{u}_\lambda$ in $C^1(\overline{\Omega})$.

Proof. (a) Let $\vartheta < \lambda \in \mathcal{L}$. Let $\bar{u}_{\lambda_n} \in S_\lambda \subseteq D_+$ be the minimal solution of (P_λ) (see Proposition 9). From Proposition 6 and its proof we know that $\vartheta \in \mathcal{L}$ and we can find $u_\vartheta \in S_\vartheta \subseteq D_+$ such that $u_\vartheta \leq \bar{u}_\lambda$ (see (16)). Therefore $\bar{u}_\vartheta \leq \bar{u}_\lambda$.

Let $\rho = ||\bar{u}_\lambda||_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)_1(iv)$. We can always take $\hat{\xi}_\rho > ||\xi||_\infty$. Then

$$\begin{aligned}& -\operatorname{div} a(D\bar{u}_\vartheta) + [\xi(z) + \hat{\xi}_\rho]\bar{u}_\vartheta^{p-1} \\ = \quad & \vartheta \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1} \\ = \quad & \lambda \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1} - (\lambda - \vartheta)\bar{u}_\vartheta^{p-1} \\ \leq \quad & \lambda \bar{u}_\vartheta^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1} - (\lambda - \vartheta) m_\vartheta^{p-1} \text{ with } m_\vartheta = \min_{\overline{\Omega}} \bar{u}_\vartheta > 0 \\ & \text{(recall that } \bar{u}_\vartheta \in D_+) \\ < \quad & \lambda \bar{u}_\lambda^{p-1} + f(z, \bar{u}_\lambda) + \hat{\xi}_\rho \bar{u}_\lambda^{p-1} \text{ (since } \bar{u}_\vartheta \leq \bar{u}_\lambda) \\ = \quad & -\operatorname{div} a(D\bar{u}_\lambda) + [\xi(z) + \hat{\xi}_\rho]\bar{u}_\lambda^{p-1} \text{ for almost all } z \in \Omega.\end{aligned}$$

Let

$$h_1(z) = \vartheta \bar{u}_\lambda^{p-1} + f(z, \bar{u}_\vartheta) + \hat{\xi}_\rho \bar{u}_\vartheta^{p-1}$$

and

$$h_2(z) = \lambda \bar{u}_\lambda^{p-1} + f(z, \bar{\lambda}) + \hat{\xi}_\rho \bar{u}_\lambda^{p-1}.$$

Evidently, $h_1, h_2 \in L^\infty(\Omega)$ (see hypothesis $H(f)_1(i)$) and

$$0 < (\lambda - \vartheta) m_\vartheta^{p-1} \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega.$$

So, we can apply Proposition 2 and conclude that $\bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+$. This proves that the mapping $\lambda \mapsto \bar{u}_\lambda$ is strictly increasing.

(b) Let $\lambda_n \to \lambda^-$ with $\lambda \in \mathcal{L}$. We set $\bar{u}_n = \bar{u}_{\lambda_n} \in S_{\lambda_n} \subseteq D_+$ for all $n \in \mathbb{N}$. Evidently, $\{\bar{u}_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$\bar{u}_n \xrightarrow{w} u_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } \bar{u}_n \to u_\lambda \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega). \tag{32}$$

We have

$$\langle A(\bar{u}_n), h\rangle + \int_\Omega \xi(z)\bar{u}_n^{p-1} h dz + \int_{\partial\Omega} \beta(z)\bar{u}_n^{p-1} h d\sigma = \int_\Omega [\lambda_n \bar{u}_n^{p-1} + f(z, \bar{u}_n)] h dz \tag{33}$$

for all $h \in W^{1,p}(\Omega),\ n \in \mathbb{N}$.

In (33) we choose $h = \bar{u}_n - u_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (32). Then

$$\begin{aligned} &\lim_{n\to\infty} \langle A(\bar{u}_n), \bar{u}_n - u_\lambda\rangle = 0, \\ \Rightarrow\quad &\bar{u}_n \to u_\lambda \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}. \end{aligned} \tag{34}$$

So, if in (32) we pass to the limit as $n \to \infty$ and use (34), then we can infer that $u_\lambda \in S_\lambda \subseteq D_+$. On account of (34) and Proposition 7 of Papageorgiou & Rădulescu [20], we can find $c_{11} > 0$ such that

$$||\bar{u}_n||_\infty \le c_{11} \text{ for all } n \in \mathbb{N}.$$

Then the nonlinear regularity theory of Lieberman [13] implies that there exist $\tau \in (0,1)$ and $c_{12} > 0$ such that

$$\bar{u}_n \in C^{1,\tau}(\overline{\Omega}) \text{ and } ||\bar{u}_n||_{C^{1,\tau}(\overline{\Omega})} \le c_{12} \text{ for all } n \in \mathbb{N}.$$

The existence of a compact embedding of $C^{1,\tau}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and (34), imply that

$$\bar{u}_n \to u_\lambda \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{35}$$

We show that $u_\lambda = \bar{u}_\lambda$. Arguing by contradiction, suppose that $u_\lambda \neq \bar{u}_\lambda$. Then we can find $z_0 \in \overline{\Omega}$ such that

$$\begin{aligned} &\bar{u}_\lambda(z_0) < u_\lambda(z_0), \\ \Rightarrow\quad &\bar{u}_\lambda(z_0) < \bar{u}_n(z_0) \text{ for all } n \ge n_0 \text{ (see (35))}, \end{aligned}$$

which contradicts (a). Therefore the mapping $\lambda \mapsto \bar{u}_\lambda$ is left continuous. □

Now we ready to show the non-admissibility of λ^*.

Proposition 12. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold, then $\lambda^* \notin \mathcal{L}$.*

Proof. Arguing by contradiction, suppose that $\lambda^* \in \mathcal{L}$. According to Proposition 10, problem (P_λ) admits a smallest positive solution $\bar{u}_* = \bar{u}_{\lambda^*} \in D_+$. Let $\lambda > \lambda^*, \eta > ||\xi||_\infty$ and consider the Carathéodory function

$$\gamma_\lambda(z,x) = \begin{cases} (\lambda+\eta)\bar{u}_*(z)^{p-1} + f(z, \bar{u}_*(z)) & \text{if } x \le \bar{u}_*(z) \\ (\lambda+\eta)x^{p-1} + f(z,x) & \text{if } \bar{u}_*(z) < x. \end{cases} \tag{36}$$

We set $\Gamma_\lambda(z,x) = \int_0^x \gamma_\lambda(z,s)ds$ and consider the C^1-functional $\tilde{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\varphi}_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u||_p^p - \int_\Omega \Gamma_\lambda(z,u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

As in the proof of Proposition 5, using hypothesis $H(f)_1(ii)$, we show that $\tilde{\varphi}_\lambda(\cdot)$ is coercive. Moreover, $\tilde{\varphi}_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\begin{aligned}
& \tilde{\varphi}_\lambda(u_\lambda) = \inf\{\tilde{\varphi}_\lambda(u) : u \in W^{1,p}(\Omega)\}, \\
\Rightarrow\ & \tilde{\varphi}'_\lambda(u_\lambda) = 0, \\
\Rightarrow\ & \langle A(u_\lambda), h\rangle + \int_\Omega [\xi(z)+\eta]|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma \\
=\ & \int_\Omega \gamma_\lambda(z, u_\lambda) h dz \qquad (37)\\
& \text{for all } h \in W^{1,p}(\Omega).
\end{aligned}$$

In (37) we choose $h = (\bar{u}_* - u_\lambda)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
& \left\langle A(u_\lambda), (\bar{u}_* - u_\lambda)^+\right\rangle + \int_\Omega [\xi(z)+\eta]|u_\lambda|^{p-2}u_\lambda(\bar{u}_* - u_\lambda)^+ dz \\
& \qquad + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda(\bar{u}_* - u_\lambda)^+ d\sigma \\
=\ & \int_\Omega [(\lambda+\eta)\bar{u}_*^{p-1} + f(z,\bar{u}_*)](\bar{u}_* - u_\lambda)^+ dz \text{ (see (36))} \\
\geq\ & \int_\Omega [(\lambda^*+\eta)\bar{u}_*^{p-1} + f(z,\bar{u}_*)](\bar{u}_* - u_\lambda)^+ dz \text{ (since } \lambda > \lambda^*) \\
=\ & \left\langle A(\bar{u}_*), (\bar{u}_* - u_\lambda)^+\right\rangle + \int_\Omega [\xi(z)+\eta]\bar{u}_*^{p-1}(\bar{u}_* - u_\lambda)^+ dz \\
& + \int_{\partial\Omega} \beta(z)\bar{u}_*^{p-1}(\bar{u}_* - u_\lambda)^+ d\sigma \\
& (\text{since } \bar{u}_* \in S_{\lambda^*}), \\
\Rightarrow\ & \bar{u}_* \leq u_\lambda.
\end{aligned}$$

Then from (36) and (37) it follows that $\bar{u}_\lambda \in S_\lambda$ and so $\lambda \in \mathcal{L}$, a contradiction. This proves that $\lambda^* \notin \mathcal{L}$. □

So, summarizing the situation for problem (P_λ) when the perturbation $f(z,\cdot)$ is $(p-1)$-sublinear, we can state the following theorem.

Theorem 3.1. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_1$ hold, then there exists $\lambda^* < +\infty$ such that*

(a) for every $\lambda \geq \lambda^*$, problem (P_λ) has no positive solutions;
(b) for every $\lambda < \lambda^*$, problem (P_λ) has at least one positive solution $u_\lambda \in D_+$;
(c) for every $\lambda < \lambda^*$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in D_+$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $(-\infty, \lambda^*)$ into $C^1(\overline{\Omega})$ is
- *strictly increasing, that is, if $\vartheta < \lambda < \lambda^*$, then*
$$\bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+;$$
- *left continuous, that is, if $\lambda_n \to \lambda^-$ and $\lambda < \lambda^*$, then $\bar{u}_{\lambda_n} \to \bar{u}_\lambda$ in $C^1(\overline{\Omega})$.*

In the special case of the p-Laplacian (that is, $a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$ with $1 < p < \infty$), we can identify λ^* as $\hat{\lambda}_1(p,\xi,\beta)$, when $f(z,x) > 0$ for almost all $z \in \Omega$, and for all $x > 0$.

So, we consider the following nonlinear Robin problem:

$$\left\{\begin{array}{l} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{p-1} + f(z,u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \text{ on } \partial\Omega, u > 0, \lambda \in \mathbb{R}, 1 < p < \infty. \end{array}\right\} \qquad (PL_\lambda)$$

Proposition 13. *Assume that hypotheses $H(\xi), H(\beta)$ hold and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

- *for almost all $z \in \Omega$, $f(z,0) = 0$ and $f(z,x) > 0$ for all $x > 0$;*
- *$f(z,x) \leq a(z)(1 + x^{p^*-1})$ for almost all $z \in \Omega$, and for all $x \geq 0$, with $a \in L^\infty(\Omega)$.*

Then for all $\lambda \geq \hat{\lambda}_1(p,\xi,\beta)$, $S_\lambda = \emptyset$.

Proof. Arguing by contradiction, suppose that $S_\lambda \neq \emptyset$ and let $u \in S_\lambda$. The nonlinear regularity theory implies that $u \in D_+$. Let $\hat{u}_1 = \hat{u}_1(p,\xi,\beta) \in D_+$ (see Proposition 4). We consider the function

$$R(\hat{u}_1, u)(z) = |D\hat{u}_1(z)|^p - |Du(z)|^{p-2}(Du(z), D\left(\frac{\hat{u}_1^p}{u^{p-1}}\right)(z))_{\mathbb{R}^N}.$$

The nonlinear Picone identity of Allegretto & Huang [2] implies that

$$\begin{aligned}
& 0 \leq R(\hat{u}_1, u)(z) \text{ for almost all } z \in \Omega, \\
\Rightarrow \quad & 0 \leq \int_\Omega R(\hat{u}_1, u) dz \\
& = ||D\hat{u}_1||_p^p - \int_\Omega |Du|^{p-2}\left(Du, D\left(\frac{\hat{u}_1^p}{u^{p-1}}\right)\right)_{\mathbb{R}^N} dz \\
& = ||D\hat{u}_1||_p^p - \int_\Omega (-\Delta_p u)\frac{\hat{u}_1^p}{u^{p-1}} dz + \int_{\partial\Omega} \beta(z) u^{p-1} \frac{\hat{u}_1^p}{u^{p-1}} d\sigma
\end{aligned}$$

(using the nonlinear Green identity, see Gasinski & Papageorgiou [8, p. 211])

$$= ||D\hat{u}_1||_p^p + \int_\Omega \xi(z)\hat{u}_1^{p-1} dz + \int_{\partial\Omega} \beta(z)\hat{u}_1^p d\sigma - \lambda - \int_\Omega f(z,u)\frac{\hat{u}_1^p}{u^{p-1}}$$

(see (PL_λ) and recall that $||\hat{u}_1||_p = 1$)

$< \mu(\hat{u}_1) - \lambda$ (recall that $f(z,x) > 0$ for almost all $z \in \Omega$, and for all $x > 0$)

$= \hat{\lambda}_1 - \lambda < 0,$

a contradiction. Therefore $S_\lambda = \emptyset$ and so $\lambda \notin \mathcal{L}$ for all $\lambda \geq \hat{\lambda}_1(p,\xi,\beta)$. □

Moreover, reasoning as in the proof of Proposition 5, via the direct method of the calculus of variations, we obtain the following result. Note that now in hypothesis $H(a)(iv)$ we take $q = p$.

Proposition 14. *If hypotheses $H(\xi), H(\beta), H(f)_1$ hold and $\lambda < \hat{\lambda}_1 = \hat{\lambda}_1(p,\xi,\beta)$, then $\lambda \in \mathcal{L}$.*

We introduce the following stronger version of hypotheses $H(f)_1$.

$H(f)_1'$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z,0) = 0$, $f(z,x) > 0$ for all $x > 0$ and hypotheses $H(f)_1'(i)$, (ii), (iii), (iv) are the same as the corresponding hypotheses $H(f)_1(i)$, (ii), (iii), (iv).

Using this stronger version of $H(f)_1$ and combining Propositions 13 and 14 we have the following theorem concerning the positive solutions of (PL_λ) as the parameter $\lambda \in \mathbb{R}$ varies.

Theorem 3.2. *If hypotheses $H(\xi), H(\beta), H(f)_1'$ hold, then*

(a) for every $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p,\xi,\beta)$, problem ($PL_\lambda$) has no positive solutions;
(b) for every $\lambda < \hat{\lambda}$, problem (PL_λ) has at least one positive solution $u_\lambda \in D_+$;

(c) *for every* $\lambda < \hat{\lambda}$, *problem* (PL_λ) *has a smallest positive solution* $\bar{u}_\lambda \in D_+$ *and the map* $\lambda \mapsto \bar{u}_\lambda$ *from* $(-\infty, \hat{\lambda}_1)$ *into* $C^1(\overline{\Omega})$ *is*
- *strictly increasing (that is,* $\vartheta < \lambda < \hat{\lambda}_1 \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+$*);*
- *left continuous.*

If we further restrict the conditions on the perturbation $f(z,x)$, we can have uniqueness for the positive solution.

The new hypotheses on $f(z,x)$ are the following:

$H(f)_1''$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z,0) = 0, f(z,x) > 0$ for all $x > 0$, hypotheses $H(f)_1''(i)$, (ii), (iii), (iv) are the same as the corresponding hypotheses $H(f)_1(i)$, (ii), (iii), (iv) and

(v) if $x - y \geq m > 0$, then $\frac{f(z,y)}{y^{p-1}} - \frac{f(z,x)}{x^{p-1}} \geq c_m > 0$ for almost all $z \in \Omega$.

Proposition 15. *If hypotheses* $H(\xi), H(\beta), H(f)_1''$ *hold and* $\lambda < \hat{\lambda}_1$, *then problem* ($P_\lambda$) *admits a unique solution* $\bar{u}_\lambda \in D_+$.

Proof. By Theorem 3.2 we already have a positive solution $\bar{u}_\lambda \in D_+$. Suppose that $\bar{u}_\lambda$ is another positive solution of (P_λ). Again we have that $\bar{v}_\lambda \in D_+$. By Proposition 2.1 of Marano & Papageorgiou [15], we can find $t > 0$ such that

$$t\bar{v}_\lambda \leq \bar{u}_\lambda. \tag{38}$$

Let $t > 0$ be the biggest real for which (38) holds. Suppose that $t < 1$. Also, let $\rho = ||\bar{u}_\lambda||_\infty$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)_1^{''}(iv)$. We can always assume that $\hat{\xi}_\rho > ||\xi||_\infty$. Also let $\bar{m}_\lambda = \min_{\overline{\Omega}} \bar{v}_\lambda > 0$. We have

$$\begin{aligned}
& -\Delta_p(t\bar{v}_\lambda) + [\xi(z) + \hat{\xi}_\rho](t\bar{v}_\lambda)^{p-1} \\
= \; & t^{p-1}(-\Delta_p\bar{v}_\lambda + [\xi(z) + \hat{\xi}_\rho]\bar{v}_\lambda^{p-1}) \\
= \; & t^{p-1}(\lambda\bar{v}_\lambda^{p-1} + f(z,\bar{v}_\lambda) + \hat{\xi}_\rho\bar{v}_\lambda^{p-1}) \text{ (since } \bar{v}_\lambda \in S_\lambda) \\
\leq \; & \lambda(t\bar{v}_\lambda)^{p-1} + f(z,t\bar{v}_\lambda) + \hat{\xi}_\rho(t\bar{v}_\lambda)^{p-1} - (1-t)\bar{m}_\lambda^{p-1} \\
& \text{(see hypothesis } H(f)_1^{''}(v) \text{ and recall that } t < 1) \\
< \; & \lambda\bar{u}_\lambda^{p-1} + f(z,\bar{u}_\lambda) + \hat{\xi}_\rho\bar{u}_\lambda^{p-1} \text{ (see (38) and hypothesis } H(f)_1^{''}(iv)) \\
= \; & -\Delta_p\bar{u}_\lambda + [\xi(z) + \hat{\xi}_\rho]\bar{u}_\lambda^{p-1} \text{ for almost all } z \in \Omega \text{ (since } \bar{u}_\lambda \in S_\lambda), \\
\Rightarrow \quad & \bar{u}_\lambda - t\bar{v}_\lambda \in \operatorname{int} C_+ \text{ (see Proposition 2)},
\end{aligned}$$

which contradicts the maximality of $t > 0$. Therefore $t \geq 1$ and we have

$$\bar{v}_\lambda \leq \bar{u}_\lambda \text{ (see (38))}.$$

Interchanging the roles of $\bar{u}_\lambda$ and $\bar{v}_\lambda$ in the above argument, we obtain

$$\begin{aligned}
& \bar{u}_\lambda \leq \bar{v}_\lambda, \\
\Rightarrow \quad & \bar{u}_\lambda = \bar{v}_\lambda.
\end{aligned}$$

This proves the uniqueness of the positive solution of problem (PL_λ). ☐

So, we can state the following existence and uniqueness theorem for problem (PL_λ).

Theorem 3.3. *If hypotheses* $H(\xi), H(\beta), H(f)_1^{''}$ *hold, then*

(a) *for every* $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p,\xi,\beta)$, *problem* ($PL_\lambda$) *has no positive solutions;*

(b) *for every* $\lambda < \hat{\lambda}_1$, *problem* ($PL_\lambda$) *has a unique positive solution* $\bar{u}_\lambda \in D_+$ *and the map* $\lambda \mapsto \bar{u}_\lambda$ *from* $(-\infty, \hat{\lambda}_1)$ *into* $C^1(\overline{\Omega})$ *is*
- *strictly increasing (that is,* $\vartheta < \lambda < \hat{\lambda}_1 \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+$*);*
- *left continuous.*

For the general nonhomogeneous problem, to have uniqueness, we need to set $\xi \equiv 0$. So, we consider the problem:

$$\left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) = \lambda u(z)^{p-1} + f(z, u(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n_a} + \beta(z) u^{p-1} = 0 \text{ on } \partial\Omega, u > 0, \lambda \in \mathbb{R}, 1 < p < \infty. \end{array} \right\} \qquad (P'_\lambda)$$

Then reasoning as in the proof of Proposition 8 (see also Fragnelli, Mugnai & Papageorgiou [7, Theorem 7]), we have uniqueness of the positive solution and we can formulate the following theorem.

Theorem 3.4. *If hypotheses* $H(a), H(\beta), H(f)_1''$ *hold, then there exists* $\lambda^* \in \mathbb{R}$ *such that*

(a) *for every* $\lambda \geq \lambda^*$, *problem* (P'_λ) *has no positive solutions;*
(b) *for every* $\lambda < \lambda^*$, *problem* (P'_λ) *has a unique positive solution* $\bar{u}_\lambda \in D_+$ *and the map* $\lambda \mapsto \bar{u}_\lambda$ *from* $(-\infty, \lambda^*)$ *into* $C^1(\overline{\Omega})$ *is*
- *strictly increasing (that is,* $\vartheta < \lambda < \lambda^* \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+$*);*
- *left continuous;*

4. $(p-1)$**-superlinear perturbation.** In this section we examine what happens in problem (P_λ) when the perturbation $f(z, \cdot)$ is $(p-1)$-superlinear. We do not assume that $f(z, \cdot)$ satisfies the usual (for "superlinear" problems) "Ambrosetti-Rabinowitz condition" (the "AR-condition" for short). Instead, we employ a less restrictive condition involving the function

$$d(z, x) = f(z, x)x - pF(z, x) \text{ for all } (z, x) \in \Omega \times \mathbb{R}.$$

In this way we incorporate in our framework $(p-1)$-superlinear functions with "slower" growth near $+\infty$, which fail to satisfy the AR-condition.

So, we introduce the following condition on the perturbation $f(z, x)$.

$H(f)_2$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for almost all $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)(1 + x^{r-1})$ for almost all $z \in \Omega$, and for all $x \geq 0$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
(ii) if $F(z, x) = \int_0^x f(z, s) ds$, then $\lim\limits_{x \to +\infty} \frac{F(z,x)}{x^p} = +\infty$ uniformly for almost all $z \in \Omega$;
(iii) if $d(z, x) = f(z, x)x - pF(z, x)$, then $d(z, x) \leq d(z, y) + \nu(z)$ for almost all $z \in \Omega$, and for all $0 \leq x \leq y$ with $\nu(\cdot) \in L^1(\Omega)$;
(iv) $\lim\limits_{x \to 0^+} \frac{f(z,x)}{x^{\tau-1}} = 0$ uniformly for almost all $z \in \Omega$, with $1 < \tau < q$, $q \leq p$ as in $H(a)(iv)$ and there exist $s \in (\tau, q)$, $\delta_0 > 0$ such that $\tilde{c}_0 x^{s-1} \leq f(z, x)$ for almost all $z \in \Omega$, $x \in [0, \delta_0]$ with $\tilde{c}_0 > 0$;
(v) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for almost all $z \in \Omega$ the mapping $x \mapsto f(z, x) + \hat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remark 4. Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, we may assume without any loss of generality, as we did in the sublinear case, that $f(z,x)=0$ for almost all $z\in\Omega$, all $x\le 0$. Hypotheses $H(f)_2(ii),(iii)$ imply that

$$\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}=+\infty \text{ uniformly for almost all } z\in\Omega. \tag{39}$$

So, the perturbation $f(z,\cdot)$ is $(p-1)$-superlinear. Usually for such problems, the superlinerity is expressed through the AR-condition, which says that there exist $\tau>p$ and $M>0$ such that

$$\begin{aligned}&0<\tau F(z,x)\le f(z,x)x \text{ for almost all } z\in\Omega,\\ &\text{and for all } x\ge M \text{ whereess} \inf_{\Omega} F(\cdot,M)>0.\end{aligned} \tag{40}$$

Here we have a unilateral version of the AR-condition, since $f(z,\cdot)_{(-\infty,0]}=0$. Integrating (39) we obtain the more general condition

$$c_{13}x^\tau\le F(z,x) \text{ for almost all } z\in\Omega, \text{ for all } x\ge M, \text{ and for some } c_{13}>0. \tag{41}$$

Evidently, (40) and (41) imply that (39) holds. Using the AR-condition (40) we can easily verify the C-condition for the energy functional. However, from (41) we see that the AR-condition is rather restrictive. It excludes from consideration superlinear functions with slower growth near $+\infty$ (see the examples below). We have replaced the AR-condition by hypotheses $H(f)_2(ii),(iii)$, which incorporate in our framework such functions. Hypothesis $H(f)_2(iii)$ is a quasi-monotonicity condition on $d(z,\cdot)$ on $\mathbb{R}_+$. This hypothesis is a slightly more general version of a condition used by Li & Yang [12], who compared this condition with other superlinearity conditions that can be found in the literature.

Examples. The following functions satisfy hypotheses $H(f)_2$. For the sake of simplicity we drop the z-dependence.

$$f_1(x)=\begin{cases}0 & \text{if } x<0\\ x^{\tau-1}-2x^{\vartheta-1} & \text{if } 0\le x\le 1\\ x^{r-1}-2x^{p-1} & \text{if } 1<x\end{cases}$$

with $1<\tau<\vartheta<p<r$

$$f_2(x)=\begin{cases}0 & \text{if } x<0\\ x^{\tau-1}-2x^{\vartheta-1} & \text{if } 0\le x\le 1\\ x^{p-1}(\ln x-1) & \text{if } 1<x\end{cases}$$

with $1<\tau<\vartheta<p$. Note that f_1 satisfies the AR-condition, whereas f_2 does not. Also, both functions may be sign-changing.

As before, we denote

$$\begin{aligned}\mathcal{L}&=\{\lambda>0: \text{problem } (P_\lambda) \text{ has a positive solution}\},\\ S_\lambda&=\text{the set of all positive solutions of problem } (P_\lambda).\end{aligned}$$

Proposition 16. *If hypotheses $H(a),H(\xi),H(\beta),H(f)_2$ hold, then $\mathcal{L}\neq\emptyset$ and $S_\lambda\subseteq D_+$.*

Proof. Let $\eta>||\xi||_\infty$ and consider the functional $\psi_\lambda: W^{1,p}(\Omega)\to\mathbb{R}$ defined by

$$\psi_\lambda(u)=\frac{1}{p}\mu(u)+\frac{\eta}{p}||u^-||_p^p-\frac{\lambda}{p}||u^+||_p^p-\int_\Omega F(z,u^+)dz \text{ for all } u\in W^{1,p}(\Omega).$$

Hypotheses $H(f)_2(i),(iv)$ imply that given $\epsilon>0$, we can find $c_{14}=c_{14}(\epsilon)>0$ such that

$$F(z,x)\le\frac{\epsilon}{p}x^\tau+c_{14}x^r \text{ for almost all } z\in\Omega, \text{ and for all } x\ge 0. \tag{42}$$

Then for $\lambda < 0$, with $|\lambda| > ||\xi||_\infty$, we have

$$\begin{aligned} \psi_\lambda(u) \quad &\geq \quad \frac{1}{p}\mu(u^-) + \frac{\eta}{p}||u^-||_p^p + \frac{1}{p}\mu(u^+) + \frac{|\lambda|}{p}||u^+||_p^p - \epsilon c_{15}||u||^\tau - c_{16}||u||^\tau \\ &\quad\quad \text{with } c_{15}, c_{16} > 0 \text{ (see (42))} \\ &\geq \quad \left[c_{17} - (\epsilon c_{15}||u||^{\tau-p} + c_{16}||u||^{r-p})\right] ||u||^p \text{ for some } c_{17} > 0. \end{aligned} \tag{43}$$

Let $k_0(t) = \epsilon c_{15} t^{\tau-p} + c_{16} t^{r-p}$ $t \geq 0$. Since $1 < \tau < p < r$, we see that $k_0(t) \to +\infty$ as $t \to 0^+$ and as $t \to +\infty$. So, we can find $t_0 > 0$ such that $k_0(t_0) = \min\limits_{t>0} k_0$. We have

$$k_0'(t_0) = 0, \qquad \Rightarrow \quad t_0 = \left[\frac{\epsilon c_{15}(p-\tau)}{c_{16}(r-p)}\right]^{\frac{1}{r-\tau}}.$$

Then $k_0(t_0) \to 0^+$ as $\epsilon \to 0^+$. So, it follows from (43) that we can find small enough $\rho \in (0,1)$ such that

$$0 < \inf\{\psi_\lambda(u) : ||u|| = \rho\} = m_\rho^\lambda. \tag{44}$$

Hypothesis $H(f)_2(ii)$ implies that if $u \in D_+$, then

$$\psi_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{45}$$

Claim 1. For every $\lambda \in \mathbb{R}, \psi_\lambda(\cdot)$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$|\psi_\lambda(u_n)| \leq M_1 \text{ for some } M_1 > 0, \text{ and for all } n \in \mathbb{N}, \tag{46}$$

$$(1 + ||u_n||)\psi_\lambda'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty. \tag{47}$$

From (47) we have

$$\begin{aligned} &|\langle A(u_n), h\rangle + \int_\Omega \xi(z)|u_n|^{p-2}u_n h dz + \int_{\partial\Omega} \beta(z)|u_n|^{p-2} h d\sigma - \eta \int_\Omega (u_n^-)^{p-1} h dz \\ &\quad - \int_\Omega \lambda (u_n^+)^{p-1} h dz - \int_\Omega f(z, u_n^+) h dz| \leq \frac{\epsilon_n ||h||}{1 + ||u_n||} \\ &\text{for all } h \in W^{1,p}(\Omega), \text{ with } \epsilon_n \to 0^+. \end{aligned} \tag{48}$$

In (48) we choose $h = -u_n^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} &\mu(u_n^-) + \eta||u_n^-||_p^p \leq \epsilon_n \text{ for all } n \in \mathbb{N}, \\ \Rightarrow \quad & c_{18}||u_n^-||^p \leq \epsilon_n \text{ for some } c_{18} > 0, \text{ and for all } n \in \mathbb{N} \text{ (recall that } \eta > ||\xi||_\infty), \\ \Rightarrow \quad & u_n^- \to 0 \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{49}$$

Next, in (48) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\mu(u_n^+) + \lambda||u_n^+||_p^p + \int_\Omega f(z, u_n^+) u_n^+ dz \leq \epsilon_n \text{ for all } n \in \mathbb{N}. \tag{50}$$

From (46) and (49) we have

$$\mu(u_n^+) - \lambda||u_n^+||_p^p - \int_\Omega pF(z, u_n^+) dz \leq M_2 \text{ for some } M_2 > 0, \text{ and for all } n \in \mathbb{N}. \tag{51}$$

We add (50), (51) and obtain

$$\int_\Omega d(z, u_n^+) dz \leq M_3 \text{ for some } M_3 > 0, \text{ and for all } n \in \mathbb{N}. \tag{52}$$

We will use (52) to show that $\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Arguing by contradiction, suppose that $||u_n^+|| \to \infty$. We set $y_n = \frac{u_n^+}{||u_n^+||}$ for all $n \in \mathbb{N}$. We have $||y_n|| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty. \tag{53}$$

First, we assume that $y \neq 0$. Let $\Omega_0 = \{z \in \Omega : y(z) = 0\}$. Then $|\Omega \backslash \Omega_0|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$) and $u_n^+(z) \to +\infty$ for almost all $z \in \Omega\backslash\Omega_0$ as $n \to \infty$. Hence hypothesis $H(f)_2(ii)$ implies that

$$\frac{F(z, u_n^+(z))}{||u_n^+||^p} = \frac{F(z, u_n^+(z))}{u_n^+(z)^p} y_n(z)^p \to +\infty \text{ for almost all } z \in \Omega\backslash\Omega_0 \text{ as } n \to \infty,$$

$$\Rightarrow \int_\Omega \frac{F(z, u_n^+)}{||u_n^+||^p} dz \to +\infty \text{ as } n \to +\infty \text{ (by Fatou's lemma).} \tag{54}$$

Corollary 1 and hypothesis $H(a)(iv)$ imply that

$$G(y) \leq c_{19}(|y|^q + |y|^p) \text{ for some } c_{19} > 0, \text{ and for all } y \in \mathbb{R}^N. \tag{55}$$

From (46) and (49), we have

$$-\int_\Omega G(Du_n^+)dz - \frac{1}{p}\int_\Omega \xi(z)(u_n^+)^p dz - \frac{1}{p}\int_{\partial\Omega} \beta(z)(u_n^+)^p d\sigma + \frac{\lambda}{p}||u_n^+||_p^p +$$
$$\int_\Omega F(z, u_n^+)dz \leq M_4 \text{ for some } M_4 > 0, \text{ and for all } n \in \mathbb{N}$$
$$\Rightarrow \int_\Omega \frac{F(z, u_n^+)}{||u_n^+||^p} dz \leq M_5 \text{ for some } M_5 > 0, \text{ and for all } n \in \mathbb{N} \tag{56}$$

(see (53), (55) and hypotheses $H(\xi), H(\beta)$).

Comparing (54) and (56), we get a contradiction.

So, we assume that $y = 0$. We consider the C^1-functional $\hat{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\psi}_\lambda(u) = \frac{c_1}{p(p-1)}||Du||_p^p + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{1}{p}\int_{\partial\Omega} \beta(z)|u|^p d\sigma + \frac{\eta}{p}||u^-||_p^p$$
$$-\frac{\lambda}{p}||u^+||_p^p - \int_\Omega F(z, u^+)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently, $\hat{\psi}_\lambda \leq \psi_\lambda$ (see Corollary 1).

We define $\vartheta_n(t) = \hat{\psi}_\lambda(tu_n^+)$ for all $t \in [0, 1]$, and for all $n \in \mathbb{N}$. Let $t_n \in [0, 1]$ be such that

$$\vartheta_n(t_n) = \max_{0\leq t\leq 1} \vartheta_n(t) = \max_{0\leq t\leq 1} \hat{\psi}_\lambda(tu_n^+) \text{ for all } n \in \mathbb{N}. \tag{57}$$

For $\gamma > 0$, let $v_n = (2\gamma)^{1/p} y_n \in W^{1,p}(\Omega)$. Evidently, $v_n \to 0$ in $L^r(\Omega)$ (see (53) and recall that we have assumed that $y = 0$). Then

$$\int_\Omega F(z, v_n)dz \to 0 \text{ as } n \to \infty. \tag{58}$$

Since $||u_n^+|| \to \infty$, we can find $n_0 \in \mathbb{N}$ such that

$$(2\gamma)^{1/p} \frac{1}{||u_n^+||} \in (0, 1) \text{ for all } n \geq n_0. \tag{59}$$

Then (57) and (59) imply that

$$\begin{aligned}
\vartheta_n(t_n) &\geq \vartheta_n\left(\frac{(2\gamma)^{1/p}}{||u_n^+||}\right) \text{ for all } n \geq n_0 \\
\Rightarrow \hat{\psi}_\lambda(t_n u_n^+) &\geq \hat{\psi}_\lambda((2\gamma)^{1/p} y_n) = \hat{\psi}_\lambda(v_n) \text{ for all } n \geq n_0 \\
&\geq \frac{2\gamma c_1}{p(p-1)}\left(||Dy_n||_p^p + \frac{p-1}{c_1}\int_\Omega [\xi(z)+\eta-\lambda] y_n^p dz\right) \\
&\quad - \left(\int_\Omega F(z, v_n) dz + \frac{\eta}{p}||v_n||_p^p\right) \\
&\geq \frac{2\gamma c_{20}}{p(p-1)} - \left(\int_\Omega F(z, v_n) dz + \frac{\eta}{p}||v_n||_p^p\right) \\
&\qquad \text{for some } c_{20} > 0, \text{ and for all } n \geq n_0.
\end{aligned} \tag{60}$$

Recall that $v_n \to 0$ in $L^p(\Omega)$. Using this fact and (58) in (60), we see that

$$\hat{\psi}_\lambda(t_n u_n^+) \geq \frac{\gamma c_{20}}{p(p-1)} \text{ for some } n \geq n_1 \geq n_0.$$

But recall that $\gamma > 0$ is arbitrary. So, it follows that

$$\hat{\psi}_\lambda(t_n u_n^+) \to +\infty \text{ as } n \to \infty. \tag{61}$$

We have $0 \leq t_n u_n^+ \leq u_n^+$ for all $n \in \mathbb{N}$. So, on account of hypothesis $H(f)_2(iii)$, we have

$$\begin{aligned}
&\int_\Omega d(z, t_n u_n^+) dz \leq \int_\Omega d(z, u_n^+) dz + ||\nu||_1 \leq M_6 \\
&\text{for some } M_6 > 0, \text{ and for all } n \in \mathbb{N} \text{ (see (52))}.
\end{aligned} \tag{62}$$

We know that

$$\begin{aligned}
&\hat{\psi}_\lambda(0) = 0 \text{ and } \hat{\psi}_\lambda(u_n^+) \leq M_7 \text{ for some } M_7 > 0 \\
&\text{(see (46), (52) and recall that } \hat{\psi}_\lambda \leq \psi_\lambda).
\end{aligned} \tag{63}$$

From (61) and (63) we infer that $t_n \in (0, 1)$ for all $n \geq n_2$. Hence we have

$$0 = t_n \frac{d}{dt}\hat{\psi}_\lambda(tu_n^+)|_{t=t_n} = \left\langle \psi'_\lambda(t_n u_n^+), t_n u_n^+ \right\rangle \text{ for all } n \geq n_2 \text{ (see (57))}. \tag{64}$$

Combining (62) and (64) we see that

$$p\hat{\psi}_\lambda(t_n u_n^+) \leq M_6 \text{ for all } n \geq n_2. \tag{65}$$

Comparing (61) and (65) we have a contradiction. Therefore

$$\begin{aligned}
&\{u_n^+\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded,} \\
\Rightarrow\quad &\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded (see (49))}.
\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega). \tag{66}$$

We return to (48) and choose $h = u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (66). Then

$$\begin{aligned}
&\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0, \\
\Rightarrow\quad &u_n \to u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}.
\end{aligned}$$

Therefore ψ_λ satisfies the C-condition and this proves the claim.

Then (44), (45) and the claim, permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_\lambda \in W^{1,p}(\Omega)$ $(\lambda < 0, |\lambda| > ||\xi||_\infty)$ such that

$$u_\lambda \in K_{\psi_\lambda} \text{ and } m_\rho^\lambda \leq \psi_\lambda(u_\lambda). \tag{67}$$

It follows from (67) that $u_\lambda \neq 0$ (see (44)) and

$$\begin{aligned}&\langle A(u_\lambda), h\rangle + \int_\Omega \xi(z)|u_\lambda|^{p-2}u_\lambda h dz + \int_{\partial\Omega} \beta(z)|u_\lambda|^{p-2}u_\lambda h d\sigma\\ &- \eta \int_\Omega (u_\lambda^-)^{p-1} h dz = \lambda \int_\Omega (u_\lambda^+)^{p-1} h dz + \int_\Omega f(z, u_\lambda^+) h dz \text{ for all } h \in W^{1,p}(\Omega).\end{aligned} \tag{68}$$

In (68) we choose $h = -u_\lambda^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}&\frac{c_1}{p-1}||Du_\lambda^-||_p^p + \int_\Omega [\xi(z) + \eta]|u_\lambda^-|^p dz \leq 0 \text{ (see Lemma 2.2)},\\ \Rightarrow \quad & u_\lambda \geq 0,\ u_\lambda \neq 0.\end{aligned}$$

It follows from (68) that u_λ is a positive solution of (P_λ), hence $\lambda \in \mathcal{L}$ and so $\mathcal{L} \neq \emptyset$. Moreover, from the nonlinear regularity theory (see [13]) and the nonlinear maximum principle (see [25]), we can deduce that $S_\lambda \subseteq D_+$. □

In the present setting, on account of hypotheses $H(f)_2(i), (iv)$, we have that

$$\lambda x^{p-1} + f(z, x) \geq \tilde{c}_0 x^{s-1} - c_{21} x^{r-1} \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0, \tag{69}$$

for some big enough $c_{21} = c_{21}(\lambda) > 0$. An inspection of the proofs of Propositions 6−−11 reveals that their conclusions remain valid in the present setting. Now, instead of (19) we use (69). So, we can state the following proposition summarizing these conclusions.

Proposition 17. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_2$ hold, then*

(a) if $\lambda \in \mathcal{L}$ and $\vartheta < \lambda$, then $\vartheta \in \mathcal{L}$;
(b) $\lambda^* = \sup \mathcal{L} < +\infty$;
(c) for every $\lambda \in \mathcal{L}$, problem (P_λ) admits a smallest element $\bar{u}_\lambda \in D_+$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L}$ into $C^1(\overline{\Omega})$ is
  - *strictly increasing (that is, $\vartheta < \lambda \in \mathcal{L} \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+$);*
  - *left continuous.*

Again we show that the critical parameter λ^* is not admissible, hence

$$\mathcal{L} = (-\infty, \lambda^*).$$

Proposition 18. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_2$ hold, then $\lambda^* \notin \mathcal{L}$.*

Proof. Arguing by contradiction, suppose that $\lambda^* \in \mathcal{L}$. Then according to Proposition 17 problem (P_λ) admits a smallest positive solution $\bar{u}_* = \bar{u}_{\lambda^*} \in D_+$.

Consider $\lambda > \lambda^*$ and, as always, let $\eta > ||\xi||_\infty$. We introduce the Carathéodory function $\hat{\gamma}_\lambda(z, x)$ define by

$$\hat{\gamma}_\lambda(z, x) = \begin{cases} (\lambda + \eta)\bar{u}_*(z)^{p-1} + f(z, \bar{u}_*(z)) & \text{if } x \leq \bar{u}_*(z) \\ (\lambda + \eta)x^{p-1} + f(z, x) & \text{if } \bar{u}_*(z) < x. \end{cases} \tag{70}$$

We set $\hat{\Gamma}_\lambda(z, x) = \int_0^x \hat{\gamma}_\lambda(z, s) ds$ and consider the C^1-functional $\tilde{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\tilde{\psi}_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u||_p^p - \int_\Omega \Gamma_\lambda(z, u) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Using (70) we can easily verify that

$$K_{\tilde{\psi}_\lambda} \subseteq [\bar{u}_*) \cap C^1(\overline{\Omega}). \tag{71}$$

As in the proof of Proposition 17 of [21] (see the Claim), we may assume that

$$\bar{u}_* \text{ is a local minimizer of } \tilde{\psi}_\lambda(\,\cdot\,). \tag{72}$$

Without any loss of generality, we assume that K_{ψ_λ} is finite (see (70), (71)). Then on account of (**??**) we can find small enough $\rho_1 \in (0,1)$ such that

$$\inf\{\tilde{\psi}_\lambda(u) : ||u - \bar{u}_*|| = \rho_1\} = \tilde{m}_1 > \tilde{\psi}_\lambda(\bar{u}_*) \text{ (see [1]).} \tag{73}$$

Also, hypothesis $H(f)_2(ii)$ implies that if $u \in D_+$, then

$$\tilde{\psi}_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{74}$$

Moreover, on account of (70), reasoning as in the proof of Proposition 16 (see the claim), we can show that

$$\tilde{\psi}_\lambda(\cdot) \text{ satisfies the C-condition.} \tag{75}$$

Then (73), (74), (75) permit the use of Theorem 2.1 (the mountain pass theorem). Hence we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{\tilde{\psi}_\lambda} \subseteq [\bar{u}_*) \cap C^1(\overline{\Omega}) \text{ (see (71))}, \ \tilde{\psi}_\lambda(u_\lambda) \geq \tilde{m}_1.$$

It follows that $u_\lambda \in S_\lambda$ and so $\lambda \in \mathcal{L}$, a contradiction. Therefore $\lambda^* \notin \mathcal{L}$. The proof is now complete. ☐

In this case for $\lambda \in \mathcal{L} = (-\infty, \lambda^*)$ we have a multiplicity result for problem (P_λ).

Proposition 19. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_2$ hold and $\lambda \in \mathcal{L} = (-\infty, \lambda^*)$, then problem (P_λ) admits at least two positive solutions*

$$u_\lambda,\ \hat{u}_\lambda \in D_+,\ u_\lambda \leq \hat{u}_\lambda,\ u_\lambda \neq \hat{u}_\lambda.$$

Proof. Since $\lambda \in \mathcal{L}$ we can find $u_\lambda \in S_\lambda \subseteq D_+$ (see Proposition 16). We may assume that u_λ is the minimal positive solution of (P_λ) produced in Proposition 17 (that is, $u_\lambda = \bar{u}_\lambda$). With $\eta > ||\xi||_\infty$, we introduce the Carathéodory function $k_\lambda(z,x)$ defined by

$$k_\lambda(z,x) = \begin{cases} (\lambda+\eta)u_\lambda(z)^{p-1} + f(z,u_\lambda(z)) & \text{if } x \leq u_\lambda(z) \\ (\lambda+\eta)x^{p-1} + f(z,x) & \text{if } u_\lambda(z) < x. \end{cases} \tag{76}$$

Let $K_\lambda(z,x) = \int_0^x k_\lambda(z,s)ds$ and consider the C^1-functional $j_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$j_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u||_p^p - \int_\Omega K_\lambda(z,u)dz.$$

Working with $j_\lambda(\cdot)$ as in the proof of Proposition 18 and using (76), we produce $\hat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$\hat{u}_\lambda \in K_{j_\lambda} \subseteq [u_\lambda) \cap C^1(\overline{\Omega}),\ \hat{u}_\lambda \notin \{0, u_\lambda\}. \tag{77}$$

It follows from (76) and (77) that $\hat{u}_\lambda \in D_+$ is the second positive solution of (P_λ). ☐

Summarizing the situation for the "superlinear" case, we can state the following result.

Theorem 4.1. *If hypotheses $H(a), H(\xi), H(\beta), H(f)_2$ hold, then there exists $\lambda^* < +\infty$ such that*

(a) for every $\lambda \geq \lambda^*$, problem (P_λ) has no positive solutions;
(b) for every $\lambda < \lambda^*$, problem (P_λ) has at least two positive solutions u_λ, $\hat{u}_\lambda \in D_+$, $u_\lambda \leq \hat{u}_\lambda$, $u_\lambda \neq \hat{u}_\lambda$;
(c) for every $\lambda < \lambda^*$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in D_+$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \hat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is
- *strictly increasing (that is, $\vartheta < \lambda \in \mathcal{L} \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in intC_+$);*
- *left continuous.*

Again, in the special case of the p-Laplacian, see problem (PL_λ) ($a(y) = |y|^{p-2}y$ for all $y \in \mathbb{R}^N$), we can identify λ^* as $\hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$, provided that $f(z, x) > 0$ for almost all $z \in \Omega$, and for all $x > 0$ and we restrict the condition near zero (that is, $H(f)_2(iv)$).

So, the new conditions on the perturbation $f(z, x)$ are the following:

$H(f)_2'$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for almost all $z \in \Omega$, $f(z, 0) = 0$, $f(z, x) > 0$ for all $x > 0$, hypotheses $H(f)_2'(i), (ii), (iii), (v)$ are the same as the corresponding hypotheses $H(f)_2'(i), (ii), (iii), (v)$ and

(iv) $\lim\limits_{x \to 0^+} \frac{f(z,x)}{x^{p-1}} = 0$ uniformly for almost all $z \in \Omega$.

From Proposition 13, we already know that for $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$ problem (PL_λ) has no positive solutions.

Proposition 20. *If hypotheses $H(\xi), H(\beta), H(f)_2'$ hold and $\lambda < \hat{\lambda}_1$, then $\lambda \in \mathcal{L}$.*

Proof. Let $\lambda \in (-\infty, \hat{\lambda}_1)$ and consider the Carathéodory function $\hat{\vartheta}_\lambda(z, x)$ defined by

$$\hat{\vartheta}_\lambda(z, x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \lambda x^{p-1} + f(z, x) & \text{if } 0 < x. \end{cases} \tag{78}$$

We set $\hat{\Theta}_\lambda(z, x) = \int_0^x \hat{\vartheta}_\lambda(z, s) ds$ and with $\eta > ||\xi||_\infty$, we consider the C^1-functional $w_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$w_\lambda(u) = \frac{1}{p}\mu(u) + \frac{\eta}{p}||u^-||_p^p - \int_\Omega \hat{\Theta}_\lambda(z, u)dz \text{ for all } u \in W^{1,p}(\Omega).$$

Hypotheses $H(f)_2'(i), (iv)$ imply that given $\epsilon > 0$, we can find $c_{22} > 0$ such that

$$F(z, x) \leq \frac{\epsilon}{p}x^p + c_{22}x^r \text{ for almost all } z \in \Omega, \text{ and for all } x \geq 0. \tag{79}$$

Then from (78) and (79), we have

$$w_\lambda(u) \geq \frac{1}{p}[\mu(u^-) + \eta||u^-||_p^p] + \frac{1}{p}[\mu(u^+) - (\lambda + \epsilon)||u^+||_p^p] - c_{22}||u^+||_r^r \text{ (see (79))}.$$

Choosing $\epsilon \in (0, \hat{\lambda}_1 - \lambda)$, we have

$$\begin{aligned} & w_\lambda(u) \geq c_{23}||u||^p - c_{24}||u||^r \text{ for some } c_{23}, c_{24} > 0, \\ \Rightarrow \quad & u = 0 \text{ local minimizer of } w_\lambda(\cdot) \text{ (recall that } r > p). \end{aligned}$$

So, we can find $\rho \in (0, 1)$ small such that

$$w_\lambda(0) = 0 < \inf\{w_\lambda(u) : ||u|| = \rho\} = m_\lambda \tag{80}$$

(see Aizicovici, Papageorgiou & Staicu [1], proof of Proposition 29).

Also, hypothesis $H(f)_2'(ii)$ implies that if $u \in D_+$, then

$$w_\lambda(tu) \to -\infty \text{ as } t \to +\infty. \tag{81}$$

Finally, from the proof of Proposition 16 (see the claim), we know that

$$w_\lambda(\cdot) \text{ satisfies the C-condition.} \tag{82}$$

Then (80), (81), (82) permit the use of Theorem 2.1 (the mountain pass theorem) and produce $u_\lambda \in W^{1,p}(\Omega)$ such that

$$u_\lambda \in K_{w_\lambda} \subseteq D_+ \cup \{0\} \text{ (see the proof of Proposition 5)}$$
$$w_\lambda(0) = 0 < m_\lambda \leq w_\lambda(u_\lambda).$$

Therefore $u_\lambda \in D_+$ is a positive solution of (PL_λ), hence $\lambda \in \mathcal{L}$. □

So, for problem (PL_λ) we can state the following theorem covering the case of a $(p-1)$-superlinear perturbation.

Theorem 4.2. *If hypotheses $H(\xi), H(\beta), H(f)_2'$ hold, then*

(a) for every $\lambda \geq \hat{\lambda}_1 = \hat{\lambda}_1(p, \xi, \beta)$, problem (PL_λ) has no positive solutions;
(b) for every $\lambda < \hat{\lambda}_1$ problem (PL_λ) has at least two positive solutions

$$u_\lambda,\ \hat{u}_\lambda \in D_+,\ u_\lambda \leq \hat{u}_\lambda,\ u_\lambda \neq \hat{u}_\lambda;$$

(c) for every $\lambda < \hat{\lambda}_1$ problem (PL_λ) has a smallest positive solution $\bar{u}_\lambda \in D_+$ and the map $\lambda \mapsto \bar{u}_\lambda$ from $\mathcal{L} = (-\infty, \hat{\lambda}_1)$ into $C^1(\overline{\Omega})$ is
- *strictly increasing (that is, $\vartheta < \lambda < \hat{\lambda}_1 \Rightarrow \bar{u}_\lambda - \bar{u}_\vartheta \in \operatorname{int} C_+$);*
- *left continuous.*

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