Continuum Mech. Thermodyn. (2019) 31:331–340
https://doi.org/10.1007/s00161-018-0731-x

ORIGINAL ARTICLE

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A polynomial way to control the decay of solutions for dipolar bodies

Received: 27 September 2018 / Accepted: 30 October 2018 / Published online: 14 November 2018

Abstract In our paper, we consider a combination of two sub-cylinders coupled by an interface in a semi-infinite cylinder. Both sub-cylinders are made of dipolar elastic materials. For one of the two sub-cylinders, we will consider the elastostatic problem, and for the other the elastodynamic problem. Thus, the spatial behaviors of the sub-cylinders are of different kind and the question arises whether the evolution of this combination can be controlled. By using a polynomial way, we prove that the decay of solutions for the two problems can be controlled.

Keywords Dipolar bodies · Elastostatics · Elastodynamics · Spatial estimates · Upper bound · Polynomial decay

1 Introduction

In the last decades, many studies have been published in which various spatial estimates on decay or growth of solutions were made. But these estimates refer to the solutions of some elliptical, parabolic or hyperbolic equations. However, some specific situations have forced the combination of different materials and the corresponding mathematical models must be based on combinations of different types of equations. Fortunately, research has proven that some combinations of two types of equations do not create any difficulty. Thus, in the case of a mixture of a parabolic equation and an elliptic equations, it was found that the spatial decay of solutions can be described by means of estimations specific to elliptical equations, see [1,2]. Analogously, in the case of the combination of hyperbolic equations with parabolic equations, it has been shown that the behavior of solutions is very similar to the behavior of solutions of parabolic equations. But, until now, there are not well-clarified the questions regarding the behavior of solutions when combining some elliptic equations with some hyperbolic equations [3,4]. In the case of an equation with delay, a spatial decay can be found in the

Communicated by Andreas Öchsner.

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paper of [5] Quintanilla. It has been shown in [6] that for the mixture of a parabolic and an elliptic equation, the spatial behavior is similar to those for elliptic equations. For other estimates on the decay of solutions see [7–10]. In this regard, our study can be considered as a first try to clarify some similar issues in the case of other combinations. Using a technique of [7], we will consider two functionals and a certain energy argument in order to control the decay of solution. Thus, we will obtain some estimates regarding the behavior of solutions for a mixed problem in the context of a semi-infinite cylinder composed of two sub-cylinders with an interface at the common boundary. Both semi-cylinders are in motion and contain a dipolar elastic material. We will consider that the mass densities of the material in the two sub-cylinders are different. As such, we can not use the known techniques, specific for the analysis of hyperbolic problems. We have approached the dipolar bodies in our study because the dipolar structures occupy a privileged place between the theories that are dedicated to the microstructure (see [11,12]). The motivation for considering these new structures is to eliminate the known contradictions: the equation of energy is a parabolic equation and the energy equation does not contain any elastic term. Therefore, the heat waves, under these conditions, will propagate at an infinite speed. To see the importance of the dipolar structure of materials, it is enough to analyze the importance given to this structure by some of the well-known researchers. For example, the studies of Mindlin [13], Green and Rivlin [14], and Fried and Gurtin [15] are very significant from this point of view. Using the Fitzpatricks method, the authors of the paper [16] elaborated new variational principles.

Other approaches to different aspects of generalized bodies can be found in [17–26].

Our work has the following structure. In Sect. 2, we define the two problems for the two sub-cylinders. Section 3 is devoted to the main results.

2 Preliminaries

Let us consider a semi-infinite cylinder $\Omega \times (0, \infty)$, the domain Ω being from the three-dimensional Euclidean space R^3 and assume that this regular region is occupied by a dipolar elastic body. Also, the content of two sub-cylinders is a dipolar elastic body. We will denote by B a two-dimensional domain obtained by a cross section in the cylinder, by a plane perpendicular to the generator of the cylinder. The boundary of B will be denoted by ∂B. Also, the plane domains, corresponding to the two sub-cylinders, will be denoted by B_1 and B_2 and their boundaries by ∂B_1 and ∂B_2, respectively. The components of the unit outward normal of ∂B will be denoted by n_i. Suppose that the boundaries ∂B, ∂B_1 and ∂B_2 are piecewise smooth curves to allow the application of the divergence theorem.

As such, synthesizing, we will use three cylinders:

$$D = B \times (0, \infty), \;\; D_1 = B_1 \times (0, \infty), \;\; D_2 = B_2 \times (0, \infty),$$

and the interface surface:

$$S = (\partial B_1 \cap \partial B_2) \times (0, \infty).$$

All points P from B will be characterized by three rectangular coordinates x_1, x_2, x_3, represented in a fixed system of rectangular Cartesian axes Ox_i, $i = 1, 2, 3$. For simplicity of writing, instead of the triplet (x_1, x_2, x_3), we will use the notation x. Except for expressly specified cases, the functions used below are dependent on (t, x), where t is the time variable and the spatial variable x is used for the position. When there is no risk of confusion, in specific cases, the time variable or/and the spatial variables of the functions are omitted. The Cartesian vector and tensor notations are also used. In the case of repeating subscripts, we will use the known rule of summation.

For the differentiation with respect to the time variable t, we will use the notation $\dot{f} = \partial f/\partial t$, that is, a point placed overhead. Also, the notation $f_{,j} = \partial f/\partial x_j$, that is, a subscript preceded by a comma, designates the partial differentiation of the respective function relative to the Cartesian coordinate x_j.

The behavior of the dipolar body will be characterized by a displacement vector having the components u_i and a dipolar displacement tensor of components φ_{ij}.

Regarding the boundary ∂B of the section in the cylinder and the boundaries ∂B_1 and ∂B_2 of the section in the sub-cylinders, we will take into account the following three situations:

(i) $\partial B \cap \partial B_1 \neq \emptyset$, and $\partial B \cap \partial B_2 \neq \emptyset$,
(ii) $\partial B \cap \partial B_1 = \emptyset$,
(iii) $\partial B \cap \partial B_2 = \emptyset$.

As strain measures, we will use the tensors ε_{ij}, γ_{ij} and χ_{ijk} which are defined by means of the strain–displacement relations (see Eringen [12]):

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \;\; \gamma_{ij}(\mathbf{u}) = u_{j,i} - \varphi_{ij}, \;\; \chi_{ijk}(\mathbf{u}) = \varphi_{jk,i}. \tag{1}$$

In previous equations, which are also called the geometric equations, $\mathbf{u}$ represents the displacement, that is, $\mathbf{u} = \left(u_i, \varphi_{ij}\right)$.

Because we restrict our considerations only to the linear theory, it is normal to consider the density of the internal energy as being a quadratic form in relation to its constitutive variables, denoted by Ψ. With the help of the principle of conservation of energy, we can develop in series, relative to the initial reference, the internal energy density.

As such, the internal energy density corresponding to the displacement $\mathbf{u}$ can be written in the following form:

$$\begin{aligned}\Psi(\mathbf{u}) = {} & \frac{1}{2}A_{ijmn}\varepsilon_{ij}(\mathbf{u})\varepsilon_{mn}(\mathbf{u}) + G_{ijmn}\varepsilon_{ij}(\mathbf{u})\gamma_{mn}(\mathbf{u}) + F_{ijmnr}\varepsilon_{ij}(\mathbf{u})\chi_{mnr}(\mathbf{u}) \\ & + \frac{1}{2}B_{ijmn}\gamma_{ij}(\mathbf{u})\gamma_{mn}(\mathbf{u}) + D_{ijmnr}\gamma_{ij}(\mathbf{u})\chi_{mnr}(\mathbf{u}) + \frac{1}{2}C_{ijkmnr}\chi_{ijk}(\mathbf{u})\chi_{mnr}(\mathbf{u}).\end{aligned} \tag{2}$$

In this way, the tensors of the stress, τ_{ij}, σ_{ij} and μ_{ijk} are obtained with the help of the internal energy density Ψ by meas of the following relations

$$\tau_{ij} = \frac{\partial \Psi}{\partial \varepsilon_{ij}}, \;\; \sigma_{ij} = \frac{\partial \Psi}{\partial \gamma_{ij}}, \;\; \mu_{ijk} = \frac{\partial \Psi}{\partial \chi_{ijk}}.$$

Having the measures of strain, we can introduce the constitutive equations. As is known, these equations express the tensors of the stress, namely, τ_{ij}, σ_{ij} and μ_{ijk}, in terms of the tensors of the deformation:

$$\begin{aligned}\tau_{ij}(\mathbf{u}) &= A_{ijmn}\varepsilon_{mn}(\mathbf{u}) + G_{mnij}\gamma_{mn}(\mathbf{u}) + F_{mnrij}\chi_{mnr}(\mathbf{u}), \\ \sigma_{ij}(\mathbf{u}) &= G_{ijmn}\varepsilon_{mn}(\mathbf{u}) + B_{ijmn}\gamma_{mn}(\mathbf{u}) + D_{ijmnr}\chi_{mnr}(\mathbf{u}), \\ \mu_{ijk}(\mathbf{u}) &= F_{ijkmn}\varepsilon_{mn}(\mathbf{u}) + D_{mnijk}\gamma_{mn}(\mathbf{u}) + C_{ijkmnr}\chi_{mnr}(\mathbf{u}).\end{aligned} \tag{3}$$

In order to achieve the results, we must impose to the strain energy density to be a positive definite quadratic form with regard to the strain tensors ε_{ij}, γ_{ij} and χ_{ijk}.

As usual, the elastic coefficients A_{ijmn}, B_{ijmn}, C_{ijkmnr}, D_{ijmn}, F_{ijkmn}, and G_{ijkmn} from Eqs. (2) and (3) are characteristics of the material and are assumed to be bounded functions on Ω, which depend only on the material points.

Furthermore, it is assumed that the above mentioned elasticity tensors satisfy the following relations of symmetry in the domain Ω:

$$A_{ijmn} = A_{mnij} = A_{jimn}, \;\; B_{ijmn} = B_{mnij}, \;\; C_{ijkmnr} = C_{mnrijk}. \tag{4}$$

We can consider that the form (2) of the internal energy density and the elastic coefficients A_{ijmn}, B_{ijmn}, C_{ijkmnr}, D_{ijmn}, F_{ijkmn}, and G_{ijkmn} are for the cylinder B_1 and the mass density of the dipolar elastic material of B_1 is ϱ. In the cylinder B_2, the mass density of the dipolar elastic material is μ and the elastic coefficients can be denoted by $\mathcal{A}_{ijmn}$, $\mathcal{B}_{ijmn}$, $\mathcal{C}_{ijkmnr}$, $\mathcal{D}_{ijmn}$, $\mathcal{F}_{ijkmn}$, and $\mathcal{G}_{ijkmn}$.

We will need the following restrictions imposed on the mass density ϱ of the cylinder B_1 and on the density μ of the cylinder B_2:

$$0 < c_1 \leq \varrho \leq c_2 < \infty, \;\; -\infty < \mu \leq c_3 < \infty, \tag{5}$$

where c_1, c_2 and c_3 are positive numbers.

In the absence of body forces and dipolar body forces, the basic equations in the sub-cylinder B_1 receive the following form

$$\begin{aligned}&\left[\left(A_{ijmn} + G_{ijmn}\right)\varepsilon_{mn} + \left(G_{mnij} + B_{ijmn}\right)\gamma_{mn} + \left(F_{mnrij} + D_{ijmnr}\right)\chi_{mnr}\right]_{,j} = \varrho\ddot{u}_i, \\ &\left(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\right)_{,i} + G_{jkmn}\varepsilon_{mn} + B_{jkmn}\gamma_{mn} + D_{jkmnr}\chi_{mnr} = I_{kr}\ddot{\varphi}_{jr}.\end{aligned} \tag{6}$$

At the same time, in the sub-cylinder B_2, we have the equations:

$$\begin{aligned}&\left[\left(\mathcal{A}_{ijmn}+\mathcal{G}_{ijmn}\right)\varepsilon_{mn}+\left(\mathcal{G}_{mnij}+\mathcal{B}_{ijmn}\right)\gamma_{mn}+\left(\mathcal{F}_{mnrij}+\mathcal{D}_{ijmnr}\right)\chi_{mnr}\right]_{,j}=\mu\ddot{u}_i,\\&\left(\mathcal{F}_{ijkmn}\varepsilon_{mn}+\mathcal{D}_{mnijk}\gamma_{mn}+\mathcal{C}_{ijkmnr}\chi_{mnr}\right)_{,i}+\mathcal{G}_{jkmn}\varepsilon_{mn}+\mathcal{B}_{jkmn}\gamma_{mn}+\mathcal{D}_{jkmnr}\chi_{mnr}=J_{kr}\ddot{\varphi}_{jr}.\end{aligned}\quad(7)$$

In order to simplify writing, in what follows, we will assume that there is a real and positive number α so that

$$\begin{aligned}&\mathcal{A}_{ijmn}=\alpha A_{ijmn},\ \mathcal{B}_{ijmn}=\alpha B_{ijmn},\ \mathcal{C}_{ijkmnr}=\alpha C_{ijkmnr},\\&\mathcal{D}_{ijmn}=\alpha D_{ijmn},\ \mathcal{F}_{ijkmn}=\alpha F_{ijkmn},\ \mathcal{G}_{ijkmn}=\alpha G_{ijkmn},\ J_{mn}=\alpha I_{mn}.\end{aligned}\quad(8)$$

By a convenient choice of α, we can ensure that at the interface between cylinder B and sub-cylinders B_1 and B_2, the solutions are smooth. For instance, if $\alpha=1$, we deduce the continuity of the traction across the interface surfaces.

In view of convention (8), the static problem in the cylinder B_2 is based on the following system of equations:

$$\begin{aligned}&\left[\left(A_{ijmn}+G_{ijmn}\right)\varepsilon_{mn}+\left(G_{mnij}+B_{ijmn}\right)\gamma_{mn}+\left(F_{mnrij}+D_{ijmnr}\right)\chi_{mnr}\right]_{,j}=0,\\&\left(F_{ijkmn}\varepsilon_{mn}+D_{mnijk}\gamma_{mn}+C_{ijkmnr}\chi_{mnr}\right)_{,i}+G_{jkmn}\varepsilon_{mn}+B_{jkmn}\gamma_{mn}+D_{jkmnr}\chi_{mnr}=0.\end{aligned}\quad(9)$$

By using the stress tensors, the dynamic problem in the cylinder B_1 can be written in the short form

$$\begin{aligned}&\left(\tau_{ij}+\sigma_{ij}\right)_{,j}=\varrho\ddot{u}_i,\\&\mu_{ijk,i}+\sigma_{jk}=I_{kr}\ddot{\varphi}_{jr},\end{aligned}\quad(10)$$

and the dynamic problem in the cylinder B_2:

$$\begin{aligned}&\left(\tau_{ij}+\sigma_{ij}\right)_{,j}=0,\\&\mu_{ijk,i}+\sigma_{jk}=0.\end{aligned}\quad(11)$$

We will be able to get the anticipated results if we impose certain conditions on the elastic coefficients. Thus, we assume that there are positive constants C_1, C_2 and C_3, so that

$$\begin{aligned}&A_{ijmn}\varepsilon_{ij}\varepsilon_{mn}\geq C_1\varepsilon_{ij}\varepsilon_{mn},\\&B_{ijmn}\gamma_{ij}\gamma_{mn}\geq C_2\gamma_{ij}\gamma_{mn},\\&C_{ijkmnr}\chi_{ijk}\chi_{mnr}\geq C_3\chi_{ijk}\chi_{mnr}.\end{aligned}\quad(12)$$

Then, we suppose that there are positive constants C_4, C_5 and C_6, so that from now on toward we have

$$\begin{aligned}&A_{i3mn}A_{i3rs}\varepsilon_{mn}\varepsilon_{rs}\leq C_4A_{ijmn}\varepsilon_{ij}\varepsilon_{mn},\\&B_{i3mn}B_{i3rs}\gamma_{mn}\gamma_{rs}\leq C_5B_{ijmn}\gamma_{ij}\gamma_{mn},\\&C_{i3kmnr}C_{i3klrs}\chi_{mnr}\chi_{lrs}\leq C_6C_{ijkmnr}\chi_{ijk}\chi_{mnr}.\end{aligned}\quad(13)$$

Also, we will need to have the following conditions of asymptotic behavior

$$\begin{aligned}&\lim_{t\to\infty}u_{i,j}=\lim_{t\to\infty}\dot{u}_{i,j}=\lim_{t\to\infty}\ddot{u}_i=0,\\&\lim_{t\to\infty}\varphi_{ij,k}=\lim_{t\to\infty}\dot{\varphi}_{ij,k}=\lim_{t\to\infty}\ddot{\varphi}_{ij}=0,\ \text{for every}\ \ (x_1,\ x_2)\in B,\end{aligned}\quad(14)$$

all of these limits taking place uniformly with respect to x_3.

In order to complete the mixed initial boundary value problem in our context, we will consider the following homogeneous initial conditions

$$\begin{aligned}&u_i(x,0)=\dot{u}_i(x,0)=0,\\&\varphi_{ij}(x,0)=\dot{\varphi}_{ij}(x,0)=0,\ x\in B,\end{aligned}\quad(15)$$

and introduce the two kinds of boundary conditions, namely

$$u_i(x,t)=0,\ \varphi_{ij}(x,t)=0,\ (x,t)\in\partial B\times(0,\infty),\quad(16)$$

and

$$u_i\left(x_1,x_2,0,t\right)=u_0\left(x_1,x_2,t\right),\ \left(x_1,x_2\right)\in B\times\{0\},\ t\in(0,\infty).\quad(17)$$

In the following we will study the behavior of the solutions of the system of Eqs. (10) and (11) which satisfy the initial conditions (15) and the boundary conditions (16) and (17).

3 Main results

First of all, we aim to obtain a decay evaluation regarding the solutions of the problem which consists of (10), (11) and (15)–(17). For the sub-cylinder B_2 in which the problem is static, we must consider some dynamical deformations. Therefore, the kinetic energy is not null. Potential difficulties that arise from a mathematical point of view will be avoided if we use the procedure proposed in the papers [12,13].

Let us introduce the notation

$$B(h) = \{(x_1, x_2, x_3) \in B : x_3 > h\}. \tag{18}$$

In order to facility obtaining the decay estimate for the solutions, we will use the following functions

$$\begin{aligned} f(h,t) &= -\int_0^t \int_{B(h)} (t-s)\left[\tau_{i3}\varepsilon_{i3} + \sigma_{i3}\gamma_{i3} + \mu_{ij3}\chi_{ij3}\right] \mathrm{d}A\mathrm{d}s, \\ g(h,t) &= -\int_0^t \int_{B(h)} (t-s)\left[\tau_{i3,k}\varepsilon_{i3,k} + \sigma_{i3,k}\gamma_{i3,k} + \mu_{ij3,k}\chi_{ij3,k}\right] \mathrm{d}A\mathrm{d}s. \end{aligned} \tag{19}$$

Now, we extend the notation from (18):

$$\begin{aligned} &D_1(h) = \{(x_1, x_2, x_3) \in D_1 : x_3 > h\}, \ D_2(h) = \{(x_1, x_2, x_3) \in D_2 : x_3 > h\}, \\ &S(h) = \{(x_1, x_2, x_3) \in S : x_3 > h\}. \end{aligned}$$

The unit normal vector to the surface S, oriented outward of sub-cylinder B_i, will be denoted by $\mathbf{n}_i$, $i = 1, 2$.

Taking into account the initial conditions (15), the boundary conditions (16) and (17) and the asymptotic conditions (14), we are led to the following four identities:

$$\begin{aligned} &-\int_0^t \int_{B_1(h)} (t-s)\left[\tau_{i3}\varepsilon_{i3} + \sigma_{i3}\gamma_{i3} + \mu_{ij3}\chi_{ij3}\right] \mathrm{d}A\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \int_{D_1(h)} \Big[\left(A_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr}\right)\varepsilon_{ij} \\ &+ \left(G_{ijmn}\varepsilon_{mn} + B_{mnij}\gamma_{mn} + D_{mnrij}\chi_{mnr}\right)\gamma_{ij} \\ &+ \left(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\right)\chi_{ijk}\Big] \mathrm{d}V\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \int_{D_1(h)} \left[\varrho u_{i,3}u_{i,3} + I_{ij}\varphi_{ik,3}\varphi_{jk,3}\right] \mathrm{d}V\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \int_{S(h)} \Big[\left(A_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr}\right)\varepsilon_{i3}\, n_{1,j} \\ &+ \left(G_{ijmn}\varepsilon_{mn} + B_{mnij}\gamma_{mn} + D_{mnrij}\chi_{mnr}\right)\gamma_{i3}\, n_{1,j} \\ &+ \left(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\right)\chi_{ij3}\, n_{1,k}\Big] \mathrm{d}A\mathrm{d}s; \end{aligned} \tag{20}$$

$$\begin{aligned} &-\int_0^t \int_{B_2(h)} (t-s)\left[\tau_{i3}\varepsilon_{i3} + \sigma_{i3}\gamma_{i3} + \mu_{ij3}\chi_{ij3}\right] \mathrm{d}A\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \int_{D_2(h)} \Big[\left(A_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr}\right)\varepsilon_{ij} \\ &+ \left(G_{ijmn}\varepsilon_{mn} + B_{mnij}\gamma_{mn} + D_{mnrij}\chi_{mnr}\right)\gamma_{ij} \\ &+ \left(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\right)\chi_{ijk}\Big] \mathrm{d}V\mathrm{d}s \\ &+ \frac{1}{2}\int_0^t \int_{S(h)} \Big[\left(A_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr}\right)\varepsilon_{i3}\, n_{2,j} \\ &+ \left(G_{ijmn}\varepsilon_{mn} + B_{mnij}\gamma_{mn} + D_{mnrij}\chi_{mnr}\right)\gamma_{i3}\, n_{2,j} \\ &+ \left(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\right)\chi_{ij3}\, n_{2,k}\Big] \mathrm{d}A\mathrm{d}s; \end{aligned} \tag{21}$$

$$
\begin{aligned}
&-\int_0^t \int_{B_1(h)} (t-s)\left[\tau_{i3,k}\varepsilon_{i3,k} + \sigma_{i3,k}\gamma_{i3,k} + \mu_{ij3,k}\chi_{ij3,k}\right] \mathrm{d}A\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \int_{D_1(h)} \left[\left(A_{ijmn}\varepsilon_{mn,k} + G_{mnij}\gamma_{mn,k} + F_{mnrij}\chi_{mnr,k}\right)\varepsilon_{ij,k}\right. \\
&\quad + \left(G_{ijmn}\varepsilon_{mn,k} + B_{mnij}\gamma_{mn,k} + D_{mnrij}\chi_{mnr,k}\right)\gamma_{ij,k} \\
&\quad \left.+ \left(F_{ijkmn}\varepsilon_{mn,l} + D_{mnijk}\gamma_{mn,l} + C_{ijkmnr}\chi_{mnr,l}\right)\chi_{ijk,l}\right] \mathrm{d}V\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \int_{D_1(h)} \left[\varrho u_{i,kk}u_{i,kk} + I_{ij}\varphi_{ir,kk}\varphi_{jr,kk}\right] \mathrm{d}V\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \int_{S(h)} \left[\left(A_{ijmn}\varepsilon_{mn,k} + G_{mnij}\gamma_{mn,k} + F_{mnrij}\chi_{mnr,k}\right)\varepsilon_{i3,k}\, n_{1,j}\right. \\
&\quad + \left(G_{ijmn}\varepsilon_{mn,k} + B_{mnij}\gamma_{mn,k} + D_{mnrij}\chi_{mnr,k}\right)\gamma_{i3,k}\, n_{1,j} \\
&\quad \left.+ \left(F_{ijkmn}\varepsilon_{mn,l} + D_{mnijk}\gamma_{mn,l} + C_{ijkmnr}\chi_{mnr,l}\right)\chi_{ij3,l}\, n_{1,k}\right] \mathrm{d}A\mathrm{d}s;
\end{aligned} \tag{22}
$$

$$
\begin{aligned}
&-\int_0^t \int_{B_1(h)} (t-s)\left[\tau_{i3,k}\varepsilon_{i3,k} + \sigma_{i3,k}\gamma_{i3,k} + \mu_{ij3,k}\chi_{ij3,k}\right] \mathrm{d}A\mathrm{d}s \\
&= \frac{1}{2}\int_0^t \int_{D_1(h)} \left[\left(A_{ijmn}\varepsilon_{mn,k} + G_{mnij}\gamma_{mn,k} + F_{mnrij}\chi_{mnr,k}\right)\varepsilon_{ij,k}\right. \\
&\quad + \left(G_{ijmn}\varepsilon_{mn,k} + B_{mnij}\gamma_{mn,k} + D_{mnrij}\chi_{mnr,k}\right)\gamma_{ij,k} \\
&\quad \left.+ \left(F_{ijkmn}\varepsilon_{mn,l} + D_{mnijk}\gamma_{mn,l} + C_{ijkmnr}\chi_{mnr,l}\right)\chi_{ijk,l}\right] \mathrm{d}V\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^t \int_{S(h)} \left[\left(A_{ijmn}\varepsilon_{mn,k} + G_{mnij}\gamma_{mn,k} + F_{mnrij}\chi_{mnr,k}\right)\varepsilon_{i3,k}\, n_{1,j}\right. \\
&\quad + \left(G_{ijmn}\varepsilon_{mn,k} + B_{mnij}\gamma_{mn,k} + D_{mnrij}\chi_{mnr,k}\right)\gamma_{i3,k}\, n_{1,j} \\
&\quad \left.+ \left(F_{ijkmn}\varepsilon_{mn,l} + D_{mnijk}\gamma_{mn,l} + C_{ijkmnr}\chi_{mnr,l}\right)\chi_{ij3,l}\, n_{1,k}\right] \mathrm{d}A\mathrm{d}s.
\end{aligned} \tag{23}
$$

Taking into account Eqs. (20) and (21), the function $f(h,t)$ from $(19)_1$ becomes

$$
\begin{aligned}
f(h,t) = &\frac{1}{2}\int_0^t \int_{D(h)} \left[\left(A_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr}\right)\varepsilon_{ij}\right. \\
&+ \left(G_{ijmn}\varepsilon_{mn} + B_{mnij}\gamma_{mn} + D_{mnrij}\chi_{mnr}\right)\gamma_{ij} \\
&\left.+ \left(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\right)\chi_{ijk}\right] \mathrm{d}V\mathrm{d}s \\
&+ \frac{1}{2}\int_0^t \int_{D_1(h)} \left[\varrho u_{i,k}u_{i,k} + I_{ij}\varphi_{ir,k}\varphi_{jr,k}\right] \mathrm{d}V\mathrm{d}s.
\end{aligned} \tag{24}
$$

Also, from $(19)_1$ and (24) we deduce

$$
\begin{aligned}
f(h,t) &\le t\left(\int_0^t \int_B \left[\tau_{i3}\tau_{i3} + \sigma_{i3}\sigma_{i3} + \mu_{ij3}\mu_{ij3}\right] \mathrm{d}A\mathrm{d}s\right)^{1/2} \\
&\quad \times \left(\int_0^t \int_B \left[\varepsilon_{i3}\varepsilon_{i3} + \gamma_{i3}\gamma_{i3} + \chi_{ij3}\chi_{ij3}\right] \mathrm{d}A\mathrm{d}s\right)^{1/2} \\
&\le t\left(\int_0^t\int_B \left(C_4 A_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + C_5 B_{ijmn}\gamma_{ij}\gamma_{mn} + C_6 C_{ijkmnr}\chi_{ijk}\chi_{mnr}\right) \mathrm{d}A\mathrm{d}s\right)^{1/2} \\
&\quad \times \left(C_P(B)\int_0^t \int_B \left[u_{i,k}u_{i,k} + \varphi_{ij,k}\varphi_{ij,k}\right] \mathrm{d}A\mathrm{d}s\right)^{1/2},
\end{aligned} \tag{25}
$$

where the constants C_4, C_5 and C_6 are defined in (13) and $C_P(B)$ is the Poincare's constant (see [4]) corresponding to the domain B, defined by

$$
\frac{1}{C_P(B)} = \min_{h\in C_0^\infty} \frac{\int_B |\nabla h|^2 \mathrm{d}A}{\int_B h^2 \mathrm{d}A}.
$$

By using Eqs. (22) and (23), the function $g(h,t)$ from $(19)_2$ becomes

$$\begin{aligned} g(h,t) = \frac{1}{2}\int_0^t \int_{D(h)} & \big[\big(A_{ijmn}\varepsilon_{mn,k} + G_{mnij}\gamma_{mn,k} + F_{mnrij}\chi_{mnr,k}\big)\varepsilon_{ij,k} \\ & + \big(G_{ijmn}\varepsilon_{mn,k} + B_{mnij}\gamma_{mn,k} + D_{mnrij}\chi_{mnr,k}\big)\gamma_{ij,k} \\ & + \big(F_{ijkmn}\varepsilon_{mn,l} + D_{mnijk}\gamma_{mn,l} + C_{ijkmnr}\chi_{mnr,l}\big)\chi_{ijk,l}\big]\,\mathrm{d}V\mathrm{d}s \\ & + \frac{1}{2}\int_0^t \int_{D_1(h)} \big[\varrho u_{i,kk}u_{i,kk} + I_{ij}\varphi_{ir,kk}\varphi_{jr,kk}\big]\,\mathrm{d}V\mathrm{d}s. \end{aligned} \tag{26}$$

From (24) and (26), by direct calculations, we can obtain the partial derivatives with respect to h, which will be useful in the following:

$$\begin{aligned} \frac{\partial f(h,t)}{\partial h} = -\frac{1}{2}\int_0^t \int_{B(h)} & \big[\big(A_{ijmn}\varepsilon_{mn} + G_{mnij}\gamma_{mn} + F_{mnrij}\chi_{mnr}\big)\varepsilon_{ij} \\ & + \big(G_{ijmn}\varepsilon_{mn} + B_{mnij}\gamma_{mn} + D_{mnrij}\chi_{mnr}\big)\gamma_{ij} \\ & + \big(F_{ijkmn}\varepsilon_{mn} + D_{mnijk}\gamma_{mn} + C_{ijkmnr}\chi_{mnr}\big)\chi_{ijk}\big]\,\mathrm{d}A\mathrm{d}s \\ & - \frac{1}{2}\int_0^t \int_{B_1(h)} \big[\varrho u_{i,k}u_{i,k} + I_{ij}\varphi_{ir,k}\varphi_{jr,k}\big]\,\mathrm{d}A\mathrm{d}s, \\ \frac{\partial g(h,t)}{\partial h} = -\frac{1}{2}\int_0^t \int_{B(h)} & \big[\big(A_{ijmn}\varepsilon_{mn,k} + G_{mnij}\gamma_{mn,k} + F_{mnrij}\chi_{mnr,k}\big)\varepsilon_{ij,k} \\ & + \big(G_{ijmn}\varepsilon_{mn,k} + B_{mnij}\gamma_{mn,k} + D_{mnrij}\chi_{mnr,k}\big)\gamma_{ij,k} \\ & + \big(F_{ijkmn}\varepsilon_{mn,l} + D_{mnijk}\gamma_{mn,l} + C_{ijkmnr}\chi_{mnr,l}\big)\chi_{ijk,l}\big]\,\mathrm{d}A\mathrm{d}s \\ & - \frac{1}{2}\int_0^t \int_{B_1(h)} \big[\varrho u_{i,kk}u_{i,kk} + I_{ij}\varphi_{ir,kk}\varphi_{jr,kk}\big]\,\mathrm{d}A\mathrm{d}s. \end{aligned} \tag{27}$$

From (25) and (27), for every arbitrary positive ε, we obtain

$$\begin{aligned} f(h,t) &\le 2t\left(-\mathcal{C}_1\frac{\partial f(h,t)}{\partial h}\right)^{1/2}\left(-\mathcal{C}_2 C_P(B)\frac{\partial g(h,t)}{\partial h}\right)^{1/2} \\ &\le -t\varepsilon\delta\left(\frac{\partial f(h,t)}{\partial h} + \frac{1}{\varepsilon^2}\frac{\partial g(h,t)}{\partial h}\right), \end{aligned} \tag{28}$$

where $\mathcal{C}_1$ is determined by C_1, C_2 and C_3 from (12), $\mathcal{C}_2$ is determined by C_4, C_5 and C_6 from (13) and

$$\delta = \sqrt{\mathcal{C}_1\mathcal{C}_2 C_P(B)}. \tag{29}$$

Theorem 1 *If $\{u_i, \varphi_{ij}\}$ is a solution of the systems of Eqs.* (10) *and* (11) *which satisfies the initial conditions* (15) *in its homogeneous form, the boundary conditions* (16) *and the asymptotic conditions* (14)*, then there is a positive constant δ of the form* (29) *so that for every arbitrary positive ε we have the following estimate*

$$f(h,t) \le \frac{t\delta\varepsilon}{h}F(0,t), \tag{30}$$

where we used the notation

$$F(h,t) = f(h,t) + \frac{1}{\varepsilon^2}g(h,t). \tag{31}$$

Proof From (28), taking into account expression of $F(h,t)$ from (31), we deduce

$$F(h,t) + \frac{1}{t\delta\varepsilon}\int_0^h f(\xi,t)\mathrm{d}\xi \le F(0,t). \tag{32}$$

Based on the elemental rule of product derivation, we deduce

$$\frac{\partial}{\partial h}(hf(h,t)) = f(h,t) + h\frac{\partial f}{\partial h},$$

so that, taking into account that the spatial derivative of $f(h,t)$ is not positive, we are led to

$$\frac{\partial}{\partial h}(hf(h,t)) \le f(h,t).$$

If we integrate this inequality on the interval $[0,h]$, we get the following inequality

$$hf(h,t) \le \int_0^h f(\xi,t)\mathrm{d}\xi \le t\delta\varepsilon F(0,t),$$

and this implies the desired inequality (30), which ends the proof of the theorem. □

Remark Clearly, the function $f(h,t)$ can be considered as a measure of the deformation, so that the estimate from (30) is in fact a polynomial decay of solution. The statement becomes more convincing if we take the particular case $t\delta\varepsilon = 1$. In this situation, the estimation (30) gets the form

$$f(h,t) \le \frac{1}{h}\left[f(0,t) + \delta^2 t^2 g(0,t)\right]. \tag{33}$$

At the end of our study, we want to obtain an upper bound for the amplitude term.

This will be expressed using the functions $f(h,t)$ and $g(h,t)$ of (19) and will be obtained by using the boundary conditions (16) and (17).

To this aim, we will use a vector field having the components $v_i(x,t)$, and a tensor field having the components $\eta_{jk}(x,t)$ defined on cylinder $D \times (0,t)$.

Theorem 2 *Suppose that the vector field $v_i(x,t)$ and the tensor field $\eta_{jk}(x,t)$ satisfy the boundary conditions (16) and (17). Assume that the vector together with its time derivatives and also the tensor and its time derivative tend to zero, as x_3 tends to ∞, uniformly with respect to x_1, x_2 and t. Then, these is an upper bound for the amplitude term.*

Proof From (10) we deduce that on B_1 we have

$$\begin{aligned}\left(\tau_{ij}+\sigma_{ij}\right)\dot{v}_{i,j} &= \left[\left(\tau_{ij}+\sigma_{ij}\right)\dot{v}_i\right]_{,j} - \varrho\ddot{u}_i\dot{v}_i,\\ \mu_{ijk}\dot{\eta}_{jk,i} + \sigma_{jk}\dot{\eta}_{jk} &= \left(\mu_{ijk}\dot{\eta}_{jk}\right)_{,i} - I_{kr}\ddot{\varphi}_{jr}\dot{\eta}_{jk},\end{aligned} \tag{34}$$

and from the dynamic problem (11) in the cylinder B_2 we deduce:

$$\begin{aligned}\left(\tau_{ij}+\sigma_{ij}\right)\dot{v}_{i,j} &= \left[\left(\tau_{ij}+\sigma_{ij}\right)\dot{v}_i\right]_{,j},\\ \mu_{ijk}\dot{\eta}_{jk,i} + \sigma_{jk}\dot{\eta}_{jk} &= \left(\mu_{ijk}\dot{\eta}_{jk}\right)_{,i}.\end{aligned} \tag{35}$$

In view of the relations (34), (35), the function $f(h,t)$ from $(19)_1$ receives the form

$$\begin{aligned}f(0,t) &= -\int_0^t\int_{B(0)}(t-s)\left[\left(\tau_{i3}+\sigma_{i3}\right)\dot{v}_i + \mu_{ij3}\dot{\eta}_{ij}\right]\mathrm{d}A\mathrm{d}s\\ &= \int_0^t\int_B (t-s)\left[\left(\tau_{ij}+\sigma_{ij}\right)\dot{v}_{i,j} + \mu_{ijk}\dot{\eta}_{jk,i}\right]\mathrm{d}V\mathrm{d}s\\ &\quad - \int_0^t\int_{B_1}(t-s)\left[\varrho\dot{u}_i\ddot{v}_i + I_{kr}\dot{\varphi}_{jr}\ddot{\eta}_{jk}\right]\mathrm{d}V\mathrm{d}s + \int_0^t\int_{B_1}\left[\varrho\dot{u}_i\dot{v}_i + I_{kr}\dot{\varphi}_{jr}\dot{\eta}_{jk}\right]\mathrm{d}V\mathrm{d}s.\end{aligned} \tag{36}$$

Using the inequality between arithmetic and geometric mean, we obtain

$$\frac{1}{2}f(0,t)\le m(t)\int_0^t\int_B\left[\dot{v}_{i,j}\dot{v}_{i,j}+\dot{\eta}_{jk,i}\dot{\eta}_{jk,i}+\ddot{v}_i\ddot{v}_i+\ddot{\eta}_{jk}\ddot{\eta}_{jk}\right]\mathrm{d}V\mathrm{d}s, \tag{37}$$

where $m(t)$ is a conveniently chosen positive function.

In view of the relations (34) and (35), by using the function $g(h,t)$ from $(19)_2$, we can obtain, in a similar manner, the following estimation

$$\frac{1}{2}g(0,t)\le m(t)\int_0^t\int_B\left[\ddot{v}_{i,j}\ddot{v}_{i,j}+\ddot{\eta}_{jk,i}\ddot{\eta}_{jk,i}+\dddot{v}_i\dddot{v}_i+\dddot{\eta}_{jk}\dddot{\eta}_{jk}\right]\mathrm{d}V\mathrm{d}s. \tag{38}$$

In the following, we will use a particular form for the vector field (v_i) and, also, for the tensor field (η_{jk}), namely,

$$v_i(x,t)=\alpha_i\,(x_1,x_2,t)\,e^{-ax_1},\;\;\eta_{jk}(x,t)=\beta_i\,(x_1,x_2,t)\,e^{-bx_1}, \tag{39}$$

where a and b are arbitrary positive constants.

By direct calculations, we deduce that

$$\begin{aligned}&\int_0^t\int_B\dot{v}_{i,j}\dot{v}_{i,j}\mathrm{d}V\mathrm{d}s=\frac{1}{2a}\int_0^t\int_D\left(\dot{\alpha}_{i,m}\dot{\alpha}_{i,m}+a^2\dot{\alpha}_i\dot{\alpha}_i\right)\mathrm{d}A\mathrm{d}s,\;\;m=1,2,\\&\int_0^t\int_B\dot{\eta}_{jk,i}\dot{\eta}_{jk,i}\mathrm{d}V\mathrm{d}s=\frac{1}{2b}\int_0^t\int_D\left(\dot{\beta}_{jk,m}\dot{\beta}_{jk,m}+b^2\dot{\beta}_{jk}\dot{\beta}_{jk}\right)\mathrm{d}A\mathrm{d}s,\;\;m=1,2.\end{aligned} \tag{40}$$

Taking into account the inequalities (40), the estimate (37) becomes

$$\begin{aligned}f(0,t)\le m(t)\Bigg[&\int_0^t\int_B\left(\frac{1}{a}\dot{\alpha}_{i,m}\dot{\alpha}_{i,m}+\frac{1}{b}\dot{\beta}_{jk,m}\dot{\beta}_{jk,m}\right)\mathrm{d}A\mathrm{d}s\\&+a\int_0^t\int_D\dot{\alpha}_i\dot{\alpha}_i\mathrm{d}A\mathrm{d}s+b\int_0^t\int_D\dot{\beta}_{jk}\dot{\beta}_{jk}\mathrm{d}A\mathrm{d}s\\&+\int_0^t\int_D\left(\frac{1}{a}\ddot{\alpha}_i\ddot{\alpha}_i+\frac{1}{b}\ddot{\beta}_{jk}\ddot{\beta}_{jk}\right)\mathrm{d}A\mathrm{d}s\Bigg].\end{aligned} \tag{41}$$

Analogously, if we consider inequalities (40), then the estimate (38) receives the form

$$\begin{aligned}g(0,t)\le m(t)\Bigg[&\int_0^t\int_B\left(\frac{1}{a}\ddot{\alpha}_{i,m}\ddot{\alpha}_{i,m}+\frac{1}{b}\ddot{\beta}_{jk,m}\ddot{\beta}_{jk,m}\right)\mathrm{d}A\mathrm{d}s\\&+a\int_0^t\int_D\ddot{\alpha}_i\ddot{\alpha}_i\mathrm{d}A\mathrm{d}s+b\int_0^t\int_D\ddot{\beta}_{jk}\ddot{\beta}_{jk}\mathrm{d}A\mathrm{d}s\\&+\int_0^t\int_D\left(\frac{1}{a}\dddot{\alpha}_i\dddot{\alpha}_i+\frac{1}{b}\dddot{\beta}_{jk}\dddot{\beta}_{jk}\right)\mathrm{d}A\mathrm{d}s\Bigg].\end{aligned} \tag{42}$$

In view of (33), we obtain the following estimate for the amplitude term

$$\begin{aligned}f(0,t)+Cg(0,t)\le m(t)\Bigg[&\int_0^t\int_B\left(\frac{1}{a}\dot{\alpha}_{i,m}\dot{\alpha}_{i,m}+\frac{1}{b}\dot{\beta}_{jk,m}\dot{\beta}_{jk,m}\right)\mathrm{d}A\mathrm{d}s\\&+a\int_0^t\int_D\dot{\alpha}_i\dot{\alpha}_i\mathrm{d}A\mathrm{d}s+b\int_0^t\int_D\dot{\beta}_{jk}\dot{\beta}_{jk}\mathrm{d}A\mathrm{d}s+\int_0^t\int_D\left(\frac{1}{a}\ddot{\alpha}_i\ddot{\alpha}_i+\frac{1}{b}\ddot{\beta}_{jk}\ddot{\beta}_{jk}\right)\mathrm{d}A\mathrm{d}s\Bigg]\\&+Cm(t)\Bigg[\int_0^t\int_B\left(\frac{1}{a}\ddot{\alpha}_{i,m}\ddot{\alpha}_{i,m}+\frac{1}{b}\ddot{\beta}_{jk,m}\ddot{\beta}_{jk,m}\right)\mathrm{d}A\mathrm{d}s\\&+a\int_0^t\int_D\ddot{\alpha}_i\ddot{\alpha}_i\mathrm{d}A\mathrm{d}s+b\int_0^t\int_D\ddot{\beta}_{jk}\ddot{\beta}_{jk}\mathrm{d}A\mathrm{d}s+\int_0^t\int_D\left(\frac{1}{a}\dddot{\alpha}_i\dddot{\alpha}_i+\frac{1}{b}\dddot{\beta}_{jk}\dddot{\beta}_{jk}\right)\mathrm{d}A\mathrm{d}s\Bigg],\end{aligned}$$

where the positive constants a and b are arbitrary, and the function $m(t)$ is computable.

Clearly, if we give two particular values for constants a and b, we will get an upper bound for the amplitude term. □

4 Conclusions

We considered a semi-infinite cylinder occupied by a material with a dipolar structure. The cylinder is composed by two sub-cylinders, one of them is a dynamic cylinder and the other is static, as their spatial behaviors are of different kind. However, we proved that the decay of solutions can be controlled. Also, we showed how it can be obtained an upper bound for the amplitude term, by using the boundary conditions.

References

1. Flavin, J.N., Knops, R.J.: Some spatial decay estimates in continuum dynamics. J. Elast. **17**, 249–264 (1987)
2. Flavin, J.N., Knops, R.J., Payne, L.E.: Energy bounds in dynamical problems for a semi-infinite elastic beam. In: Eason, G., Ogden, R.W. (eds.) Elasticity: Mathematical Methods and Applications, pp. 101–111. Ellis-Horwood, Chichester (1990)
3. Flavin, J.N., Rionero, S.: Qualitative Estimates for Partial Differential Equations. An Introduction. CRC Press, Boca Raton (1996)
4. Horgan, C.O., Quintanilla, R.: Spatial decay of transient end effects in functionally graded heat conducting materials. Q. Appl. Math. **59**, 529542 (2001)
5. Quintanilla, R.: Spatial estimates for an equation with delay. Z. Angew. Math. Phys. **61**(2), 381–388 (2010)
6. Quintanilla, R.: Spatial stability for the quasi-static problem of thermo-elasticity. J. Elast. **76**, 93105 (2004)
7. Quintanilla, R., Saccomandi, G.: Quasistatic anti-plane motion in the simplest theory of nonlinear viscoelasticity. Nonlinear Anal. Ser. B Real World Appl. **9**, 14991517 (2008)
8. Chirita, S.: On the spatial decay estimates in certain time-dependent problems of continuum mechanics. Arch. Mech. **47**, 755–771 (1995)
9. Horgan, C.O., Payne, L.E., Wheeler, L.T.: Spatial decay estimates in transient heat conduction. Q. Appl. Math. **42**, 119–127 (1984)
10. Nunziato, J.W.: On the spatial decay of solutions in the nonlinear theory of heat conduction. J. Math. Anal. Appl. **48**, 687–698 (1974)
11. Eringen, A.C.: Theory of thermo-microstretch elastic solids. Int. J. Eng. Sci. **28**, 1291–1301 (1990)
12. Eringen, A.C.: Microcontinuum Field Theories. Springer, New York (1999)
13. Mindlin, R.D.: Micro-structure in linear elasticity. Arch. Ration. Mech. Anal. **16**, 51–78 (1964)
14. Green, A.E., Rivlin, R.S.: Multipolar continuum mechanics. Arch. Ration. Mech. Anal. **17**, 113–147 (1964)
15. Fried, E., Gurtin, M.E.: Thermomechanics of the interface between a body and its environment. Contin. Mech. Thermodyn. **19**(5), 253–271 (2007)
16. Vallée, C., Rădulescu, V., Atchonouglo, K.: New variational principles for solving extended Dirichlet–Neumann problems. J. Elast. **123**, 1–18 (2016)
17. Marin, M.: Weak solutions in elasticity of dipolar porous materials. Math. Probl. Eng. **1–8**, 158908 (2008)
18. Marin, M., Öchsner, A.: The effect of a dipolar structure on the Holder stability in Green–Naghdi thermoelasticity. Contin. Mech. Thermodyn. **29**(6), 1365–1374 (2017)
19. Marin, M., Agarwal, R.P., Mahmoud, S.R.: Modeling a microstretch thermo-elastic body with two temperatures. Abstr. Appl. Anal. **2013**, 1–7 (2013)
20. Marin, M.: An approach of a heat-flux dependent theory for micropolar porous media. Meccanica **51**(5), 1127–1133 (2016)
21. Abbas, I.A.: A GN model based upon two-temperature generalized thermoelastic theory in an unbounded medium with a spherical cavity. Appl. Math. Comput. **245**, 108–115 (2014)
22. Abbas, I.A.: Eigenvalue approach for an unbounded medium with a spherical cavity based upon two-temperature generalized thermoelastic theory. J. Mech. Sci. Technol. **28**(10), 4193–4198 (2014)
23. Othman, M.I.A.: State space approach to generalized thermoelasticity plane waves with two relaxation times under the dependence of the modulus of elasticity on reference temperature. Can. J. Phys. **81**(12), 1403–1418 (2003)
24. Sharma, J.N., Othman, M.I.A.: Effect of rotation on generalized thermo-viscoelastic Rayleigh–Lamb waves. Int. J. Solids Struct. **44**(13), 4243–4255 (2007)
25. Craciun, E.M., Carabineanu, A., Peride, N.: Antiplane interface crack in a pre-stressed fiber-reinforced elastic composite. Comput. Mater. Sci. **43**(1), 184–189 (2008)
26. Sadowski, T., Craciun, E.M., Rabaea, A., Marsavina, L.: Mathematical modeling of three equal collinear cracks in an orthotropic solid. Meccanica **51**(2), 329–339 (2016)