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EXISTENCE AND UNIQUENESS OF BLOW-UP SOLUTIONS FOR A CLASS OF LOGISTIC EQUATIONS

FLORICA-CORINA ŞT. CÎ RSTEA

School of Communications and Informatics, Victoria University of Technology, PO Box 14428, Melbourne City MC, Victoria 8001, Australia
florica@matilda.vu.edu.au

VICENŢIU D. RĂDULESCU

Department of Mathematics, University of Craiova, 1100 Craiova, Romania
vicrad@yahoo.com

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This paper is dedicated with esteem to Professor Viorel Barbu on his 60th birthday

Let f be a non-negative C^1-function on $[0,\infty)$ such that $f(u)/u$ is increasing and $\int_1^\infty 1/\sqrt{F(t)}dt < \infty$, where $F(t) = \int_0^t f(s)ds$. Assume $\Omega \subset \mathbf{R}^N$ is a smooth bounded domain, a is a real parameter and $b \geq 0$ is a continuous function on $\bar{\Omega}$, $b \not\equiv 0$. We consider the problem $\Delta u + au = b(x)f(u)$ in Ω and we prove a necessary and sufficient condition for the existence of positive solutions that blow-up at the boundary. We also deduce several existence and uniqueness results for a related problem, subject to homogeneous Dirichlet, Neumann or Robin boundary condition.

Keywords: ???

1. Introduction and the Main Results

Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega\,, \tag{1}$$

where Ω is a smooth bounded domain in $\mathbf{R}^N$, $N \geq 3$. Let a be a real parameter and $b \in C^{0,\mu}(\bar{\Omega})$, $0 < \mu < 1$, such that $b \geq 0$ and $b \not\equiv 0$ in Ω. Set

$$\Omega_0 = \int \{x \in \Omega : b(x) = 0\}$$

and suppose, throughout, that $\bar{\Omega}_0 \subset \Omega$ and $b > 0$ on $\Omega \backslash \bar{\Omega}_0$. Assume that $f \in C^1[0,\infty)$ satisfies (A_1) $f \geq 0$ and $f(u)/u$ is increasing on $(0,\infty)$.

Following Alama and Tarantello [1], define by H_∞ the Dirichlet Laplacian on Ω_0 as the unique self-adjoint operator associated to the quadratic form $\psi(u) = \int_\Omega |\nabla u|^2 dx$ with form domain

$$H^1_D(\Omega_0) = \{u \in H^1_0(\Omega) : u(x) = 0 \quad \text{for a.e. } x \in \Omega \backslash \Omega_0\}\,.$$

If $\partial\Omega_0$ satisfies the exterior cone condition then, according to [1], $H_D^1(\Omega_0)$ coincides with $H_0^1(\Omega_0)$ and H_∞ is the classical Laplace operator with Dirichlet condition on $\partial\Omega_0$.

Let $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of H_∞ in Ω_0. We understand $\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$.

Set $\mu_0 := \lim_{u\searrow 0} \frac{f(u)}{u}$, $\mu_\infty := \lim_{u\to\infty} \frac{f(u)}{u}$, and denote by $\lambda_1(\mu_0)$ (respectively, $\lambda_1(\mu_\infty)$) the first eigenvalue of the operator $H_{\mu_0} = -\Delta + \mu_0 b$ (respectively, $H_{\mu_\infty} = -\Delta + \mu_\infty b$) in $H_0^1(\Omega)$. Recall that $\lambda_1(+\infty) = \lambda_{\infty,1}$.

Alama and Tarantello [1] proved that problem (1) subject to the Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega \tag{2}$$

has a positive solution u_a if and only if $a \in (\lambda_1(\mu_0), \lambda_1(\mu_\infty))$. Moreover, u_a is the unique positive solution for (1) + (2) (see [20, Theorem A (bis)]). We shall refer to the combination of (1) + (2) as problem (E_a).

Our first aim is to give a corresponding necessary and sufficient condition, but for the existence of *large* (or *explosive*) solutions of (1). A solution u of (1) such that $u \geq 0$ in Ω and $u(x) \to \infty$ as $\mathrm{dist}(x, \partial\Omega) \to 0$ will be called a large solution. Cf. Corollary A.2 in the Appendix, if such a solution exists, then it is *positive* even if f satisfies a weaker condition than (A_1), namely

$(A_1)'$ $f(0) = 0$, $f' \geq 0$ and $f > 0$ on $(0, \infty)$.

Problems related to large solutions have a long history and are studied by many authors and in many contexts. Singular value problems of this type go back to the pioneering work [29] on the equation $\Delta u = e^u$ in the space, and were later studied under the general form $\Delta u = f(u)$ in N-dimensional domains. We refer only to [22–6, 11, 15, 16, 21, 22, 24–26], and [31]. We also point out the paper [30], where there are studied large solutions of the problem

$$\Delta u = K(x) u^{(N+2)/(N-2)}$$

in a ball, in particular for questions of existence, uniqueness and boundary behaviour.

Keller [20] and Osserman [27] supplied a necessary and sufficient condition on f for the existence of large solutions to (1) when $a \equiv 0$, $b \equiv 1$ and f is assumed to fulfill $(A_1)'$. More precisely, f must satisfy the Keller–Osserman condition (see [20, 27]),

(A_2) $\int_1^\infty \frac{dt}{\sqrt{F(t)}} < \infty$, where $F(t) = \int_0^t f(s)ds$.

Keeping this in mind and using Theorem A.1 in the Appendix we find that our problem (1) can have large solutions only if the Keller–Osserman condition (A_2) is fulfilled (see Remark 3.1). Furthermore, when this really happens, our first result gives the maximal interval for the parameter a that ensures the existence of large solutions to problem (1). More precisely, we prove

Theorem 1.1. *Assume that f satisfies conditions (A_1) and (A_2). Then problem* (1) *has a large solution if and only if $a \in (-\infty, \lambda_{\infty,1})$.*

We point out that our framework in the above result includes the case when b vanishes at some points on $\partial\Omega$, or even if $b \equiv 0$ on $\partial\Omega$. In this sense, our result responds to a question raised to one of us by Professor Haim Brezis in Paris, May 2001.

Denote by $\mathcal{D}$ and $\mathcal{R}$ the boundary operators

$$\mathcal{D}u := u \quad \text{and} \quad \mathcal{R}u := \partial_\nu u + \beta(x)u\,,$$

where ν is the unit outward normal to $\partial\Omega$, and $\beta \in C^{1,\mu}(\partial\Omega)$ is non-negative. Hence, $\mathcal{D}$ is the *Dirichlet* boundary operator and $\mathcal{R}$ is either the *Neumann* boundary operator, if $\beta \equiv 0$, or the *Robin* boundary operator, if $\beta \not\equiv 0$. Throughout this work, $\mathcal{B}$ can define any of these boundary operators.

Note that the Robin condition $\mathcal{R} = 0$ relies essentially to heat flow problems in a body with constant temperature in the surrounding medium. More generally, if α and β are smooth functions on $\partial\Omega$ such that $\alpha, \beta \geq 0$, $\alpha + \beta > 0$, then the boundary condition $Bu = \alpha\partial_\nu u + \beta u = 0$ represents the exchange of heat at the surface of the reactant by Newtonian cooling. Moreover, the boundary condition $Bu = 0$ is called isothermal (Dirichlet) condition if $\alpha \equiv 0$, and it becomes an adiabatic (Neumann) condition if $\beta \equiv 0$. An intuitive meaning of the condition $\alpha + \beta > 0$ on $\partial\Omega$ is that, for the diffusion process described by problem (1), either the reflection phenomenon or the absorption phenomenon may occur at each point of the boundary.

If $f(u) = u^p$ $(p > 1)$, the semilinear elliptic problem

$$\begin{cases} \Delta u + au = b(x)u^p & \text{in } \Omega\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

is basic population model (see, e.g. [18]) and is also related to some prescribed curvature problems in Riemannian geometry (see, e.g. [28] and [19]). The existence of positive solutions of (3) has been intensively studied; see for example [1, 2, 12, 13, 17] and [28].

If b is positive on $\bar{\Omega}$ then (3) is known as the logistic equation and it has a unique positive solution if and only if $a > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

We are now concerned with the following boundary blow-up problem

$$\begin{cases} \Delta u + au = b(x)f(u) & \text{in } \Omega\backslash\bar{\Omega}_0\,, \\ \mathcal{B}u = 0 & \text{on } \partial\Omega\,, \\ u = \infty & \text{on } \partial\Omega_0\,, \end{cases} \tag{4}$$

where $b > 0$ on $\partial\Omega$, while $\bar{\Omega}_0$ is non-empty, connected and with smooth boundary. Here, $u = \infty$ on $\partial\Omega_0$ means that $u(x) \to \infty$ as $x \in \Omega\backslash\bar{\Omega}_0$ and $d(x) := \mathrm{dist}(x, \Omega_0) \to 0$.

The question of existence and uniqueness of positive solutions for problem (4) in the case of pure superlinear power in the non-linearity is treated by Du–Huang [16]. Our next results extend their previous paper to the case of much more general non-linearities of Keller–Osserman type.

In the following, by $(\tilde{A}_1)$ we mean that (A_1) is fulfilled and there exists $\lim_{u\to\infty}(F/f)'(u) := \gamma$. Then, $\gamma \geq 0$. Moreover, $\gamma \leq 1/2$ if, in addition, (A_2) is satisfied (see Lemma 4.1).

We prove

Theorem 1.2. *Let* $(\tilde{A}_1)$ *and* (A_2) *hold. Then, for any* $a \in \mathbf{R}$, *problem* (4) *has a minimal* (*respectively, maximal*) *positive solution* $\underline{U}_a$ (*respectively,* $\bar{U}_a$).

In proving Theorem 1.2 we rely on an appropriate comparison principle (see Lemma 2.3) which allows us to prove that $(u_n)_{n\geq 1}$ is non-decreasing, where u_n is the unique positive solution of problem (31) (in Lemma 5.1) with $\Phi \equiv n$. The minimal positive solution of (4) will be obtained as the limit of the sequence $(u_n)_{n\geq 1}$. Note that, since $b = 0$ on $\partial\Omega_0$, the main difficulty is related to the construction of an upper bound of this sequence (see Lemma 5.2) which must fit to our general framework. To overcome it, we find an equivalent criterion to the Keller–Osserman condition (A_2) (see Lemma 4.2). Next, we deduce the maximal positive solution of (4) as the limit of the non-increasing sequence $(v_m)_{m\geq m_1}$ provided m_1 is large so that $\Omega_{m_1} \subset\subset \Omega$. We denoted by v_m the minimal positive solution of (4) with Ω_0 replaced by

$$\Omega_m := \{x \in \Omega : d(x) < 1/m\}, \quad m \geq m_1 \,. \tag{5}$$

The next question is whether one can conclude the uniqueness of positive solutions of problem (4). We recall first what is already known in this direction. When $f(u) = u^p$, $p > 1$, Du–Huang [16] proved the uniqueness of solution to problem (4) and established its behavior near $\partial\Omega_0$, under the assumption

$$\lim_{d(x)\searrow 0} \frac{b(x)}{[d(x)]^\tau} = c \quad \text{for some positive constants } \tau,\ c > 0\,. \tag{6}$$

We shall give a general uniqueness result provided that b and f satisfy the following assumptions:

(B_1) $\lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))} = c$ for some constant $c > 0$, where $0 < k \in C^1(0,\delta_0)$ is increasing and satisfies

(B_2) $K(t) = \frac{\int_0^t \sqrt{k(s)}ds}{\sqrt{k(t)}} \in C^1[0,\delta_0)$, for some $\delta_0 > 0$.

Assume there exist $\zeta > 0$ and $t_0 \geq 1$ such that

(A_3) $f(\xi t) \leq \xi^{1+\zeta} f(t)$, $\forall\, \xi \in (0,1)$, $\forall\, t \geq t_0/\xi$

(A_4) the mapping $(0,1] \ni \xi \longmapsto A(\xi) = \lim_{u\to\infty} \frac{f(\xi u)}{\xi f(u)}$ is a continuous positive function.

Our uniqueness result is

Theorem 1.3. *Assume the conditions* $(\tilde{A}_1)$ *with* $\gamma \neq 0$, (A_3), (A_4), (B_1) *and* (B_2) *hold. Then, for any* $a \in \mathbf{R}$, *problem* (4) *has a unique positive solution* U_a. *Moreover,*

$$\lim_{d(x)\searrow 0} \frac{U_a(x)}{h(d(x))} = \xi_0\,,$$

where h *is defined by*

$$\int_{h(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = \int_0^t \sqrt{k(s)}ds\,, \quad \forall\, t \in (0, \delta_0) \tag{7}$$

and ξ_0 *is the unique positive solution of* $A(\xi) = \frac{K'(0)(1-2\gamma)+2\gamma}{c}$.

Remark 1.1. (a) $(A_1) + (A_3) \Rightarrow (A_2)$. Indeed, $\lim_{u\to\infty} \frac{f(u)}{u^{1+\zeta}} > 0$ since $\frac{f(t)}{t^{1+\zeta}}$ is non-decreasing for $t \geq t_0$.

(b) $K'(0)(1-2\gamma)+2\gamma \in (0,1]$ when $(\tilde{A}_1)$ with $\gamma \neq 0$, (A_2), (B_1) and (B_2) hold (see Lemma 4.3).

(c) The function $(0,\infty) \ni \xi \longmapsto A(\xi) \in (0,\infty)$ is bijective when (A_3) and (A_4) hold (see Lemma 6.1).

Among the non-linearities f that satisfy the assumptions of Theorem 1.3 we note: (i) $f(u) = u^p$, $p > 1$; (ii) $f(u) = u^p \ln(u+1)$, $p > 1$; (iii) $f(u) = u^p \arctan u$, $p > 1$.

Theorem 2.8 in [16] follows by applying Theorem 1.3 with $f(u) = u^p$, $p > 1$ and $k(t) = t^\tau$ for $t > 0$. However, our result proves the uniqueness for a larger class of functions b than in [16]. Indeed, if (B_1) is satisfied with $k(t) = e^{-1/t}$ for $t > 0$, then the uniqueness remains despite of (6) which is not valid.

The above results also apply to problems on Riemannian manifolds if Δ is replaced by the Laplace–Beltrami operator

$$\Delta_B = \frac{1}{\sqrt{c}}\frac{\partial}{\partial x_i}\left(\sqrt{c}a_{ij}(x)\frac{\partial}{\partial x_i}\right), \quad c := \det(a_{ij})\,,$$

with respect to the metric $ds^2 = c_{ij}dx_i dx_j$, where (c_{ij}) is the inverse of (a_{ij}). In this case our results apply to concrete problems arising in Riemannian geometry. For instance, (cf. Loewner–Nirenberg [24] and Li [23]) if Ω is replaced by the standard N-sphere (S^N, g_0), Δ is the Laplace–Beltrami operator Δ_{g_0}, $a = N(N-2)/4$, and $f(u) = (N-2)/[4(N-1)]u^{(N+2)/(N-2)}$, we find the prescribing scalar curvature equation on S^N.

2. Comparison Principles

Throughout this section, we assume that f is continuous on $(0,\infty)$ and $\frac{f(u)}{u}$ is increasing on $(0,\infty)$.

Lemma 2.1. *Assume* ω *is a bounded domain and* $p \in C^{0,\mu}(\bar{\omega})$ *is a positive function in* ω.

If $u_1,\, u_2 \in C^2(\omega)$ are positive functions in ω and

$$\Delta u_1 + au_1 - p(x)f(u_1) \le 0 \le \Delta u_2 + au_2 - p(x)f(u_2) \quad \text{in } \omega \tag{8}$$

$$\limsup_{\operatorname{dist}(x,\partial\omega)\to 0} (u_2 - u_1)(x) \le 0 \tag{9}$$

then $u_1 \ge u_2$ in ω.

Proof. We use the same method as in the proof of Lemma 1.1 in Marcus–Veron [26] (see also [16, Lemma 2.1]), that goes back to Benguria–Brezis–Lieb [7].

By (8) we obtain, for any non-negative function $\phi \in H^1(\omega)$ with compact support in ω,

$$\begin{aligned} &\int_\omega (\nabla u_1 \cdot \nabla\phi - au_1\phi + p(x)f(u_1)\phi)dx \ge 0 \\ &\qquad \ge \int_\omega (\nabla u_2 \cdot \nabla\phi - au_2\phi + p(x)f(u_2)\phi)dx\,. \end{aligned} \tag{10}$$

Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

$$\omega_+(\varepsilon_1,\varepsilon_2) = \{x \in \omega : u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1\}\,.$$

$$v_i = (u_i + \varepsilon_i)^{-1}((u_2+\varepsilon_2)^2 - (u_1+\varepsilon_1)^2)^+, \quad i = 1,2\,.$$

Notice that $v_i \in H^1_{\mathrm{loc}}(\omega)$ and, in view of (9), it has compact support in ω. Using (10) with $\phi = v_i$ and taking into account the fact that v_i vanishes outside $\omega_+(\varepsilon_1,\varepsilon_2)$ we find

$$\begin{aligned} &-\int_{\omega_+(\varepsilon_1,\varepsilon_2)} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)dx \\ &\quad \ge \int_{\omega_+(\varepsilon_1,\varepsilon_2)} p(x)(f(u_2)v_2 - f(u_1)v_1)dx + a\int_{\omega_+(\varepsilon_1,\varepsilon_2)} (u_1v_1 - u_2v_2)dx\,. \end{aligned} \tag{11}$$

A simple computation shows that the integral in the left-hand side of (11) equals

$$-\int_{\omega_+(\varepsilon_1,\varepsilon_2)} \left(\left|\nabla u_2 - \frac{u_2+\varepsilon_2}{u_1+\varepsilon_1}\nabla u_1\right|^2 + \left|\nabla u_1 - \frac{u_1+\varepsilon_1}{u_2+\varepsilon_2}\nabla u_2\right|^2\right) dx \le 0\,.$$

Passing to the limit as $0 < \varepsilon_2 < \varepsilon_1 \to 0$, the first term in the right hand-side of (11) converges to

$$\int_{\omega_+(0,0)} p(x)\left(\frac{f(u_2)}{u_2} - \frac{f(u_1)}{u_1}\right)(u_2^2 - u_1^2)dx\,,$$

while the other term converges to 0. Hence, we avoid a contradiction only in the case that $\omega_+(0,0)$ has measure 0, which means that $u_1 \ge u_2$ on ω. □

With the same arguments Lemma 2.1 can be written in the following more general form.

Lemma 2.2. *Let ω be a bounded domain. Assume that p, q, r are $C^{0,\mu}$-functions on $\bar{\omega}$ such that $r \geq 0$ and $p > 0$ in ω. If u_1, $u_2 \in C^2(\omega)$ are positive functions in ω and*

$$\Delta u_1 + q(x)u_1 - p(x)f(u_1) + r(x) \leq 0$$
$$\leq \Delta u_2 + q(x)u_2 - p(x)f(u_2) + r(x) \quad \text{in } \omega \tag{12}$$
$$\limsup_{\text{dist}(x,\partial\omega)\to 0} (u_2 - u_1)(x) \leq 0 \tag{13}$$

then $u_1 \geq u_2$ in ω.

The next result extends Lemma 2.1 in Du–Huang [16].

Lemma 2.3. *Assume $\omega \subset\subset \Omega$ and $p \in C^{0,\mu}(\bar{\Omega}\backslash\omega)$ is a positive function in $\Omega\backslash\bar{\omega}$. If u_1, $u_2 \in C^2(\bar{\Omega}\backslash\bar{\omega})$ are positive functions in $\Omega\backslash\bar{\omega}$ and*

$$\Delta u_1 + au_1 - p(x)f(u_1) \leq 0 \leq \Delta u_2 + au_2 - p(x)f(u_2) \quad \text{in } \Omega\backslash\bar{\omega} \tag{14}$$

$$\mathcal{B}u_1 \geq 0 \geq \mathcal{B}u_2 \quad \text{on } \partial\Omega\,; \qquad \limsup_{\text{dist}(x,\partial\omega)\to 0} (u_2 - u_1)(x) \leq 0\,, \tag{15}$$

then $u_1 \geq u_2$ on $\bar{\Omega}\backslash\bar{\omega}$.

Proof. We distinguish 2 cases:

Case 1. $\mathcal{B} = \mathcal{D}$. The assertion is an easy consequence of Lemma 2.1.
Case 2. $\mathcal{B} = \mathcal{R}$. Let ϕ_1, ϕ_2 be two non-negative C^2-functions on $\bar{\Omega}\backslash\omega$ vanishing near $\partial\omega$.

Multiplying in (14) the first inequality (respectively, the second one) by ϕ_1 (respectively, ϕ_2) and applying integration by parts together with (15) we deduce that

$$-\int_{\tilde{\Omega}} (\nabla u_2 \cdot \nabla\phi_2 - \nabla u_1 \cdot \nabla\phi_1)dx - \int_{\partial\Omega} \beta(x)(u_2\phi_2 - u_1\phi_1)dS(x)$$
$$\geq \int_{\tilde{\Omega}} p(x)(f(u_2)\phi_2 - f(u_1)\phi_1)dx + a\int_{\tilde{\Omega}}(u_1\phi_1 - u_2\phi_2)dx\,, \tag{16}$$

where $\tilde{\Omega} := \Omega\backslash\bar{\omega}$. Let $\varepsilon_1 > \varepsilon_2 > 0$ and denote

$$\Omega_+(\varepsilon_1, \varepsilon_2) = \{x \in \tilde{\Omega} : u_2(x) + \varepsilon_2 > u_1(x) + \varepsilon_1\}\,.$$
$$v_i = (u_i + \varepsilon_i)^{-1}((u_2 + \varepsilon_2)^2 - (u_1 + \varepsilon_1)^2)^+, \quad i = 1, 2\,.$$

Since v_i can be approximated closely in the $H^1 \cap L^\infty$-topology on $\bar{\Omega}\backslash\omega$ by non-negative C^2-functions vanishing near $\partial\omega$, it follows that (16) holds for v_i taking place of ϕ_i. Since v_i vanishes outside the set $\Omega_+(\varepsilon_1, \varepsilon_2)$ relation (16) becomes

$$-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (\nabla u_2 \cdot \nabla v_2 - \nabla u_1 \cdot \nabla v_1)dx - \int_{\partial\Omega} \beta(x)(u_2 v_2 - u_1 v_1)dS(x)$$
$$\geq \int_{\Omega_+(\varepsilon_1,\varepsilon_2)} p(x)(f(u_2)v_2 - f(u_1)v_1)dx + a\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (u_1 v_1 - u_2 v_2)dx\,. \tag{17}$$

As $\varepsilon_1 \to 0$ (recall that $\varepsilon_1 > \varepsilon_2 > 0$) the second term on the left hand-side of (17) converges to 0. From now on, the course of the proof is the same as in Lemma 2.1. This completes the proof. $\square$

3. Large Solutions of Problem (1)

Remark 3.1. Assuming (A_1), problem (1) can have large solutions only if f satisfies the Keller–Osserman condition (A_2).

Suppose, *a priori*, that problem (1) has a large solution u_∞. Set $\tilde{f}(u) = |a|u + \|b\|_\infty f(u)$ for $u \geq 0$. Notice that $\tilde{f} \in C^1[0,\infty)$ satisfies $(A_1)'$. For any $n \geq 1$, consider the problem

$$\begin{cases} \Delta u = \tilde{f}(u) & \text{in } \Omega\,, \\ u = n & \text{on } \partial\Omega\,, \\ u \geq 0 & \text{in } \Omega\,. \end{cases}$$

By Theorem A.1, this problem has a unique solution, say u_n, which, moreover, is positive in $\bar{\Omega}$. Applying Lemma 2.2 for $q \equiv -|a|$, $p \equiv \|b\|_\infty$, $r \equiv 0$ and $\omega = \Omega$ we obtain

$$0 < u_n \leq u_{n+1} \leq u_\infty \quad \text{in } \Omega, \quad \forall\, n \geq 1\,.$$

Thus, for every $x \in \Omega$, we can define $\bar{u}(x) = \lim_{n\to\infty} u_n(x)$. Moreover, since (u_n) is uniformly bounded on every compact subset of Ω, standard elliptic regularity arguments show that $\bar{u}$ is a positive large solution of the problem $\Delta u = \tilde{f}(u)$. It follows that $\tilde{f}$ satisfies the Keller–Osserman condition (A_2). Then, by (A_1), $\mu_\infty := \lim_{u\to\infty} f(u)/u > 0$ which yields $\lim_{u\to\infty} \tilde{f}(u)/f(u) = |a|/\mu_\infty + \|b\|_\infty < \infty$. Consequently, our claim follows.

Typical examples of non-linearities satisfying (A_1) and (A_2) are:

(i) $f(u) = e^u - 1$;
(ii) $f(u) = u^p,\ p > 1$;
(iii) $f(u) = u[\ln(u+1)]^p$, $p > 2$.

Remark 3.2. We have $\mu_\infty := \lim_{u\to\infty} f(u)/u = \lim_{u\to\infty} f'(u) = \infty$.

Indeed, by l'Hospital's rule, $\lim_{u\to\infty} F(u)/u^2 = \mu_\infty/2$. But, by (A_2), we deduce that $\mu_\infty = \infty$. Then, by (A_1) we find that $f'(u) \geq f(u)/u$ for any $u > 0$, which shows that $\lim_{u\to\infty} f'(u) = \infty$.

Proof of Theorem 1.1. A. Necessary condition. Let u_∞ be a large solution of problem (1). Corollary A.2 implies that u_∞ is positive. Suppose $\lambda_{\infty,1}$ is finite. Arguing by contradiction, let us assume $a \geq \lambda_{\infty,1}$. Set $\lambda \in (\lambda_1(\mu_0), \lambda_{\infty,1})$ and denote by u_λ the unique positive solution of problem (E_a) with $a = \lambda$. We have

$$\begin{cases} \Delta(Mu_\infty) + \lambda_{\infty,1}(Mu_\infty) \leq b(x) f(Mu_\infty) & \text{in } \Omega\,, \\ Mu_\infty = \infty & \text{on } \partial\Omega\,, \\ Mu_\infty \geq u_\lambda & \text{in } \Omega\,, \end{cases}$$

where $M := \max\{\max_{\bar{\Omega}} u_\lambda / \min_\Omega u_\infty; 1\}$. By the sub-super solution method we conclude that problem (E_a) with $a = \lambda_{\infty,1}$ has at least a positive solution (between u_λ and Mu_∞). But this is a contradiction. So, necessarily, $a \in (-\infty, \lambda_{\infty,1})$.

B. Sufficient condition. This will be proved with the aid of several results. We assume, until the end of this Section, that f satisfies (A_1) and (A_2). $\square$

Lemma 3.1. *Let ω be a smooth bounded domain in $\mathbf{R}^N$. Assume p, q, r are $C^{0,\mu}$-functions on $\bar{\omega}$ such that $r \geq 0$ and $p > 0$ in $\bar{\omega}$. Then for any non-negative function $0 \not\equiv \Phi \in C^{0,\mu}(\partial\omega)$ the boundary value problem*

$$\begin{cases} \Delta u + q(x)u = p(x)f(u) - r(x) & \text{in } \omega\,, \\ u > 0 & \text{in } \omega\,, \\ u = \Phi & \text{on } \partial\omega\,, \end{cases} \tag{18}$$

has a unique solution.

Proof. By Lemma 2.2, problem (18) has at most a solution. The existence of a positive solution will be obtained by device of sub and super-solutions.

Set $p_0 := \inf_\omega p > 0$. Define $\bar{f}(u) = p_0 f(u) - \|q\|_\infty u - \bar{r}$, where $\bar{r} := \sup_\omega r + 1 > 0$. Let t_1 be the unique positive solution of the equation $\bar{f}(u) = 0$. By Remark 3.2 we derive that $\lim_{u\to\infty} \frac{\bar{f}(u)}{f(u)} = p_0 > 0$. Combining this with (A_2), we conclude that the function $\phi(w) = \bar{f}(w + t_1)$ defined for $w \geq 0$ satisfies the assumptions of Theorem III in [20]. It follows that there exists a positive large solution for the equation $\Delta w = \phi(w)$ in ω. Thus the function $\bar{u}(x) = w(x) + t_1$, for all $x \in \omega$, is a positive large solution of the problem

$$\Delta u + \|q\|_\infty u = p_0 f(u) - \bar{r} \quad \text{in } \omega\,. \tag{19}$$

Applying Theorem A.1, the boundary value problem

$$\begin{cases} \Delta u = \|q\|_\infty u + \|p\|_\infty f(u) & \text{in } \omega\,, \\ u > 0 & \text{in } \omega\,, \\ u = \Phi & \text{on } \partial\omega\,, \end{cases} \tag{20}$$

has a unique classical solution $\underline{u}$. By Lemma 2.2, we find that $\underline{u} \leq \bar{u}$ in ω and $\underline{u}$ (respectively, $\bar{u}$) is a positive sub-solution (respectively, super-solution) of problem (18). It follows that (18) has a unique solution. $\square$

Under the assumptions of Lemma 3.1 we obtain the following result which generalizes [26, Lemma 1.3].

Corollary 3.1. *There exists a positive large solution of the problem*

$$\Delta u + q(x)u = p(x)f(u) - r(x) \quad \text{in } \omega\,. \tag{21}$$

Proof. Set $\Phi = n$ and let u_n be the unique solution of (18). By Lemma 2.2, $u_n \leq u_{n+1} \leq \bar{u}$ in ω, where $\bar{u}$ denotes a large solution of (19). Thus $\lim_{n\to\infty} u_n(x) =$

$u_\infty(x)$ exists and is a positive large solution of (21). Furthermore, every positive large solution of (21) dominates u_∞, i.e. the solution u_∞ is the *minimal large solution*. This follows from the definition of u_∞ and Lemma 2.2. $\square$

Lemma 3.2. *If $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ is a non-negative function and $b > 0$ on $\partial\Omega$, then the boundary value problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & in\ \Omega\,, \\ u > 0 & in\ \Omega\,, \\ u = \Phi & on\ \partial\Omega\,, \end{cases} \tag{22}$$

has a solution if and only if $a \in (-\infty, \lambda_{\infty,1})$. Moreover, in this case, the solution is unique.

Proof. The first part follows exactly in the same way as the proof of Theorem 1.1 (necessary condition).

For the sufficient condition, fix $a < \lambda_{\infty,1}$ and let $\lambda_{\infty,1} > \lambda_* > \max\{a, \lambda_1(\mu_0)\}$. Let u_* be the unique positive solution of (E_a) with $a = \lambda_*$.

Let Ω_i $(i = 1, 2)$ be subdomains of Ω such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ and $\Omega\backslash\bar\Omega_1$ is smooth.

We define $u_+ \in C^2(\Omega)$ as a positive function in Ω such that $u_+ \equiv u_\infty$ on $\Omega\backslash\Omega_2$ and $u_+ \equiv u_*$ on Ω_1. Here u_∞ denotes a positive large solution of (21) for $p(x) = b(x)$, $r(x) = 0$, $q(x) = a$ and $\omega = \Omega\backslash\bar\Omega_1$. Using Remark 3.2 and the fact that $b_0 := \inf_{\Omega_2\backslash\Omega_1} b$ is positive, it is easy to check that if $C > 0$ is large enough then $\bar v_\Phi = Cu_+$ satisfies

$$\begin{cases} \Delta\bar v_\Phi + a\bar v_\Phi \le b(x)f(\bar v_\Phi) & \text{in } \Omega\,, \\ \bar v_\Phi = \infty & \text{on } \partial\Omega\,. \\ \bar v_\Phi \ge \max_{\partial\Omega}\Phi & \text{in } \Omega\,. \end{cases}$$

By Theorem A.1 in the Appendix, there exists a unique classical solution $\underline{v}_\Phi$ of the problem

$$\begin{cases} \Delta\underline{v}_\Phi = |a|\underline{v}_\Phi + \|b\|_\infty f(\underline{v}_\Phi) & \text{in } \Omega\,, \\ \underline{v}_\Phi > 0 & \text{in } \Omega\,, \\ \underline{v}_\Phi = \Phi & \text{on } \partial\Omega\,. \end{cases}$$

It is clear that $\underline{v}_\Phi$ is a positive sub-solution of (22) and $\underline{v}_\Phi \le \max_{\partial\Omega}\Phi \le \bar v_\Phi$ in Ω. Therefore, by the sub-super solution method, problem (22) has at least a solution v_Φ between $\underline{v}_\Phi$ and $\bar v_\Phi$. Next, the uniqueness of solution to (22) can be obtained by using essentially the same technique as in [10, Theorem 1.1] or [9, Appendix II]. $\square$

Proof of Theorem 1.1 completed. Fix $a \in (-\infty, \lambda_{\infty,1})$. Two cases may occur:

Case 1. $b > 0$ on $\partial\Omega$. Denote by v_n the unique solution of (22) with $\Phi \equiv n$. For $\Phi \equiv 1$, set $v := \underline{v}_\Phi$ and $V := \bar v_\Phi$, where $\underline{v}_\Phi$ and $\bar v_\Phi$ are defined in the proof of

Lemma 3.2. The sub and super-solutions method combined with the uniqueness of solution of (22) shows that $v \le v_n \le v_{n+1} \le V$ in Ω. Hence $v_\infty(x) := \lim_{n\to\infty} v_n(x)$ exists and is a positive large solution of (1).

Case 2. $b \ge 0$ on $\partial\Omega$. Let z_n $(n \ge 1)$ be the unique solution of (18) for $p \equiv b+1/n$, $r \equiv 0$, $q \equiv a$, $\Phi \equiv n$ and $\omega = \Omega$. By Lemma 2.1, (z_n) is non-decreasing. Moreover, (z_n) is uniformly bounded on every compact subdomain of Ω. Indeed, if $K \subset \Omega$ is an arbitrary compact set, then $d := \mathrm{dist}(K, \partial\Omega) > 0$. Choose $\delta \in (0, d)$ small enough so that $\bar{\Omega}_0 \subset C_\delta$, where $C_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$. Since $b > 0$ on ∂C_δ, Case 1 allows us to define z_+ as a positive large solution of (1) for $\Omega = C_\delta$. Using Lemma 2.1 for $p \equiv b+1/n$ and $\omega = C_\delta$ we obtain $z_n \le z_+$ in C_δ, for all $n \ge 1$. So, (z_n) is uniformly bounded on K. By the monotonicity of (z_n), we conclude that $z_n \to \underline{z}$ in $L^\infty_{\mathrm{loc}}(\Omega)$. Finally, standard elliptic regularity arguments lead to $z_n \to \underline{z}$ in $C^{2,\mu}(\Omega)$. This completes the proof of Theorem 1.1. □

4. Auxiliary Results

The main purpose of this section is to provide an equivalent criterion to the Keller–Osserman condition (A_2). To our best knowledge there are no results of this type. We point out that, throughout this Section, a significant role plays the set $\mathcal{G}$ defined by

$$\mathcal{G} = \left\{ g : \exists\, \delta > 0 \text{ such that } g \in C^2(0,\delta),\ g'' > 0 \text{ on } (0,\delta)\,, \right.$$
$$\left. \lim_{t\searrow 0} g(t) = \infty \text{ and } \exists \lim_{t\searrow 0} \frac{g'(t)}{g''(t)} \right\}.$$

Note that $\mathcal{G} \not\equiv \emptyset$. We see, for example, that $e^\Theta \subset \mathcal{G}$ where

$$\Theta = \{\theta : \theta \in C^2(0,\infty),\ \theta \text{ is convex on } (0,\infty) \text{ and } \lim_{t\searrow 0} \theta(t) = \infty\}\,.$$

Obviously, $\Theta \not\equiv \emptyset$. Let $\theta \in \Theta$ be arbitrary. Since θ' is non-decreasing on $(0,\infty)$ and $\lim_{t\searrow 0} \theta(t) = \infty$, it follows that $\lim_{t\searrow 0} \theta'(t) = -\infty$. Then,

$$\left| \frac{\theta'(t)}{(\theta'(t))^2 + \theta''(t)} \right| \le \frac{1}{|\theta'(t)|} \to 0 \quad \text{as } t \searrow 0$$

which proves that $e^\theta \in \mathcal{G}$.

Remark 4.1. $\lim_{t\searrow 0} \frac{g(t)}{g''(t)} = \lim_{t\searrow 0} \frac{g'(t)}{g''(t)} = 0$ for any function $g \in \mathcal{G}$.

Indeed, if $g \in \mathcal{G}$ is chosen arbitrarily, then

$$\lim_{t\searrow 0} g'(t) = -\infty\,, \quad \lim_{t\searrow 0} \ln g(t) = \infty \quad \text{and} \quad \lim_{t\searrow 0} \ln|g'(t)| = \infty\,. \tag{23}$$

L'Hospital's rule and (23) imply that $\lim_{t\searrow 0} \frac{g(t)}{g'(t)} = \lim_{t\searrow 0} \frac{g'(t)}{g''(t)} = 0$.

Lemma 4.1. *Assume* $(\tilde{A}_1)$. *Then, the following hold*:

(i) $\gamma \geq 0$.
(ii) $\gamma \leq 1/2$ *provided that* (A_2) *is fulfilled.*

Proof. (i) If we suppose $\gamma < 0$, then there exists $s_1 > 0$ such that

$$\left(\frac{F}{f}\right)'(u) \leq \frac{\gamma}{2} < 0 \quad \text{for any } u \geq s_1\,.$$

Integrating this inequality over (s_1, ∞) we obtain a contradiction. It follows that $\gamma \geq 0$.

(ii) Let (A_2) be satisfied. Using the definition of γ, we find $\lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)} = 1-\gamma$. By Remark 3.2 and L'Hospital's rule we obtain

$$\lim_{u\to\infty} \frac{F(u)}{f^2(u)} \overset{\frac{\infty}{\infty}}{=} \lim_{u\to\infty} \frac{1}{2f'(u)} = 0$$

and

$$0 \leq \lim_{u\to\infty} \frac{\frac{\sqrt{F(u)}}{f(u)}}{\int_u^\infty \frac{ds}{\sqrt{F(s)}}} \overset{\frac{0}{0}}{=} -\frac{1}{2} + \lim_{u\to\infty} \frac{F(u)f'(u)}{f^2(u)} = \frac{1}{2} - \gamma\,. \tag{24}$$

This concludes our proof. $\square$

Lemma 4.2. *Assume* $(\tilde{A}_1)$. *Then the Keller–Osserman growth condition* (A_2) *holds if and only if*

(A_g) $\lim_{t\searrow 0} \frac{tf(g(t))}{g''(t)} = \infty$ *for some function* $g \in \mathcal{G}$.

Proof. A. Necessary condition. Since (A_2) holds, we can define the positive function g as follows

$$\int_{g(t)}^\infty \frac{ds}{\sqrt{F(s)}} = t^\vartheta \quad \text{for all } t > 0\,, \quad \text{where } \vartheta \in \left(\frac{3}{2}, \infty\right) \text{ is arbitrary}\,. \tag{25}$$

Obviously, $g \in C^2(0,\infty)$ and $\lim_{t\searrow 0} g(t) = \infty$. We claim that $g \in \mathcal{G}$ and condition (A_g) is fulfilled. To argue this, we divide our argument into three steps:

Step 1. $\lim_{t\searrow 0} \frac{g'(t)}{t^{2\vartheta-1}f(g(t))} = \vartheta(\gamma - \frac{1}{2})$.

We derive twice relation (25) and obtain

$$g'(t) = -\vartheta t^{\vartheta-1}\sqrt{F(g(t))}\,, \tag{26}$$

$$\begin{aligned} g''(t) &= \frac{\vartheta-1}{t} g'(t) + \frac{\vartheta^2}{2} t^{2\vartheta-2} f(g(t)) \\ &= \frac{\vartheta^2}{2} t^{2\vartheta-2} f(g(t)) \left(\frac{2(\vartheta-1)}{\vartheta^2} \frac{g'(t)}{t^{2\vartheta-1}f(g(t))} + 1\right). \end{aligned} \tag{27}$$

By using (26) and (24) we find

$$\lim_{t\searrow 0}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}=\lim_{t\searrow 0}\frac{-\vartheta t^{\vartheta-1}\sqrt{F(g(t))}}{t^{2\vartheta-1}f(g(t))}$$

$$=\lim_{t\searrow 0}-\vartheta\frac{\frac{\sqrt{F(g(t))}}{f(g(t))}}{\int_{g(t)}^{\infty}\frac{ds}{\sqrt{F(s)}}}=\lim_{u\to\infty}-\vartheta\frac{\frac{\sqrt{F(u)}}{f(u)}}{\int_{u}^{\infty}\frac{ds}{\sqrt{F(s)}}}=\vartheta\left(\gamma-\frac{1}{2}\right).$$

Step 2. $g''>0$ on $(0,\delta)$ for δ small enough.

Since $\gamma\geq 0$, by using Step 1 we find

$$\lim_{t\searrow 0}\frac{2(\vartheta-1)}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}=\frac{2(\vartheta-1)}{\vartheta}\left(\gamma-\frac{1}{2}\right)\geq\frac{1}{\vartheta}-1>-1\,. \tag{28}$$

In view of (27), the assertion of this step follows.

Step 3. $\lim_{t\searrow 0}\frac{g'(t)}{g''(t)}=0$ and $\lim_{t\searrow 0}\frac{tf(g(t))}{g''(t)}=\infty$.

Taking into account (27) and (28) we find

$$\lim_{t\searrow 0}\frac{g'(t)}{g''(t)}=\lim_{t\searrow 0}\frac{2t}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}\frac{1}{\frac{2(\vartheta-1)}{\vartheta^2}\frac{g'(t)}{t^{2\vartheta-1}f(g(t))}+1}=0$$

and, for any $t\in(0,\delta)$ where $\delta>0$ is given by Step 2, we have

$$\frac{tf(g(t))}{g''(t)}=\frac{tf(g(t))}{\frac{\vartheta-1}{t}g'(t)+\frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t))}\geq\frac{tf(g(t))}{\frac{\vartheta^2}{2}t^{2\vartheta-2}f(g(t))}=\frac{2}{\vartheta^2t^{2\vartheta-3}}\,.$$

Sending t to 0, the claim of Step 3 is proved.

B. Sufficient condition. Let $g\in\mathcal{G}$ be chosen so that (A_g) is fulfilled. By L'Hospital's rule we find

$$\lim_{t\searrow 0}\frac{(g'(t))^2}{F(g(t))}=2\lim_{t\searrow 0}\frac{g''(t)}{f(g(t))}=0\,.$$

We choose $\delta>0$ small enough such that $g'(s)<0$ and $g''(s)>0$ for all $s\in(0,\delta)$. It follows that

$$\int_{g(\delta)}^{\infty}\frac{dt}{\sqrt{F(t)}}=\lim_{t\searrow 0}\int_{g(\delta)}^{g(t)}\frac{ds}{\sqrt{F(s)}}=\lim_{t\searrow 0}\int_{t}^{\delta}\frac{-g'(s)ds}{\sqrt{F(g(s))}}\leq\delta\sup_{t\in(0,\delta)}\frac{-g'(t)}{\sqrt{F(g(t))}}<\infty\,.$$

Hence, the growth condition (A_2) holds. ◻

Lemma 4.3. *Assume that* $(\tilde{A}_1)$ *with* $\gamma\neq 0$, (A_2), (B_1) *and* (B_2) *are fulfilled. Then, the following hold*:

(i) $K'(0)(1-2\gamma)+2\gamma\in(0,1]$.
(ii) $h\in\mathcal{G}$, *where* h *is the function defined by* (7).

Proof. (i) Since $\gamma \neq 0$, by Lemma 4.1 we find $0 < \gamma \leq 1/2$. Therefore, the claim of (i) follows if we prove that $K'(0) \in [0,1]$. To this aim, we remark that $K(0) = 0$. Suppose that $K(0) \neq 0$. Then, we obtain

$$\lim_{t \searrow 0} \left[\ln \left(\int_0^t \sqrt{k(s)} ds \right) \right]' (t) = \frac{1}{K(0)} \in (0, \infty) ,$$

which contradicts the fact that $\lim_{t\searrow 0} \ln(\int_0^t \sqrt{k(s)}ds) = -\infty$. So, $K(0) = 0$. This produces $K'(0) \geq 0$. Since $K \in C_1[0, \delta_0)$, we have

$$K'(0) = \lim_{t \searrow 0} \left(\frac{\int_0^t \sqrt{k(s)} ds}{\sqrt{k(t)}} \right)'$$

so that

$$\lim_{t \searrow 0} \frac{k'(t) \int_0^t \sqrt{k(s)} ds}{k^{3/2}(t)} = 2 \left(1 - \lim_{t \searrow 0} \left(\frac{\int_0^t \sqrt{k(s)} ds}{\sqrt{k(t)}} \right)' \right) = 2(1 - K'(0)) . \tag{29}$$

Hence, $K'(0) \leq 1$. Indeed, assuming the contrary, relation (29) yields $k'(t) < 0$ for $t \in (0, \tilde{\delta})$ for some $0 < \tilde{\delta} < \delta_0$. But this is impossible, since $\lim_{t\searrow 0} k(t) = 0$ and $k > 0$ on $(0, \delta_0)$.

(ii) Using the definition of h, we deduce that $h \in C^2(0, \delta_0)$ and $\lim_{t\searrow 0} h(t) = \infty$. Then, by twice deriving relation (7), we find

$$h'(t) = -\sqrt{k(t)} \sqrt{2F(h(t))}, \quad \forall \, t \in (0, \delta_0) ,$$

respectively,

$$\begin{aligned} h''(t) &= k(t) f(h(t)) - \frac{1}{\sqrt{2}} \frac{\sqrt{F(h(t))}}{\sqrt{k(t)}} k'(t) \\ &= k(t) f(h(t)) \left(1 - \frac{k'(t) \int_0^t \sqrt{k(s)} ds}{k^{3/2}(t)} \frac{\frac{\sqrt{F(h(t))}}{f(h(t))}}{\int_{h(t)}^{\infty} \frac{ds}{\sqrt{F(s)}}} \right) . \end{aligned}$$

Using (24) and (29), we obtain

$$\begin{aligned} \lim_{t \searrow 0} \frac{h'(t)}{h''(t)} &= \frac{-2}{K'(0)(1-2\gamma) + 2\gamma} \lim_{t \searrow 0} \frac{\frac{\sqrt{F(h(t))}}{f(h(t))}}{\int_{h(t)}^{\infty} \frac{ds}{\sqrt{F(s)}}} \lim_{t \searrow 0} \frac{\int_0^t \sqrt{k(s)} ds}{\sqrt{k(t)}} \\ &= \frac{2\gamma - 1}{K'(0)(1-2\gamma) + 2\gamma} K(0) = 0 . \end{aligned}$$

and

$$\lim_{t \searrow 0} \frac{h''(t)}{k(t) f(h(t))} = K'(0)(1-2\gamma) + 2\gamma > 0 \tag{30}$$

which shows that h'' is positive on $(0, \delta_1)$ for some $\delta_1 > 0$. This concludes our proof. $\square$

5. Proof of Theorem 1.2

We start with the following result.

Lemma 5.1. *Assume $b > 0$ on $\partial\Omega$. If (A_1) and (A_2) hold, then for any positive function $\Phi \in C^{2,\mu}(\partial\Omega_0)$ and $a \in \mathbf{R}$ the problem*

$$\begin{cases} \Delta u + au = b(x)f(u) & \textit{in } \Omega\backslash\bar{\Omega}_0\,, \\ \mathcal{B}u = 0 & \textit{on } \partial\Omega\,, \\ u = \Phi & \textit{on } \partial\Omega_0\,, \end{cases} \tag{31}$$

has a unique positive solution.

Proof. In view of Lemma 2.3 we find that (31) has at most a positive solution. To prove the existence of a positive solution to (31) we shall use the sub and super-solution method.

Let $\omega \subset\subset \Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0\backslash\bar{\omega}$ is greater than a. Let $p \in C^{0,\mu}(\bar{\Omega})$ be such that $p(x) = b(x)$ for $x \in \bar{\Omega}\backslash\Omega_0$, $p(x) = 0$ for $x \in \bar{\Omega}_0\backslash\omega$ and $p(x) > 0$ for $x \in \omega$. By virtue of Lemma 3.2, problem

$$\begin{cases} \Delta u + au = p(x)f(u) & \text{in } \Omega\,, \\ u = 1 & \text{on } \partial\Omega\,, \end{cases}$$

has a unique positive solution u_1.

We choose Ω_1 and Ω_2 two subdomains of Ω such that $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$.

Define $u^* \in C^2(\bar{\Omega}\backslash\Omega_0)$ so that $u^* \equiv 1$ on $\bar{\Omega}\backslash\Omega_2$, $u^* \equiv u_1$ on $\bar{\Omega}_1\backslash\Omega_0$ and $m_* := \min_{\bar{\Omega}\backslash\Omega_0} u^* > 0$.

Claim. *For $\ell \geq 1$ large enough, ℓu^* is a super-solution for problem* (31).

We first observe that

$$\begin{aligned} -\Delta(\ell u^*) &= \ell a u_1 - \ell p(x) f(u_1) \\ &\geq a(\ell u^*) - b(x)f(\ell u^*) \quad \text{for } x \in \bar{\Omega}_1\backslash\bar{\Omega}_0 \quad \text{and} \quad \ell \geq 1\,. \end{aligned} \tag{32}$$

Denote by $M^* := \sup_{\Omega\backslash\Omega_1}(au^* + \Delta u^*)$ and $b_0 := \min_{\bar{\Omega}\backslash\Omega_1} b > 0$. By Remark 3.2, we obtain that there exists $\ell_1 \geq 1$ such that

$$f(\ell m_*) \geq \frac{\ell M^*}{b_0} \quad \text{for all } \ell \geq \ell_1\,.$$

For $x \in \Omega\backslash\bar{\Omega}_1$ and $\ell \geq \ell_1$ we have

$$b(x)f(\ell u^*) \geq b_0 f(\ell m_*) \geq \ell(au^* + \Delta u^*)$$

which can be rewritten as

$$-\Delta(\ell u^*) \geq a(\ell u^*) - b(x)f(\ell u^*) \quad \text{for } x \in \Omega\backslash\bar{\Omega}_1 \quad \text{and} \quad \ell \geq \ell_1\,. \tag{33}$$

By (32) and (33) it follows that

$$-\Delta(\ell u^*) \geq a(\ell u^*) - b(x)f(\ell u^*) \quad \text{in } \Omega\backslash\bar{\Omega}_0\,, \quad \text{for any } \ell \geq \ell_1\,.$$

On the other hand,

$$\mathcal{B}(\ell u^*) \geq \ell \min\left\{1, \min_{x\in\partial\Omega} \beta(x)\right\} \geq 0 \quad \text{on } \partial\Omega\,, \quad \text{for every } \ell > 0\,.$$

By taking $\ell \geq \max\{\max_{\partial\Omega_0} \Phi/m_*; \ell_1\}$ the claim follows.

Set $\bar{b} := \sup_\Omega b$. By Theorem A.1, the boundary value problem

$$\begin{cases} \Delta u_* = \bar{b} f(u_*) + |a| u_* & \text{in } \Omega\backslash\bar{\Omega}_0\,, \\ u_* = 0 & \text{on } \partial\Omega\,, \\ u_* = \Phi & \text{on } \partial\Omega_0\,, \end{cases} \tag{34}$$

has a unique non-negative solution, which is positive in $\Omega\backslash\bar{\Omega}_0$. Since $u_* = 0$ on $\partial\Omega$ we find that $\mathcal{R}u_* = \partial_\nu u_* \leq 0$ on $\partial\Omega$. It is easy to see that u_* is a sub-solution of (31) and $u_* \leq \ell u^*$ in $\bar{\Omega}\backslash\Omega_0$ for ℓ large enough. The conclusion of Lemma 5.1 follows now by the sub-super solution method. □

Corollary 5.1. *If Ω_0 is replaced by Ω_m defined in* (5), *then the statement of Lemma* 5.1 *holds.*

Proof. The proof is very easy in this case. The construction of the sub-solution is made as before, while the super-solution can be chosen any number $\ell \geq 1$ large enough. □

We now come back to the proof of Theorem 1.2, that will be divided into two steps:

Step 1. Existence of the minimal positive solution for problem (4).

For any $n \geq 1$, let u_n be the unique positive solution of problem (31) with $\Phi \equiv n$. By Lemma 2.3, $u_n(x)$ increases with n for all $x \in \bar{\Omega}\backslash\bar{\Omega}_0$. Moreover, we prove

Lemma 5.2. *The sequence $(u_n(x))_n$ is bounded from above by some function $V(x)$ which is uniformly bounded on all compact subsets of $\bar{\Omega}\backslash\bar{\Omega}_0$.*

Proof. Let b^* be a C^2-function on $\bar{\Omega}\backslash\Omega_0$ such that

$$0 < b^*(x) \leq b(x) \quad \forall\, x \in \bar{\Omega}\backslash\bar{\Omega}_0\,.$$

For x bounded away from $\partial\Omega_0$ is not a problem to find such a function b^*. For x satisfying $0 < d(x) < \delta$ with $\delta > 0$ small such that $x \to d(x)$ is a C^2-function, we can take

$$b^*(x) = \int_0^{d(x)} \int_0^t \Big[\min_{d(z)\geq s} b(z)\Big] ds dt\,.$$

Let $g \in \mathcal{G}$ be a function such that (A_g) holds. The existence of g is guaranteed by Lemma 4.2. Since $b^*(x) \to 0$ as $d(x) \searrow 0$, we deduce, by Remark 4.1 and (A_1), the existence of some $\delta > 0$ such that for all $x \in \Omega$ with $0 < d(x) < \delta$ and $\xi > 1$

$$\frac{b^*(x) f(g(b^*(x))\xi)}{g''(b^*(x))\xi} > \sup_{\bar{\Omega}\backslash\Omega_0} |\nabla b^*|^2 + \frac{g'(b^*(x))}{g''(b^*(x))} \inf_{\bar{\Omega}\backslash\Omega_0} (\Delta b^*) + a\frac{g(b^*(x))}{g''(b^*(x))}\,.$$

Here, $\delta > 0$ is taken sufficiently small so that $g'(b^*(x)) < 0$ and $g''(b^*(x)) > 0$ for all x with $0 < d(x) < \delta$.

For $n_0 \geq 1$ fixed, define V^* as follows

(i) $V^*(x) = u_{n_0}(x) + 1$ for $x \in \bar{\Omega}$ and near $\partial\Omega$;
(ii) $V^*(x) = g(b^*(x))$ for x satisfying $0 < d(x) < \delta$;
(iii) $V^* \in C^2(\bar{\Omega}\backslash\bar{\Omega}_0)$ is positive on $\bar{\Omega}\backslash\bar{\Omega}_0$.

We show that for $\xi > 1$ large enough the upper bound of the sequence $(u_n(x))_n$ can be taken as $V(x) = \xi V^*(x)$. Since

$$\mathcal{B}V(x) = \xi\mathcal{B}V^*(x) \geq \xi \min\{1, \beta(x)\} \geq 0\,,$$
$$\forall\, x \in \partial\Omega \quad \text{and} \quad \lim_{d(x)\searrow 0}[u_n(x) - V(x)] = -\infty < 0\,,$$

to conclude that $u_n(x) \leq V(x)$ for all $x \in \bar{\Omega}\backslash\bar{\Omega}_0$ it is sufficient to show, by virtue of Lemma 2.3, that

$$-\Delta V(x) \geq aV(x) - b(x)f(V(x))\,, \quad \forall\, x \in \Omega\backslash\bar{\Omega}_0\,. \tag{35}$$

For $x \in \Omega$ satisfying $0 < d(x) < \delta$ and $\xi > 1$ we have

$$\begin{aligned}-\Delta V(x) - aV(x) + b(x)f(V(x)) &= -\xi\Delta g(b^*(x)) - a\xi g(b^*(x)) + b(x)f(g(b^*(x))\xi)\\ &\geq \xi g''(b^*(x))\left(-\frac{g'(b^*(x))}{g''(b^*(x))}\Delta b^*(x) - |\nabla b^*(x)|^2\right.\\ &\quad\left. -\ a\frac{g(b^*(x))}{g''(b^*(x))} + b^*(x)\frac{f(g(b^*(x))\xi)}{g''(b^*(x))\xi}\right) > 0\,.\end{aligned}$$

For $x \in \Omega$ satisfying $d(x) \geq \delta$,

$$-\Delta V(x) - aV(x) + b(x)f(V(x)) = \xi\left(-\Delta V^*(x) - aV^*(x) + b(x)\frac{f(\xi V^*(x))}{\xi}\right) \geq 0$$

for ξ sufficiently large. In the last inequality, we have used (iii) and Remark 3.2. It follows that (35) is fulfilled provided ξ is large enough. This finishes the proof of the lemma. □

By Lemma 5.2, $\underline{U}_a(x) \equiv \lim_{n\to\infty} u_n(x)$ exists, for any $x \in \bar{\Omega}\backslash\bar{\Omega}_0$. Moreover, $\underline{U}_a$ is a positive solution of (4). Using Lemma 2.3 once more, we find that any positive solution u of (4) satisfies $u \geq u_n$ on $\bar{\Omega}\backslash\bar{\Omega}_0$, for all $n \geq 1$. Hence $\underline{U}_a$ is the minimal positive solution of (4).

Proof of Theorem 1.2 completed.

Step 2. Existence of the maximal positive solution for problem (4).

Lemma 5.3. *If Ω_0 is replaced by Ω_m defined in* (5), *then problem* (4) *has a minimal positive solution provided that* (A_1) *and* (A_2) *are fulfilled.*

Proof. The argument used here (more easier, since $b > 0$ on $\bar{\Omega}\backslash\Omega_m$) is similar to that in Step 1. The only difference which appears in the proof (except the replacement of Ω_0 by Ω_m) is related to the construction of $V^*(x)$ for x near $\partial\Omega_m$. Here, instead of Lemma 4.2 we use our Theorem 1.1 which says that, for any $a \in \mathbf{R}$, there exists a positive large solution $u_{a,\infty}$ of problem (1) in the domain $\Omega\backslash\bar{\Omega}_m$. We define $V^*(x) = u_{a,\infty}(x)$ for $x \in \Omega\backslash\bar{\Omega}_m$ and near $\partial\Omega_m$. For $\xi > 1$ and $x \in \Omega\backslash\bar{\Omega}_m$ near $\partial\Omega_m$ we have

$$\begin{aligned} -\Delta V(x) - aV(x) + b(x)f(V(x)) &= -\xi\Delta V^*(x) - a\xi V^*(x) + b(x)f(\xi V^*(x)) \\ &= b(x)[f(\xi V^*(x)) - \xi f(V^*(x)] \geq 0\,. \end{aligned}$$

This completes the proof. □

Let v_m be the minimal positive solution for the problem considered in the statement of Lemma 5.3. By Lemma 2.3, $v_m \geq v_{m+1} \geq u$ on $\bar{\Omega}\backslash\bar{\Omega}_m$, where u is any positive solution of (4). Hence $\bar{U}_a(x) := \lim_{m\to\infty} v_m(x) \geq u(x)$. A regularity and compactness argument shows that $\bar{U}_a$ is a positive solution of (4). Consequently, $\bar{U}_a$ is the maximal positive solution. This concludes the proof of Theorem 1.2. □

6. Proof of Theorem 1.3

By (A_4) we deduce that the mapping $(0,\infty) \ni \xi \longmapsto A(\xi) = \lim_{u\to\infty} \frac{f(\xi u)}{\xi f(u)}$ is a continuous positive function, since $A(1/\xi) = 1/A(\xi)$ for any $\xi \in (0,1)$. Moreover, we claim

Lemma 6.1. *The function $A : (0,\infty) \to (0,\infty)$ is bijective, provided that (A_3) and (A_4) are fulfilled.*

Proof. By the continuity of A, we see that the surjectivity of A follows if we prove that $\lim_{\xi\searrow 0} A(\xi) = 0$. To this aim, let $\xi \in (0,1)$ be fixed. Using (A_3) we find

$$\frac{f(\xi u)}{\xi f(u)} \leq \xi^\zeta\,, \quad \forall\, u \geq \frac{t_0}{\xi}$$

which yields $A(\xi) \leq \xi^\zeta$. Since $\xi \in (0,1)$ is arbitrary, it follows that $\lim_{\xi\searrow 0} A(\xi) = 0$.

We now prove that the function $\xi \longmapsto A(\xi)$ is increasing on $(0,\infty)$ which concludes our lemma. Let $0 < \xi_1 < \xi_2 < \infty$ be chosen arbitrarily. Using assumption (A_3) once more, we obtain

$$f(\xi_1 u) = f\left(\frac{\xi_1}{\xi_2}\xi_2 u\right) \leq \left(\frac{\xi_1}{\xi_2}\right)^{1+\zeta} f(\xi_2 u)\,, \quad \forall\, u \geq t_0\frac{\xi_2}{\xi_1}\,.$$

It follows that

$$\frac{f(\xi_1 u)}{\xi_1 f(u)} \leq \left(\frac{\xi_1}{\xi_2}\right)^\zeta \frac{f(\xi_2 u)}{\xi_2 f(u)}\,, \quad \forall\, u \geq t_0\frac{\xi_2}{\xi_1}\,.$$

Passing to the limit as $u \to \infty$ we find

$$A(\xi_1) \le \left(\frac{\xi_1}{\xi_2}\right)^{\zeta} A(\xi_2) < A(\xi_2)\,,$$

which finishes the proof. $\square$

Proof of Theorem 1.3 completed. By Lemma 4.3, $h \in \mathcal{G}$. Set $\Pi(\xi) = \lim_{d(x)\searrow 0} b(x)\frac{f(h(d(x))\xi)}{h''(d(x))\xi}$, for any $\xi > 0$. Using (B_1) and (30), we find

$$\begin{aligned}\Pi(\xi) &= \lim_{d(x)\searrow 0} \frac{b(x)}{k(d(x))}\frac{k(d(x))f(h(d(x)))}{h''(d(x))}\frac{f(h(d(x))\xi)}{\xi f(h(d(x)))}\\ &= c\lim_{t\searrow 0}\frac{k(t)f(h(t))}{h''(t)}\lim_{u\to\infty}\frac{f(\xi u)}{\xi f(u)} = \frac{c}{K'(0)(1-2\gamma)+2\gamma}A(\xi)\,.\end{aligned}$$

This and Lemma 6.1 imply that the function $\Pi : (0,\infty) \to (0,\infty)$ is bijective. Let ξ_0 be the unique positive solution of $\Pi(\xi) = 1$, that is $A(\xi_0) = \frac{K'(0)(1-2\gamma)+2\gamma}{c}$.

For $\varepsilon \in (0, 1/4)$ arbitrary, we denote $\xi_1 = \Pi^{-1}(1-4\varepsilon)$, respectively $\xi_2 = \Pi^{-1}(1+4\varepsilon)$.

Using Remark 4.1, (B_1) and the regularity of $\partial\Omega_0$, we can choose $\delta > 0$ small enough such that

(i) $\mathrm{dist}(x, \partial\Omega_0)$ is a C^2 function on the set $\{x \in \Omega : \mathrm{dist}(x, \partial\Omega_0) \le 2\delta\}$;
(ii) $|\frac{h'(s)}{h''(s)}\Delta d(x) + a\frac{h(s)}{h''(s)}| < \varepsilon$ and $h''(s) > 0$ for all $s \in (0, 2\delta)$ and x satisfying $0 < d(x) < 2\delta$;
(iii) $(\Pi(\xi_2) - \varepsilon)\frac{h''(d(x))\xi_2}{f(h(d(x))\xi_2)} \le b(x) \le (\Pi(\xi_1) + \varepsilon)\frac{h''(d(x))\xi_1}{f(h(d(x))\xi_1)}$, for every x with $0 < d(x) < 2\delta$.
(iv) $b(y) < (1+\varepsilon)b(x)$, for every x, y with $0 < d(y) < d(x) < 2\delta$.

Let $\sigma \in (0, \delta)$ be arbitrary. We define $\underline{v}_\sigma(x) = h(d(x) + \sigma)\xi_1$, for any x with $d(x) + \sigma < 2\delta$, respectively $\bar{v}_\sigma(x) = h(d(x) - \sigma)\xi_2$ for any x with $\sigma < d(x) < 2\delta$.

Using (ii), (iv) and the first inequality in (iii), when $\sigma < d(x) < 2\delta$, we obtain (since $|\nabla d(x)| \equiv 1$)

$$\begin{aligned}&-\Delta\bar{v}_\sigma(x) - a\bar{v}_\sigma(x) + b(x)f(\bar{v}_\sigma(x))\\ &\quad= \xi_2\Bigg(-h'(d(x)-\sigma)\Delta d(x) - h''(d(x)-\sigma)\\ &\qquad - ah(d(x)-\sigma) + \frac{b(x)f(h(d(x)-\sigma)\xi_2)}{\xi_2}\Bigg)\\ &\quad= \xi_2 h''(d(x)-\sigma)\Bigg(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x)\\ &\qquad - a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 + \frac{b(x)f(h(d(x)-\sigma)\xi_2)}{h''(d(x)-\sigma)\xi_2}\Bigg)\end{aligned}$$

$$\geq \xi_2 h''(d(x)-\sigma)\Big(-\frac{h'(d(x)-\sigma)}{h''(d(x)-\sigma)}\Delta d(x)$$

$$-a\frac{h(d(x)-\sigma)}{h''(d(x)-\sigma)} - 1 + \frac{\Pi(\xi_2)-\varepsilon}{1+\varepsilon}\Big) \geq 0$$

for all x satisfying $\sigma < d(x) < 2\delta$.

Similarly, using (ii), (iv) and the second inequality in (iii), when $d(x)+\sigma < 2\delta$ we find

$$-\Delta\underline{v}_\sigma(x) - a\underline{v}_\sigma(x) + b(x)f(\underline{v}_\sigma(x))$$

$$= \xi_1 h''(d(x)+\sigma)\Big(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x)$$

$$-a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} - 1 + \frac{b(x)f(h(d(x)+\sigma)\xi_1)}{h''(d(x)+\sigma)\xi_1}\Big)$$

$$\leq \xi_1 h''(d(x)+\sigma)\Big(-\frac{h'(d(x)+\sigma)}{h''(d(x)+\sigma)}\Delta d(x)$$

$$-a\frac{h(d(x)+\sigma)}{h''(d(x)+\sigma)} - 1 + (1+\varepsilon)(\Pi(\xi_1)+\varepsilon)\Big) \leq 0\,,$$

for all x satisfying $d(x)+\sigma < 2\delta$.

Define $\Omega_\delta \equiv \{x \in \Omega : d(x) < \delta\}$. Let $\omega \subset\subset \Omega_0$ be such that the first Dirichlet eigenvalue of $(-\Delta)$ in the smooth domain $\Omega_0\backslash\bar{\omega}$ is strictly greater than a. Denote by w a positive large solution to the following problem

$$-\Delta w = aw - p(x)f(w) \quad \text{in } \Omega_\delta\,,$$

where $p \in C^{0,\mu}(\bar{\Omega}_\delta)$ satisfies $0 < p(x) \leq b(x)$ for $x \in \bar{\Omega}_\delta\backslash\bar{\Omega}_0$, $p(x) = 0$ on $\bar{\Omega}_0\backslash\omega$ and $p(x) > 0$ for $x \in \omega$. The existence of w is guaranteed by our Theorem 1.1.

Suppose that u is an arbitrary solution of (4) and let $v := u+w$. Then v satisfies

$$-\Delta v \geq av - b(x)f(v) \quad \text{in } \Omega_\delta\backslash\bar{\Omega}_0\,.$$

Since

$$v|_{\partial\Omega_0} = \infty > \underline{v}_\sigma|_{\partial\Omega_0} \quad \text{and} \quad v|_{\partial\Omega_\delta} = \infty > \underline{v}_\sigma|_{\partial\Omega_\delta}\,,$$

by Lemma 2.3 we find

$$u + w \geq \underline{v}_\sigma \quad \text{on } \Omega_\delta\backslash\bar{\Omega}_0\,. \tag{36}$$

Similarly

$$\bar{v}_\sigma + w \geq u \quad \text{on } \Omega_\delta\backslash\bar{\Omega}_\sigma\,. \tag{37}$$

Letting $\sigma \to 0$ in (36) and (37), we deduce

$$h(d(x))\xi_2 + 2w \geq u + w \geq h(d(x))\xi_1\,, \quad \forall\, x \in \Omega_\delta\backslash\bar{\Omega}_0\,.$$

Since w is uniformly bounded on $\partial\Omega_0$, it follows that

$$\xi_1 \le \liminf_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} \le \limsup_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} \le \xi_2 \,. \tag{38}$$

Letting $\varepsilon \to 0$ in (38) and looking at the definition of ξ_1 respectively ξ_2 we find

$$\lim_{d(x)\searrow 0} \frac{u(x)}{h(d(x))} = \xi_0 \,. \tag{39}$$

This behavior of the solution will be speculated in order to prove that problem (4) has a unique solution. Indeed, let u_1, u_2 be two positive solutions of (4). For any $\varepsilon > 0$, denote $\tilde{u}_i = (1+\varepsilon)u_i$, $i = 1, 2$. By virtue of (39) we get

$$\lim_{d(x)\searrow 0} \frac{u_1(x) - \tilde{u}_2(x)}{h(d(x))} = \lim_{d(x)\searrow 0} \frac{u_2(x) - \tilde{u}_1(x)}{h(d(x))} = -\varepsilon\xi_0 < 0$$

which implies

$$\lim_{d(x)\searrow 0} [u_1(x) - \tilde{u}_2(x)] = \lim_{d(x)\searrow 0} [u_2(x) - \tilde{u}_1(x)] = -\infty \,.$$

On the other hand, since $\frac{f(u)}{u}$ is increasing for $u > 0$, we obtain

$$-\Delta\tilde{u}_i = -(1+\varepsilon)\Delta u_i = (1+\varepsilon)(au_i - b(x)f(u_i)) \ge a\tilde{u}_i - b(x)f(\tilde{u}_i) \quad \text{in } \Omega\backslash\bar{\Omega}_0 \,,$$

$$\mathcal{B}\tilde{u}_i = \mathcal{B}u_i = 0 \quad \text{on } \partial\Omega \,.$$

So, by Lemma 2.3,

$$u_1(x) \le \tilde{u}_2(x), \quad u_2(x) \le \tilde{u}_1(x), \quad \forall\, x \in \Omega\backslash\bar{\Omega}_0 \,.$$

Letting $\varepsilon \to 0$, we obtain $u_1 \equiv u_2$. The proof of Theorem 1.3 is complete. □

Remark 6.1. Assume that f satisfies (A_1) and (A_3). Then problem (1) with $a \equiv 0$, $b \equiv 1$ has a unique large solution $\tilde{u}$. Moreover, $\tilde{u}$ satisfies the asymptotic condition (see [5, Theorems 2.3 and 2.4])

$$\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} \frac{\tilde{u}(x)}{\Gamma(\mathrm{dist}(x,\partial\Omega))} = 1 \,,$$

where Γ is the function defined as

$$\int_{\Gamma(t)}^{\infty} \frac{ds}{\sqrt{2F(s)}} = t \,, \quad \forall\, t > 0 \,.$$

Let $\Omega_1 \subset\subset \Omega$ be a connected subdomain, with smooth boundary such that $\bar{\Omega}_0 \subset \Omega_1$. Theorem 1.3 yields

Corollary 6.1. *Let (A_4) be added to the assumptions of Remark* 6.1. *Then, for any $a \in \mathbf{R}$, problem* (4) *with $b \equiv 1$ on $\partial\Omega_1$ and Ω_0 replaced by Ω_1, has a unique positive solution U_a. Moreover, U_a behaves on $\partial\Omega_1$ exactly in the same manner as $\tilde{u}$ on $\partial\Omega$, i.e.*

$$\lim_{\mathrm{dist}(x,\partial\Omega_1)\to 0} \frac{U_a(x)}{\Gamma(\mathrm{dist}(x,\partial\Omega_1))} = 1 \,.$$

Proof. By Remark 1.1(a), we can apply the argument of Lemma 5.3 to deduce the existence of a positive solution for problem considered here. Concerning the uniqueness, we remark that (B_1) and (B_2) are fulfilled by taking $c = 1$ and $k \equiv 1$ on $(0, \infty)$. It follows that h defined by (7) coincides with Γ. But $\Gamma'(t) = -\sqrt{2F(\Gamma(t))}$ and $\Gamma''(t) = f(\Gamma(t))$ for any $t \in (0, \infty)$. Thus, we obtain $\Gamma \in \mathcal{G}$ (without calling Lemma 4.3) and $\Pi(\xi) = A(\xi)$ for all $\xi > 0$. So, by Lemma 6.1, $\Pi : (0, \infty) \to (0, \infty)$ is bijective. From now on, we proceed as in the proof of Theorem 1.3 remaining only to replace h by Γ and Ω_0 by Ω_1. □

Appendix A.

The following result has been applied several times in the paper and it is mentioned without proof in Marcus [25]. For the convenience of the reader we give in what follows a complete proof of this result.

Theorem A.1. *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain. Assume $0 \not\equiv p \in C^{0,\mu}(\bar{\Omega})$ is non-negative and $f \in C^1[0, \infty)$ is a positive, non-decreasing function on $(0, \infty)$ such that $f(0) = 0$. If $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ is non-negative, then the boundary value problem*

$$\begin{cases} \Delta u = p(x) f(u) & \text{in } \Omega\,, \\ u = \Phi & \text{on } \partial\Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \end{cases} \tag{A.1}$$

has a unique classical solution, which is positive in Ω.

Remark A.1. The conclusion of Theorem A.1 has been established in [11, Theorem 5] when Φ is assumed to be positive on $\partial\Omega$. Our approach for proving the positivity of solution was essentially based on this assumption and it fails when the zero set of Φ is non-empty.

Under the same assumptions on p and f as in the statement of Theorem A.1 we have

Corollary A.1 (Strong maximum principle). *Let Ω be a non-empty domain in $\mathbf{R}^N$. If u is a non-negative classical solution of the equation $\Delta u = p(x)f(u)$ in Ω then the following alternative holds: either $u \equiv 0$ in Ω or u is positive in Ω.*

Proof. If $u \not\equiv 0$ in Ω, then there exists $x_0 \in \Omega$ such that $u(x_0) > 0$. We claim that $u > 0$ in Ω. Arguing by contradiction, let us assume that $u(x_1) = 0$ for some $x_1 \in \Omega$. Let $\omega \subset\subset \Omega$ be a bounded smooth domain such that $x_1 \in \omega$ and $x_0 \in \partial\omega$. Set $p_0 := 1 + \sup_\omega p > 0$ and consider the problem

$$\begin{cases} \Delta v = p_0 f(v) & \text{in } \omega\,, \\ v = u \not\equiv 0 & \text{on } \partial\omega\,, \\ v \geq 0 & \text{in } \omega\,. \end{cases} \tag{A.2}$$

By Theorem A.1, this problem has a unique solution v_0 which, moreover, is positive in ω. It is clear that 0 (respectively, u) is sub-solution (respectively, super-solution) for problem (A.2). So, there exists a solution v_1 of (A.2) satisfying $0 \le v_1 \le u$. By uniqueness we deduce that $v_1 = v_0 > 0$ in ω. It follows that $u \ge v_0 > 0$ in ω. But this is impossible since $u(x_1) = 0$. □

Corollary A.2. *Let $\Omega \subset \mathbf{R}^N$ be a bounded smooth domain. If u_1 is a non-negative classical solution of the equation $\Delta u + au = p(x)f(u)$ in Ω such that $u_1 \not\equiv 0$ on $\partial\Omega$ then u_1 is positive in Ω.*

Proof. Let $\Phi \in C^{0,\mu}(\partial\Omega)$ be such that $\Phi \not\equiv 0$ and $0 \le \Phi \le u_1$ on $\partial\Omega$. Consider the problem

$$\begin{cases} \Delta u = |a|u + \|p\|_\infty f(u) & \text{in } \Omega\,, \\ u = \Phi & \text{on } \partial\Omega\,, \\ u \ge 0 & \text{in } \Omega\,. \end{cases} \tag{A.3}$$

By Theorem A.1, this problem has a unique solution, say u_0 and, moreover, $u_0 > 0$ in Ω. But u_1 is supersolution for problem (A.3), so $u_1 \ge u_0 > 0$ in Ω and our claim is proved. □

Proof of Theorem A.1. We first observe that $u_- = 0$ is a sub-solution of (A.1), while $u^+ = n$ is a super-solution of (A.1) if n is large enough. Hence problem (A.1) has at least a solution u_Φ.

Then, taking into account the regularity of p and f, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u_\Phi \in C^2(\Omega) \cap C(\bar{\Omega})$. The fact that u_Φ is the unique classical solution to (A.1) follows in the same way as in [11, Theorem 5].

We state in what follows two proofs for the positivity of u_Φ: the first one relies essentially on Theorem 1.20 in [14] while the second proof offers a more easier and direct approach. □

First proof: Set $M := \max_{\bar{\Omega}} p$. Let u_* be the unique non-negative classical solution of the problem

$$\begin{cases} \Delta u_* = Mf(u_*) & \text{in } \Omega\,, \\ u_* = \Phi & \text{on } \partial\Omega\,. \end{cases}$$

To conclude that $u_\Phi > 0$ in Ω it is enough to show that $u_\Phi \ge u_* > 0$ in Ω. Since $f \in C^1[0, \infty)$ we have

$$\lim_{u \to 0^+} \frac{u^2}{F(u)} = \lim_{u \to 0^+} \frac{2u}{f(u)} = \frac{2}{f'(0)} > 0 \tag{A.4}$$

which implies immediately that $\int_{0^+}^1 \frac{du}{\sqrt{F(u)}} = \infty$. By applying Theorem 1.20 in Diaz [14], we conclude that $u_* > 0$ in Ω.

We now prove that $u_\Phi \geq u_*$ in Ω. To this aim, fix $\varepsilon > 0$. We claim that

$$u_*(x) \leq u_\Phi(x) + \varepsilon(1+|x|^2)^{-1/2} \quad \text{for any } x \in \Omega\,. \tag{A.5}$$

Assume the contrary. Since $u_{*|\partial\Omega} = u_{\Phi|\partial\Omega} = \Phi$ we deduce that

$$\max_{x\in\bar{\Omega}}\{u_*(x) - u_\Phi(x) - \varepsilon(1+|x|^2)^{-1/2}\}$$

is achieved in Ω. At that point we have

$$\begin{aligned} 0 &\geq \Delta(u_*(x) - u_\Phi(x) - \varepsilon(1+|x|^2)^{-1/2}) \\ &= Mf(u_*(x)) - p(x)f(u_\Phi(x)) - \varepsilon\Delta(1+|x|^2)^{-1/2} \\ &\geq p(x)(f(u_*(x)) - f(u_\Phi(x))) + \varepsilon(N-3)(1+|x|^2)^{-3/2} + 3\varepsilon(1+|x|^2)^{-5/2} > 0\,, \end{aligned}$$

which is a contradiction. Since $\varepsilon > 0$ is chosen arbitrarily, inequality (A.5) implies $u_\Phi \geq u_*$ in Ω. □

Second proof: Since $\Phi \not\equiv 0$, there exists $x_0 \in \Omega$ such that $u_\Phi(x_0) > 0$. To conclude that $u_\Phi > 0$ in Ω it is sufficient to prove that $u_\Phi > 0$ on $B(x_0;\bar{r})$ where $\bar{r} = \operatorname{dist}(x_0, \partial\Omega)$. Without loss of generality we can assume $x_0 = 0$. By the continuity of u_Φ, there exists $\underline{r} \in (0,\bar{r})$ such that $u_\Phi(x) > 0$ for all x with $|x| \leq \underline{r}$. So, $\min_{|x|=\underline{r}} u_\Phi(x) =: \rho > 0$. We define

$$M := \max_{\bar{\Omega}} p\,, \quad \eta := \int_\rho^{\rho+1} \frac{dt}{f(t)} \quad \text{and} \quad \nu(\varepsilon) := \int_\varepsilon^{\rho+1} \frac{dt}{f(t)} \quad \text{for } 0 < \varepsilon < \rho\,.$$

It remains to show that $u_\Phi > 0$ in $A(\underline{r},\bar{r})$, where

$$A(\underline{r},\bar{r}) := \{x \in \mathbf{R}^N : \underline{r} < |x| < \bar{r}\}\,.$$

For this aim, we need the following lemma.

Lemma A.1. *For $\varepsilon > 0$ small enough, the problem*

$$\begin{cases} -\Delta v = M & \text{in } A(\underline{r},\bar{r})\,, \\ v(x) = \eta & \text{as } |x| = \underline{r}\,, \\ v(x) = \nu(\varepsilon) & \text{as } |x| = \bar{r}\,, \end{cases} \tag{A.6}$$

has a unique solution, which is increasing in $A(\underline{r},\bar{r})$.

Proof. By the maximum principle, the problem (A.6) has a unique solution. Moreover, v is radially symmetric in $A(\underline{r},\bar{r})$, namely $v(x) = v(r)$, $r = |x|$. The function v satisfies

$$v''(r) + \frac{N-1}{r}v'(r) = -M\,, \quad \underline{r} < r < \bar{r}\,.$$

Integrating twice this relation we find

$$v(r) = -\frac{M}{2N}r^2 - \frac{C_1}{N-2}r^{2-N} + C_2\,, \quad \underline{r} < r < \bar{r}\,,$$

where C_1 and C_2 are real constants. The boundary conditions $v(\underline{r}) = \eta$ and $v(\bar{r}) = \nu(\varepsilon)$ imply

$$C_1 = \left(\nu(\varepsilon) - \eta + \frac{M}{2N}(\bar{r}^2 - \underline{r}^2)\right) \frac{N-2}{\underline{r}^{2-N} - \bar{r}^{2-N}}\,.$$

From (A.4) we deduce that $\nu(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Thus, taking $\varepsilon > 0$ sufficiently small, C_1 becomes large enough to ensure that $v'(r) > 0$ for all $r \in (\underline{r}, \bar{r})$. □

Set $\varepsilon > 0$ sufficiently small such that the conclusion of Lemma A.1 holds. Let $\underline{u}$ be the function defined implicitely as follows

$$\int_{\underline{u}(x)+\varepsilon}^{\rho+1} \frac{dt}{f(t)} = v(x) \quad \text{for all } x \in A(\underline{r}, \bar{r})\,. \tag{A.7}$$

It is easy to check that

$$\begin{cases} \Delta\underline{u} \geq Mf(\underline{u}+\varepsilon) \geq p(x)f(\underline{u}) & \text{in } A(\underline{r}, \bar{r})\,, \\ \underline{u}(x) = \rho - \varepsilon < u_\Phi(x) & \text{as } |x| = \underline{r}\,, \\ \underline{u}(x) = 0 \leq u_\Phi(x) & \text{as } |x| = \bar{r}\,. \end{cases}$$

Using the maximum principle (as in the proof of (A.5)) we deduce that $\underline{u} \leq u_\Phi$ in $A(\underline{r}, \bar{r})$. By (A.7) and Lemma A.1 we deduce that $\underline{u}$ decreases in $A(\underline{r}, \bar{r})$. Thus, $\underline{u} > 0$ in $A(\underline{r}, \bar{r})$. This completes the proof. □

The positiveness of the solution in Theorem A.1 follows essentially by the assumption $f \in C^1$ on $[0, \infty)$. We show in what follows that if f is not differentiable at the origin, then problem (A.1) has a unique solution that is not necessarily positive in Ω. However, in this case, the positiveness of the solution may depend on c and on the geometry of Ω. Indeed, let us consider the problem

$$\begin{cases} \Delta u = \sqrt{u} & \text{in } \Omega\,, \\ u \geq 0 & \text{in } \Omega\,, \\ u = c & \text{on } \partial\Omega\,, \end{cases} \tag{A.8}$$

where $c > 0$ is a constant.

In order to justify the uniqueness, let u_1, u_2 be two solutions of (A.8). It is sufficient to show that $u_1 \leq u_2$ in Ω. Set $\omega = \{x \in \Omega; u_1(x) > u_2(x)\}$ and assume that $\omega \neq \emptyset$. Then $\Delta(u_1 - u_2) = \sqrt{u_1} - \sqrt{u_2} > 0$ in ω and $u_1 - u_2 = 0$ on $\partial\omega$. The maximum principle implies $u_1 - u_2 \leq 0$ in ω which yields a contradiction.

The existence of a solution follows after observing that $u_- = 0$ (respectively $u_+ = c$) are sub-solution (respectively super-solution) for our problem.

The following example illustrates that in certain situations the unique solution of the problem (A.8) may vanish.

Example A.1. Set $\Omega = B(0,1) \subset \mathbf{R}^N$ and $w(x) = a|x|^4$. If $c \le \frac{1}{(4N+8)^2}$, let us choose a so that $c \le a \le \frac{1}{(4N+8)^2}$. It follows that

$$\begin{cases} \Delta w = (4N+8)a|x|^2 \le \sqrt{a}|x|^2 = \sqrt{w} & \text{in } \Omega\,, \\ w = a \ge c & \text{on } \partial\Omega\,. \end{cases}$$

This means that w is a super-solution of (A.8). Since $w(0) = 0$ then, necessarily, $u(0) = 0$.

The next example shows that in some cases, depending on c and on $\operatorname{diam}\Omega$, the unique solution of (A.8) is positive.

Example A.2. Suppose that Ω can be included in a ball $B(x_0, R)$ with $R \le R_c := 2\sqrt{[4]c\sqrt{N+2}}$. Define $w(x) = a|x - x_0|^4$, where a is chosen so that $\frac{\sqrt{c}}{R^2} \ge \sqrt{a} \ge \frac{1}{4N+8}$. Then w satisfies

$$\begin{cases} \Delta w = (4N+8)a|x - x_0|^2 \ge \sqrt{a}|x - x_0|^2 = \sqrt{w} & \text{in } \Omega\,, \\ w = a|x - x_0|^4 \le c & \text{on } \partial\Omega \end{cases}$$

which shows that w is a sub-solution of (A.8). We conclude that $u(x) \ge w(x) > 0$, for any $x \in \Omega\backslash\{x_0\}$.

If $\operatorname{diam}\Omega < 2R \le 2R_c$, there exist two points x_0 and x_1 such that Ω can be included in each of the balls $B(x_0, R)$ and $B(x_1, R)$. Using the previous conclusion we have

$$u(x) \ge a \max\{|x - x_0|^4, |x - x_1|^4\} \ge a \left|\frac{x_1 - x_0}{2}\right|^4 > 0\,.$$

Choosing $a = \frac{c}{R^4}$, $|x_1 - x_0| = 2R - \operatorname{diam}\Omega$ and $R = R_c$, we find

$$u(x) \ge \frac{c}{R^4}\left(\frac{2R - \operatorname{diam}\,\Omega}{2}\right)^4 = c\left(1 - \frac{\operatorname{diam}\Omega}{2R}\right)^4 > 0\,, \quad \forall\, x \in \Omega\,.$$

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