Research Article

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Combined Effects of Concave-Convex Nonlinearities in a Fourth-Order Problem with Variable Exponent

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Abstract: We study two classes of nonhomogeneous elliptic problems with Dirichlet boundary condition and involving a fourth-order differential operator with variable exponent and power-type nonlinearities. The first result of this paper establishes the existence of a nontrivial weak solution in the case of a small perturbation of the right-hand side. The proof combines variational methods, including the Ekeland variational principle and the mountain pass theorem of Ambrosetti and Rabinowitz. Next we consider a very related eigenvalue problem and we prove the existence of nontrivial weak solutions for large values of the parameter. The direct method of the calculus of variations, estimates of the levels of the associated energy functional and basic properties of the Lebesgue and Sobolev spaces with variable exponent have an important role in our arguments.

Keywords: Nonhomogeneous Elliptic Problem, Variable Exponent, Dirichlet Boundary Condition, Ekeland Variational Principle

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1 Introduction

In a pioneering paper, A. Ambrosetti, H. Brezis and G. Cerami [1] initiated the qualitative analysis of semilinear Dirichlet elliptic problems that involve concave and convex nonlinearities. They proved several existence, multiplicity and nonexistence results and developed powerful topological and variational methods for the study of such nonlinear problems. In particular, they studied the effects of small perturbations for the existence of solutions. In [13, 17] related existence results are established in the case of elliptic problems with variable exponents and Dirichlet boundary condition (see [26, 28] for further developments and related properties). The main purpose of this paper is to complete the results of L. Kong [13] and to prove the existence of a family of eigenvalues in a neighborhood of the origin. We also refer to the related papers [10, 18, 27, 29, 30]. Additional results on higher-order problems or nonlinear partial differential equations with variable exponent can be found in the papers by G. Autuori, F. Colasuonno and P. Pucci [3], Z. Chen [5], F. Colasuonno and P. Pucci [6], A. Kratohvil and I. Necas [14], V. Lubyshev [16], P. Pucci and Q. Zhang [24].

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Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Consider the following nonhomogeneous eigenvalue problem:

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{q(x)-2} u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $p, q : \overline{\Omega} \to \mathbb{R}$ are continuous functions and $\Delta_{p(x)}$ denotes the $p(x)$-*Laplace operator*, which is defined by

$$\Delta_{p(x)} u := \operatorname{div}(|\nabla u|^{p(x)-2} \nabla u).$$

Problem (1.1) is studied in [17] (see also [28, Section 2.3.1]) in a subcritical setting under the basic assumption

$$1 < \min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x) < \max_{x \in \overline{\Omega}} q(x).$$

Under this hypothesis, the main result in [17] establishes that there exists $\lambda^* > 0$ such that problem (1.1) has at least one nontrivial solution for all $\lambda \in (0, \lambda^*)$. Since the associated energy functional does not have a mountain pass geometry (see A. Ambrosetti and P. Rabinowitz [2]), the proof relies essentially on the Ekeland variational principle, see [9]. We point out that the original proof of the mountain pass theorem is based on several powerful deformation techniques developed by R. Palais and S. Smale [20, 21], who developed the main ideas of the Morse theory in the abstract framework of differential topology on infinite-dimensional Riemann manifolds. A simpler proof of the mountain pass theorem is due to H. Brezis and L. Nirenberg [4], who used a pseudo-gradient lemma, a perturbation argument and the Ekeland variational principle.

The study initiated in [17, 28] was continued by L. Kong [13] in the framework of the $p(x)$-*biharmonic operator* $\Delta^2_{p(x)}$, namely

$$\Delta^2_{p(x)} u := \Delta(|\Delta u|^{p(x)-2} \Delta u).$$

Consider the fourth-order nonlinear elliptic equation with variable exponent and Dirichlet boundary condition

$$\begin{cases} \Delta^2_{p(x)} u + a(x)|u|^{p(x)-2} u = \lambda w(x) f(u), & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.2}$$

where $a(x)$ and $w(x)$ are nonnegative potentials and the nonlinear term f behaves like

$$f(u) = |u|^{\gamma(x)-2} u - |u|^{\beta(x)-2} u,$$

where γ, $\beta > 1$ are continuous functions, and we assume the basic hypothesis

$$\gamma(x) < \beta(x) < p(x) \quad \text{for all } x \in \overline{\Omega}. \tag{1.3}$$

The main result in [13] asserts that there exists $\lambda^* > 0$ such that problem (1.2) has at least one nontrivial solution for all $\lambda \in (0, \lambda^*)$.

In the present paper, we establish several existence results for problems related to (1.2) but under some basic assumptions different from (1.3).

We consider the nonlinear problem

$$\begin{cases} \Delta^2_{p(x)} u + a|u|^{p(x)-2} u = \lambda(|u|^{\gamma(x)-2} - |u|^{\beta(x)-2}) u, & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where λ is a positive parameter and $a \geq 0$. Under two different assumptions, we show that problem (1.4) has at least one nontrivial solution if the positive parameter λ is *small enough*. The proof relies on the Ekeland variational principle and the mountain pass theorem. We refer to J. Garcia Azorero and I. Peral Alonso [11] who applied the mountain pass theorem to obtain the existence of a nodal (that is, sign-changing) solution in a related quasilinear setting.

The situation changes if we consider a problem very close to (1.4). Let us consider the following eigenvalue nonlinear Dirichlet problem:

$$\begin{cases} \Delta^2_{p(x)} u + a|u|^{p(x)-2}u = \lambda |u|^{\gamma(x)-2} - |u|^{\beta(x)-2}u, & x \in \Omega, \\ u = \Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{1.5}$$

In this case, we establish a sufficient condition for the existence of nontrivial solutions provided that the parameter λ is *large enough*. The proof is based on the direct method of the calculus of variations.

In Section 2 we recall some basic definitions and properties concerning the basic function spaces with variable exponent. We refer to the recent monographs of L. Diening, P. Hästo, P. Harjulehto and M. Ruzicka [8] and V. Rădulescu and D. Repovš [28] for related properties of Lebesgue and Sobolev spaces with variable exponents. The main results are stated in Section 3 of this paper. Final comments and some open problems are given in Section 4.

2 Function Spaces with Variable Exponent

Consider the set

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega});\ p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $p \in C_+(\overline{\Omega})$ we define

$$p^+ = \sup_{x\in\Omega} p(x) \quad \text{and} \quad p^- = \inf_{x\in\Omega} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) = \Big\{u;\ u \text{ is measurable and } \int_\Omega |u(x)|^{p(x)}\,dx < \infty\Big\}.$$

This vector space is a Banach space if it is endowed with the *Luxemburg norm*, which is defined by

$$|u|_{p(x)} = \inf\Big\{\mu > 0;\ \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)}\,dx \le 1\Big\}.$$

The function space $L^{p(x)}(\Omega)$ is reflexive if and only if $1 < p^- \le p^+ < \infty$. Continuous functions with compact support are dense in $L^{p(x)}(\Omega)$ if $p^+ < \infty$.

Let $L^{q(x)}(\Omega)$ denote the conjugate space of $L^{p(x)}(\Omega)$, where $1/p(x) + 1/q(x) = 1$. If $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, then the following Hölder-type inequality holds:

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

Moreover, if $p_j \in C_+(\overline{\Omega})$ ($j = 1, 2, 3$) and

$$\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1,$$

then, for all $u \in L^{p_1(x)}(\Omega)$, $v \in L^{p_2(x)}(\Omega)$, $w \in L^{p_3(x)}(\Omega)$,

$$\Big|\int_\Omega uvw\,dx\Big| \le \Big(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\Big)|u|_{p_1(x)}|v|_{p_2(x)}|w|_{p_3(x)}.$$

The inclusion between Lebesgue spaces also generalizes the classical framework, namely if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1 \le p_2$ in Ω, then there exists the continuous embedding

$$L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega).$$

If k is a positive integer number and $p \in C_+(\overline{\Omega})$, we define the variable exponent Sobolev space by

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega);\ D^\alpha u \in L^{p(x)}(\Omega) \text{ for all } |\alpha| \le k\}.$$

Here $\alpha = (\alpha_1, \dots, \alpha_N)$ is a multi-index, $|\alpha| = \sum_{i=1}^N \alpha_i$ and

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1}^{\alpha_1} \dots \partial_{x_N}^{\alpha_N}}.$$

On $W^{k,p(x)}(\Omega)$ we consider the norm

$$\|u\|_{k,p(x)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)}.$$

Then $W^{k,p(x)}(\Omega)$ is a reflexive and separable Banach space. Let $W_0^{k,p(x)}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$.

Consider the function space E defined by

$$E = W_0^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega).$$

Then E is a separable and reflexive Banach space if it is equipped with the norm

$$\|u\|_E = \|u\|_{1,p(x)} + \|u\|_{2,p(x)}.$$

The norms $\|u\|_E$ and $|\Delta u|_{p(x)}$ are equivalent (cf. [13, p. 251]).

If a is a positive number, define, for all $u \in E$,

$$\|u\|_a = \inf\Big\{\lambda > 0;\ \int_\Omega \Big(\Big|\frac{\Delta u}{\lambda}\Big|^{p(x)} + a\Big|\frac{u}{\lambda}\Big|^{p(x)}\Big) dx \le 1\Big\}.$$

Then $\|u\|_a$ is well-defined and it is a norm which is equivalent with the norms $\|u\|_E$ and $|\Delta u|_{p(x)}$ in E.

Let $\varrho_a : E \to \mathbb{R}$ be the *modular function* defined by

$$\varrho_a(u) = \int_\Omega (|\Delta u|^{p(x)} + a|u|^{p(x)})\, dx.$$

If (u_n), $u \in E$, then the following properties are true:

$$\|u\|_a > 1 \quad \Rightarrow \quad \|u\|_a^{p^-} \le \varrho_a(u) \le \|u\|_a^{p^+}, \tag{2.1}$$

$$\|u\|_a < 1 \quad \Rightarrow \quad \|u\|_a^{p^+} \le \varrho_a(u) \le \|u\|_a^{p^-}, \tag{2.2}$$

$$\|u_n - u\|_E \to 0 \quad \Leftrightarrow \quad \varrho_a(u_n - u) \to 0.$$

Let $p^*(x)$ denote the *critical Sobolev exponent*, namely

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-2p(x)} & \text{if } 2p(x) < N, \\ +\infty & \text{if } 2p(x) \ge N. \end{cases}$$

We point out that if $p, q \in C_+(\overline{\Omega})$ and $q(x) < p^\star(x)$ for all $x \in \overline{\Omega}$, then the embedding $E \hookrightarrow L^{q(x)}(\Omega)$ is compact, see [13, Proposition 1.3].

The variable exponent Lebesgue and Sobolev spaces are generalizations of the classical Lebesgue and Sobolev spaces, replacing the constant exponent p with an exponent function $p(\cdot)$. These spaces have been the subject of constant interest since the beginning of the 20th century both as function spaces with intrinsic interest and for their applications to problems arising in nonlinear partial differential equations and the calculus of variations. We refer to the monographs [7, 8, 28] for related properties of these spaces and their history.

3 The Main Results and Related Properties

We say that λ is an *eigenvalue* of problem (1.4) if there exists $u \in E \setminus \{0\}$ such that, for all $v \in E$,

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx + a \int_\Omega |u|^{p(x)-2} uv dx = \lambda \int_\Omega (|u|^{\gamma(x)-2} uv - |u|^{\beta(x)-2} uv) dx.$$

If λ is an eigenvalue of problem (1.4), the corresponding function $u \in E \setminus \{0\}$ is a *weak solution* of problem (1.4).

We study problem (1.4) under one of the following hypotheses:

$$1 < \gamma(x) < \min\{p(x), \beta(x)\} < \max\{p(x), \beta(x)\} < p^*(x) \quad \text{for all } x \in \overline{\Omega} \tag{3.1}$$

or

$$1 < \min\{p(x), \beta(x)\} < \max\{p(x), \beta(x)\} < \gamma(x) < p^*(x) \quad \text{for all } x \in \overline{\Omega}. \tag{3.2}$$

The energy functional associated to problem (1.4) is defined as

$$\mathcal{E}_\lambda(u) = \int_\Omega \frac{1}{p(x)} (|\Delta u|^{p(x)} + a|u|^{p(x)}) dx - \lambda \int_\Omega \Big[\frac{|u|^{\gamma(x)}}{\gamma(x)} - \frac{|u|^{\beta(x)}}{\beta(x)} \Big] dx \quad \text{for all } u \in E.$$

Hypothesis (3.1) implies that $\mathcal{E}_\lambda$ is well-defined, of class C^1, and

$$\langle \mathcal{E}_\lambda'(u), v \rangle = \int_\Omega (|\Delta u|^{p(x)-2} \Delta u \Delta v + a|u|^{p(x)-2} uv) dx - \lambda \int_\Omega (|u|^{\gamma(x)-2} - |u|^{\beta(x)-2}) uv dx \quad \text{for all } v \in E.$$

The first result of this paper is the following.

Theorem 3.1. *Assume that one of the hypotheses* (3.1) *or* (3.2) *is satisfied. Then there exists a positive number* λ^* *such that for all* $\lambda \in (0, \lambda^*)$ *problem* (1.4) *has at least one nontrivial weak solution.*

We are then concerned with the study of problem (1.5). We say that λ is an *eigenvalue* of problem (1.5) if there exists $u \in E \setminus \{0\}$ such that

$$\int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta v dx + a \int_\Omega |u|^{p(x)-2} uv dx = \lambda \int_\Omega |u|^{\gamma(x)-2} uv dx - \int_\Omega |u|^{\beta(x)-2} uv dx \quad \text{for all } v \in E.$$

Theorem 3.2. *Assume that the hypothesis* (3.1) *is satisfied. Then there exists a positive number* λ^{**} *such that for all* $\lambda \in (\lambda^{**}, \infty)$ *problem* (1.5) *has at least one nontrivial weak solution.*

We point out that hypothesis (3.1) implies that problem (1.4) does not have a mountain pass geometry. More precisely, $\mathcal{E}_\lambda$ satisfies one of the geometric hypotheses of the mountain pass theorem, namely the existence of a "mountain" between two prescribed "villages". However, the second geometric assumption of the mountain pass theorem is not fulfilled because this "valley" is close to the first "village" and not across the chain of mountains, as requested by the mountain pass theorem. For this reason the existence of the solution follows with different arguments and only for small perturbations (in terms of λ). An interesting *open problem* is to provide a complete description for *all* values of the positive parameter λ.

We remark that Theorem 3.1 establishes a property related to [13, Theorem 2.1]. However, our result is based on the assumption (3.1), which is more general than the corresponding hypothesis (2.1) in [13].

The proofs of Theorems 3.1 and 3.2 use some ideas developed in [17, 27, 28] in the framework of $p(x)$-Laplace operators and extended in [13] to biharmonic operators with variable exponent.

3.1 Existence of a Mountain and a Village

We are first concerned with the proof of Theorem 3.1 if the hypothesis (3.1) is fulfilled.

We have $\mathcal{E}_\lambda(0) = 0$. We first establish the following auxiliary property.

Lemma 3.3. *There exists a positive number λ^* such that for all $\lambda \in (0, \lambda^*)$ there are positive numbers r and η such that $\mathcal{E}_\lambda(u) \geq r$ for all $u \in E$ with $\|u\| = \eta$.*

Proof. We observe that

$$\begin{aligned} \mathcal{E}_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega (|\Delta u|^{p(x)} + a|u|^{p(x)})dx - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} dx + \frac{\lambda}{\beta^+} \int_\Omega |u|^{\beta(x)} dx \\ &= \frac{1}{p^+} \varrho_a(u) - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} dx + \frac{\lambda}{\beta^+} \int_\Omega |u|^{\beta(x)} dx \\ &\geq \frac{1}{p^+} \varrho_a(u) - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} dx. \end{aligned}$$

Fix $\eta \in (0, 1)$ and assume that $\|u\|_E = \eta$. Using relation (2.2), we obtain

$$\mathcal{E}_\lambda(u) \geq \frac{1}{p^+} \|u\|_E^{p^+} - \frac{\lambda}{\gamma^-} \int_\Omega |u|^{\gamma(x)} dx.$$

Since the embedding $E \hookrightarrow L^{\gamma(x)}(\Omega)$ is continuous, there exists $C_1 > 0$ such that

$$\mathcal{E}_\lambda(u) \geq \frac{1}{p^+} \|u\|_E^{p^+} - \lambda C_1 \|u\|_E^{\gamma^-} = \frac{\eta^{p^+}}{p^+} - \lambda C_1 \eta^{\gamma^-} \quad \text{for all } u \in E.$$

Now, taking λ^* sufficiently small, we deduce that for all $\lambda \in (0, \lambda^*)$ there exists $r > 0$ such $\mathcal{E}_\lambda(u) \geq r$ for all $u \in E$ with $\|u\|_E = \eta$. □

Next, we establish the existence of a valley near the origin.

Lemma 3.4. *There exist $v \in E$ and $t_0 > 0$ such that $\mathcal{E}_\lambda(tv) < 0$ for all $t \in (0, t_0)$.*

Proof. Fix $v \in E \setminus \{0\}$ such that $v \geq 0$. For all $t \in (0, 1)$ we have

$$\begin{aligned} \mathcal{E}_\lambda(tv) &= \int_\Omega \frac{t^{p(x)}}{p(x)} (|\Delta v|^{p(x)} + a v^{p(x)}) dx - \lambda \int_\Omega \frac{t^{\gamma(x)}}{\gamma(x)} v^{\gamma(x)} dx + \lambda \int_\Omega \frac{t^{\beta(x)}}{\beta(x)} v^{\beta(x)} dx \\ &\leq \frac{t^{p^-}}{p^-} \varrho_a(v) - \lambda \frac{t^{\gamma^+}}{\gamma^+} \int_\Omega v^{\gamma(x)} dx + \lambda \frac{t^{\beta^-}}{\beta^-} \int_\Omega v^{\beta(x)} dx \\ &= C_1 t^{p^-} + C_2 t^{\beta^-} - C_3 t^{\gamma^+}, \end{aligned}$$

where C_1, C_2, C_3 are positive numbers.

Using hypothesis (3.1), we deduce that $\mathcal{E}_\lambda(tv) < 0$, provided that $t > 0$ is sufficiently small. □

3.2 A Compactness Condition Versus a Variational Principle

We recall that a sequence $(u_n) \subset E$ is a *Palais–Smale sequence* if

$$\mathcal{E}_\lambda(u_n) = O(1) \quad \text{and} \quad \|\mathcal{E}'_\lambda(u_n)\|_{E^*} = o(1) \quad \text{as } n \to \infty.$$

Since the right-hand side of equation (1.4) does not satisfy the Ambrosetti–Rabinowitz condition, we cannot deduce that $\mathcal{E}_\lambda$ satisfies the Palais–Smale condition, that is, any Palais–Smale sequence is relatively compact. However, we prove in what follows that there is a suitable *bounded* Palais–Smale sequence that contains a strongly convergent subsequence.

Returning to Lemma 3.3, we have

$$\inf_{u \in \partial B} \mathcal{E}_\lambda(u) \geq r > 0, \tag{3.3}$$

where

$$B := \{u \in E;\ \|u\|_a < \eta\}.$$

By Lemma 3.4, there exists $v \in E$ such that

$$\mathcal{E}_\lambda(tv) < 0 \quad \text{for all } t > 0 \text{ small enough.} \tag{3.4}$$

Set

$$m := \inf_{u \in \overline{B}} \mathcal{E}_\lambda(u).$$

Then m is finite and using relation (3.4), we deduce that $m < 0$. By (3.3) it follows that

$$\inf_{u \in \partial B} \mathcal{E}_\lambda(u) - \inf_{u \in \overline{B}} \mathcal{E}_\lambda(u) > 0.$$

Fix $\varepsilon > 0$ such that

$$\varepsilon < \inf_{u \in \partial B} \mathcal{E}_\lambda(u) - \inf_{u \in \overline{B}} \mathcal{E}_\lambda(u).$$

The functional $\mathcal{E}_\lambda$ restricted to the complete metric space $\overline{B}$ satisfies the hypotheses of the Ekeland variational principle. A straightforward computation as in [28, pp. 46–47] shows that there exists a *bounded sequence* $(u_n) \subset B$ such that

$$\mathcal{E}_\lambda(u_n) \to m \quad \text{and} \quad \|\mathcal{E}_\lambda'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

So, up to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } E, \\ u_n &\to u_0 \quad \text{in } L^{\gamma(x)}(\Omega), \\ u_n &\to u_0 \quad \text{in } L^{\beta(x)}(\Omega). \end{aligned}$$

We claim that, in fact,

$$u_n \to u_0 \quad \text{in } E.$$

Using the second information in relation (3.5), we deduce that, for all $\varphi \in E$,

$$\begin{aligned} &\int_\Omega [|\Delta u_n|^{p(x)-2} \Delta u_n \Delta(u_n - u_0) + a|u_n|^{p(x)-2} u_n(u_n - u_0)]\,dx \\ &\qquad - \lambda \int_\Omega (|u_n|^{\gamma(x)-2} - |u_n|^{\beta(x)-2}) u_n(u_n - u_0)\,dx \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

By [13, Lemma 2.1 (b)], the operator $\mathcal{E}_\lambda' : E \to E^*$ is an operator of type (S_+). Thus we obtain that $u_n \to u_0$ in E, which is our claim. So, by (3.5),

$$\mathcal{E}_\lambda(u_0) = m < 0 \quad \text{and} \quad \mathcal{E}_\lambda'(u_0) = 0.$$

We conclude that u_0 is a nontrivial weak solution of problem (1.4). Thus each $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem (1.4). The proof of Theorem 3.1 is now complete, provided that hypothesis (3.1) is fulfilled.

We are now concerned with the related property if condition (3.2) is satisfied. We first observe that under this new hypothesis, the conclusion of Lemma 3.3 remains true. Next, since condition (3.2) implies that the dominating term in the right-hand side of problem (1.4) is $|u|^{\gamma(x)-2}u$, we prove in what follows the existence of a valley *across* the chain of mountains.

Lemma 3.5. *There exist $v \in E$ and $t_0 > 0$ such that $\mathcal{E}_\lambda(tv) < 0$ for all $t > t_0$.*

Proof. Fix $v \in E \setminus \{0\}$ such that $v \geq 0$. By (2.1) we deduce that for all $t > 1$ we have

$$\begin{aligned} \mathcal{E}_\lambda(tv) &= \int_\Omega \frac{t^{p(x)}}{p(x)} (|\Delta v|^{p(x)} + a v^{p(x)})\,dx - \lambda \int_\Omega \frac{t^{\gamma(x)}}{\gamma(x)} v^{\gamma(x)}\,dx + \lambda \int_\Omega \frac{t^{\beta(x)}}{\beta(x)} v^{\beta(x)}\,dx \\ &\leq \frac{t^{p^+}}{p^-} \varrho_a(v) - \lambda \frac{t^{\gamma^-}}{\gamma^+} \int_\Omega v^{\gamma(x)}\,dx + \lambda \frac{t^{\beta^+}}{\beta^-} \int_\Omega v^{\beta(x)}\,dx \\ &= C_4 t^{p^-} + C_5 t^{\beta^+} - C_6 t^{\gamma^-}, \end{aligned}$$

where C_4, C_5, C_6 are positive numbers.

Using hypothesis (3.1), we have $\gamma^- > \max\{p^-, \beta^+\}$. It follows that $\mathcal{E}_\lambda(tv) < 0$, provided that $t > 0$ is sufficiently large. □

3.3 Verification of the Palais–Smale Condition

We recall that the energy functional $\mathcal{E}_\lambda : E \to \mathbb{R}$ satisfies the Palais–Smale condition if any sequence $(u_n) \subset E$ such that

$$\mathcal{E}_\lambda(u_n) = O(1) \quad \text{and} \quad \|\mathcal{E}'_\lambda(u_n)\|_{E^*} = o(1) \quad \text{as } n \to \infty, \tag{3.6}$$

is relatively compact.

Let $(u_n) \subset E$ be a sequence such that relation (3.6) is fulfilled.

We claim that (u_n) is bounded in E.

Arguing by contradiction, we suppose that the sequence (u_n) is unbounded in E. Without loss of generality, we can assume that $\|u_n\|_a > 1$ for all $n \geq 1$. Using relation (3.6), we have

$$\begin{aligned} O(1) + o(\|u_n\|) &= \mathcal{E}_\lambda(u_n) - \frac{1}{\gamma^-}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle \\ &= \int_\Omega \frac{1}{p(x)}(|\Delta u_n|^{p(x)} + a|u_n|^{p(x)})dx - \lambda \int_\Omega \Big[\frac{|u_n|^{\gamma(x)}}{\gamma(x)} - \frac{|u_n|^{\beta(x)}}{\beta(x)}\Big]dx \\ &\quad - \frac{1}{\gamma^-}\int_\Omega (|\Delta u_n|^{p(x)} + a|u_n|^{p(x)})dx - \frac{\lambda}{\gamma^-}\int_\Omega (|u_n|^{\gamma(x)} - |u_n|^{\beta(x)})dx. \end{aligned}$$

By relation (2.1) we deduce that

$$\begin{aligned} O(1) + o(\|u_n\|) &= \mathcal{E}_\lambda(u_n) - \frac{1}{\gamma^-}\langle \mathcal{E}'_\lambda(u_n), u_n\rangle \\ &\geq \Big(\frac{1}{p^+} - \frac{1}{\gamma^-}\Big)\int_\Omega (|\Delta u_n|^{p^-} + a|u_n|^{p^-})dx \\ &\quad + \lambda \int_\Omega \Big(\frac{1}{\beta(x)} - \frac{1}{\gamma^-}\Big)|u_n|^{\beta(x)}dx + \lambda \int_\Omega \Big(\frac{1}{\gamma^-} - \frac{1}{\gamma(x)}\Big)|u_n|^{\gamma(x)}dx. \end{aligned}$$

Using now the hypothesis (3.2), we conclude that

$$O(1) + o(\|u_n\|) \geq \Big(\frac{1}{p^+} - \frac{1}{\gamma^-}\Big)\|u_n\|_a^{p^-} \quad \text{as } n \to \infty.$$

Since $\gamma^- > p^+$, it follows that

$$\|u_n\|_a = O(1) \quad \text{as } n \to \infty.$$

This shows that (u_n) is bounded in E, thus our claim. So, up to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } E, \\ u_n &\to u_0 \quad \text{in } L^{\gamma(x)}(\Omega), \\ u_n &\to u_0 \quad \text{in } L^{\beta(x)}(\Omega). \end{aligned}$$

We show in what follows that

$$u_n \to u_0 \quad \text{in } E.$$

Using the second information in relation (3.6), we deduce that for all $\varphi \in E$

$$\begin{aligned} &\int_\Omega [|\Delta u_n|^{p(x)-2}\Delta u_n \Delta\varphi + a|u_n|^{p(x)-2}u_n\varphi]dx \\ &\quad - \lambda \int_\Omega (|u_n|^{\gamma(x)-2} - |u_n|^{\beta(x)-2})u_n\varphi dx \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

With the same arguments as in the first case and since $\mathcal{E}'_\lambda : E \to E^*$ is an operator of type (S_+), we conclude that $u_n \to u_0$ in E, which shows that the Palais–Smale condition is satisfied. At this stage it is enough to apply the mountain pass theorem in order to obtain a nontrivial weak solution of problem (1.4) for all $\lambda > 0$, provided that the condition (3.2) is satisfied.

The proof of Theorem 3.1 is complete.

3.4 Proof of Theorem 3.2

The energy functional associated to problem (1.5) is defined as

$$\mathcal{J}_\lambda(u) = \int_\Omega \frac{1}{p(x)}(|\Delta u|^{p(x)} + a|u|^{p(x)})dx - \lambda \int_\Omega \frac{|u|^{\gamma(x)}}{\gamma(x)}dx + \int_\Omega \frac{|u|^{\beta(x)}}{\beta(x)}dx \quad \text{for all } u \in E.$$

We show that $\mathcal{J}_\lambda$ is coercive, namely

$$\mathcal{J}_\lambda(u) \to +\infty \quad \text{as } \|u\|_a \to \infty.$$

Indeed, for all $u \in E$ with $\|u\|_a > 1$ we have

$$\begin{aligned} \mathcal{J}_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega (|\Delta u|^{p^-} + a|u|^{p^-})dx - \frac{\lambda}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx + \frac{1}{\beta^+}\int_\Omega |u|^{\beta(x)}dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{\lambda}{\gamma^-}\int_\Omega |u|^{\gamma(x)}dx \\ &\geq \frac{1}{p^+}\|u\|_a^{p^-} - \frac{c\lambda}{\gamma^-}\|u\|_a^{\gamma^+}, \end{aligned}$$

where c is the best constant of the continuous embedding $E \hookrightarrow L^{\gamma(x)}(\Omega)$. By hypothesis (3.1) we have $p^- > \gamma^+$, which infers that the energy functional $\mathcal{J}_\lambda$ is coercive.

Let (v_n) be a minimizing sequence of the functional $\mathcal{J}_\lambda$ in E. Since $\mathcal{J}_\lambda$ is coercive, we deduce that (v_n) is a bounded sequence. So, up to a subsequence, we can assume that

$$\begin{aligned} v_n &\rightharpoonup v_0 \quad \text{in } E, \\ v_n &\to v_0 \quad \text{in } L^{\gamma(x)}(\Omega), \\ v_n &\to v_0 \quad \text{in } L^{\beta(x)}(\Omega). \end{aligned}$$

Using now the lower semicontinuity of $\mathcal{J}_\lambda$ (see [13, Lemma 2.1 (a)]), we deduce that v_0 is a global minimizer of $\mathcal{J}_\lambda$ on E. It remains to prove that $v_0 \neq 0$. We have $\mathcal{J}_\lambda(0) = 0$. Thus it is enough to show that

$$\inf\{\mathcal{J}_\lambda(v);\ v \in E\} < 0 \quad \text{for } \lambda \text{ big enough.}$$

Indeed, let us consider the following constrained minimization problem:

$$\lambda^{**} := \inf\Big\{\int_\Omega \frac{1}{p(x)}(|\Delta w|^{p(x)} + a|w|^{p(x)})dx + \int_\Omega \frac{|w|^{\beta(x)}}{\beta(x)}dx;\ w \in E \text{ and } \int_\Omega \frac{|w|^{\gamma(x)}}{\gamma(x)}dx = 1\Big\}. \tag{3.7}$$

If $(w_n) \subset E$ is an arbitrary minimizing sequence of problem (3.7) then (w_n) is bounded. Thus, up to subsequence, (w_n) converges weakly in E and strongly in $L^{\gamma(x)}(\Omega)$ and $L^{\beta(x)}(\Omega)$ to some w_0 satisfying

$$\int_\Omega \frac{|w_0|^{\gamma(x)}}{\gamma(x)}dx = 1$$

and

$$\lambda^{**} = \int_\Omega \frac{1}{p(x)}(|\Delta w_0|^{p(x)} + a|w_0|^{p(x)})dx + \int_\Omega \frac{|w_0|^{\beta(x)}}{\beta(x)}dx > 0.$$

We conclude that

$$\mathcal{J}_\lambda(w_0) = \lambda^{**} - \lambda < 0 \quad \text{for all } \lambda > \lambda^{**},$$

hence w_0 is a nontrivial weak solution of problem (1.5). The proof of Theorem 3.2 is complete.

4 Final Comments

The analysis of the proofs of Theorems 3.1 and 3.2 shows that the results remain true if the left-hand side of problems (1.4) and (1.5) is replaced with

$$\Delta^2_{p(x)}u + \alpha|u|^{p(x)-2}u,$$

where α is a real number such that the operator $\Delta^2_{p(x)}u + \alpha|u|^{p(x)-2}u$ is *coercive* in E, hence there is some $C > 0$ such that, for all $u \in E$,

$$\int_\Omega (|\Delta u|^{p(x)} + \alpha|u|^{p(x)})\,dx \geq C\varrho_a(u).$$

Even more, we expect that the results established in this paper are true for more general operators, say Leray–Lions operators with variable exponents. We refer here to the pioneering paper of J. Leray and J.-L. Lions [15].

The existence properties established in Theorems 3.1 and 3.2 remain valid if the *bounded* domain Ω is replaced with an *unbounded* domain with boundary $\partial\Omega$. In such a case local arguments are used, see F. Gazzola and V. Rădulescu [12, p. 59].

We point out that the results of this paper can be extended in a nonsmooth multi-valued setting, namely under weaker assumptions on the right-hand side of problems (1.4) and (1.5), which imply that the associated energy functionals are no longer of class C^1. This corresponds to *variational-hemivariational inequalities*. We refer to D. Motreanu and V. Rădulescu [19] for a related inequality problem.

Problems (1.4) and (1.5) have been studied in this paper in the subcritical case, which corresponds to the basic assumption that the growth of the variable exponents β, γ and p is inferior than the critical exponent $p^*(x)$ for all $x \in \overline{\Omega}$. This hypothesis is crucial in order to ensure related compact embeddings of E into Lebesgue spaces with variable exponent. A very interesting *open problem* is to study problems (1.4) and (1.5) in the following *almost critical* setting: there exists $x_0 \in \Omega$ such that

$$\max\{p(x), \beta(x), \gamma(x)\} < p^*(x) \quad \text{for all } x \in \overline{\Omega} \setminus \{x_0\} \quad \text{and} \quad \max\{p(x_0), \beta(x_0), \gamma(x_0)\} = p^*(x_0).$$

We believe that a very interesting research subject is to study problems (1.4) and (1.5) if the biharmonic operator with variable exponent $\Delta^2_{p(x)}u$ is replaced by an operator with several variable exponents, for instance

$$\Delta((|\Delta u|^{p_1(x)-2} + |\Delta u|^{p_2(x)-2})\Delta u).$$

We conclude with a very interesting *open problem* concerning (1.4) under the hypothesis (3.2). We have applied in our proof the standard mountain pass theorem of A. Ambrosetti and P. Rabinowitz [2]. This pioneering result corresponds to mountains of *positive altitude*. The degenerate case is associated with mountains of *zero altitude* and was established by P. Pucci and J. Serrin [22, 23] (see also Rădulescu [25] for an overview of these results). We suggest to formulate the optimal assumptions for the right-hand side of equation (1.4) in order to study this problem in the degenerate case of mountains of zero altitude.

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