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Bifurcation analysis for degenerate problems with mixed regime and absorption

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We are concerned with the study of a bifurcation problem driven by a degenerate operator of Baouendi–Grushin type. Due to its degenerate structure, this differential operator has a mixed regime. Studying the combined effects generated by the absorption and the

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reaction terms, we establish the bifurcation behavior in two cases. First, if the absorption nonlinearity is dominating, then the problem admits solutions only for high perturbations of the reaction. In the case when the reaction dominates the absorption term, we prove that the problem admits nontrivial solutions for all the values of the parameter. The analysis developed in this paper is associated with patterns describing transonic flow restricted to subsonic regions.

Keywords: Baouendi–Grushin vector field; degenerate differential operator; nonlinear bifurcation problem; perturbation; transonic flow.

Mathematics Subject Classification 2020: 35J70, 35M10, 35M32, 58E07, 58J55, 76H05

1. Introduction

This paper deals with the bifurcation analysis of solutions for a class quasilinear degenerate of mixed-type. Problems with mixed regime are strictly connected with the analysis of nonlinear patterns and stationary waves for transonic flow models. We refer to the pioneering work of Morawetz [21–23] on the theory of transonic fluid flow — referring to partial differential equations that possess both elliptic and hyperbolic region — and this remains the most fundamental mathematical work on this subject. The flow is supersonic in the elliptic region, while a shock wave is created at the boundary between the elliptic and hyperbolic regions. In the 1950s, Morawetz used functional-analytic methods to study boundary value problems for such transonic problems.

To the best of our knowledge, the study of elliptic-problems has been initiated by Tricomi [31] who studied problems driven by the operator

$$T := \frac{\partial^2}{\partial x^2} + x\frac{\partial^2}{\partial y^2} \quad \text{if } (x,y) \in \mathbb{R}^2.$$

We remark that the Tricomi operator is degenerate in the region $\Omega_0 := \{(x,y) \in \mathbb{R}^2 : \ x = 0\}$. The Tricomi operator has important applications in the theory of planar transonic flow, see Manwell [16].

There are many generalizations of the Tricomi operator T, which can be obtained by substituting x with a function $g(x)$. For instance, Baouendi [3] and Grushin [13] studied the degenerate operator

$$\frac{\partial^2}{\partial x^2} + x^{2r}\frac{\partial^2}{\partial y^2}, \quad r \in \mathbb{N}. \tag{1}$$

We observe that the Baouendi–Grushin operator can be viewed as the Tricomi operator for transonic flow restricted to subsonic regions.

In higher dimensions, let us consider the operator $G_{2r} := \Delta_x + |x|^{2r}\Delta_y$, where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and $n + m = N$. This operator can be seen as the N-dimensional analogue of (1). If $z = (x,y) \in \mathbb{R}^N$, we notice that the operator G_{2r} can be rewritten, with a suitable choice of function a_α, in the form

$$\mathcal{L}u := \sum_{|\alpha|} D_z^\alpha(a_\alpha(z,u)).$$

This operator has been studied in a pioneering paper by Mitidieri and Pohozaev [19]. A related differential operator was studied by Bahrouni *et al.* [2] in the framework of double-phase transonic flow problems with variable growth.

We give in what follows some details about the differential operator that describes our problem. Assume that the Euclidean space $\mathbb{R}^N$ $(N \geq 2)$ can be written as $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ and for all $z \in \Omega \subset \mathbb{R}^N$ (Ω is an arbitrary open set) we denote $z = (x, y)$, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Let γ be a nonnegative real number. Then the differential operator that describes the problem studied in this paper involves the sub-elliptic gradient, which is the N-dimensional vector field given by

$$\nabla_\gamma := (\nabla_x, |x|^\gamma \nabla_y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n),$$

where $|x|$ (respectively, $|y|$) is the Euclidean norm in $\mathbb{R}^m$ (respectively, $\mathbb{R}^n$) and with the corresponding Baouendi–Grushin vector fields

$$X_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, m$$

and

$$Y_j = |x|^\gamma \frac{\partial}{\partial y_j}, \quad j = 1, \ldots, n.$$

Then the Baouendi–Grushin operator on $\Omega \subset \mathbb{R}^N = \mathbb{R}^{m+n}$ is defined by

$$\Delta_\gamma = \sum_{i=1}^{m} X_i^2 + \sum_{j=1}^{n} Y_j^2 = \Delta_x + |x|^{2\gamma} \Delta_y = \nabla_\gamma \cdot \nabla_\gamma,$$

where Δ_x and Δ_y are the Laplace operators in the variables x and y, respectively.

We point out that the Baouendi–Grushin operator for an even positive integer γ is a sum of squares of C^∞ vector fields satisfying the Hörmander condition

$$\operatorname{rank} \operatorname{Lie}[X_1, \ldots, X_m, Y_1, \ldots, Y_n] = N.$$

We can define on $\mathbb{R}^N = \mathbb{R}^{m+n}$ the anisotropic dilation δ_λ attached to Δ_γ as

$$\delta_\lambda(x, y) := (\lambda x, \lambda^{1+\gamma} y),$$

for $\lambda > 0$.

An important element in the mathematical analysis of PDEs driven by the Baouendi–Grushin operator is the *homogeneous dimension* with respect to the dilation δ_λ, which is defined by

$$Q = m + (1 + \gamma)n. \tag{2}$$

We notice that $Q = N + \gamma n > N \geq 2$.

By the change of variables formula for the Lebesgue measure we obtain

$$d \circ \delta_\lambda(x, y) = \lambda^Q dx dy.$$

It follows that

$$X_i(\delta_\lambda) = \lambda\delta_\lambda(X_i), \quad Y_j(\delta_\lambda) = \lambda\delta_\lambda(Y_j),$$

and hence

$$\nabla_\gamma \circ \delta_\lambda = \lambda\delta_\lambda\nabla_\gamma.$$

Assuming that $\gamma = 1/2$, then the corresponding Baouendi–Grushin operator is intimately connected to the sub-Laplacians in groups of Heisenberg type, see Garofalo and Lanconelli [11].

Finally, we point out that if $f = f(r) \in C^2(0,\infty)$ and define $u = f(r)$ then we have the following useful formula:

$$\Delta_\gamma u = \frac{|x|^{2\gamma}}{r^{2\gamma}}\left(f'' + \frac{Q-1}{r}\,f'\right).$$

In this paper, we use the Baouendi–Grushin differential operator in a quasilinear nonradial setting.

Some of the abstract methods used in this paper are developed in the recent monograph by Papageorgiou *et al.* [26].

2. Statement of the Problem

Our purpose in this paper is to study a class of degenerate bifurcation problems described by a nonlinear differential operator of mixed-type.

Let $\Omega \subset \mathbb{R}^N = \mathbb{R}^{m+n}$ $(N \geq 2)$ be a bounded domain with smooth boundary. We suppose that Ω intersects the degeneracy set $[x = 0]$, that is

$$\Omega \cap \{(0, y) : y \in \mathbb{R}^m\} \neq \emptyset.$$

We are concerned with the study of the following degenerate bifurcation problem:

$$\begin{cases} -\Delta_x u - |x|^{2\gamma}\Delta_y u + |u|^{q-2}u = \lambda\,|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{3}$$

A related problem was studied for exterior domains and for the p-Laplace operator by Yu [32] and by Filippucci *et al.* [8].

Throughout this paper, we denote by $\|\cdot\|_s$ the norm in the Lebesgue space $L^s(\Omega)$ for $1 \leq s \leq \infty$.

We shall distinguish two cases in the analysis developed in this paper. We first consider the case where the absorption term $|u|^{q-2}u$ in problem (3) is dominating with respect to the reaction $|u|^{r-2}u$. Mathematically, this corresponds to $q > r$. In this case, due to the dominance of the absorption, we prove that solutions exist only for high perturbations of the reaction, that is, for sufficiently big values of the parameter λ. Next, if the reaction is the dominating nonlinearity (that is, if $r > q$)

we prove that the influence of the reaction is sufficient to guarantee solutions for all values of the positive parameter λ.

Let $\mathcal{X}$ be the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_1 = \|\nabla_x u\|_2 + |\,\||x|^\gamma \nabla_y u\|_2.$$

Let Q^* denote the Sobolev critical exponent associated with the homogeneous dimension Q defined in (2), that is, $Q^* = 2Q/(Q-2)$. We observe that $2 < Q^* < 2^*$.

We assume throughout this paper that $\gamma > 0$ and $q, r \in (2, Q^*)$.

We say that $u \in \mathcal{X}\backslash\{0\}$ is a solution of problem (3) if for all $v \in \mathcal{X}$, we have

$$\int_\Omega \nabla_x u \nabla_x v dxdy + \int_\Omega |x|^{2\gamma} \nabla_y u \nabla_y v dxdy + \int_\Omega |u|^{q-2} uv dxdy$$
$$= \lambda \int_\Omega |u|^{r-2} uv dxdy. \tag{4}$$

In this case, we say that λ is an eigenvalue of problem (3) and the corresponding $u \in \mathcal{X}\backslash\{0\}$ is an eigenfunction of (3). These terms are in accordance with the related notions introduced by Fučik *et al.* [10, p. 117] in the context of *nonlinear* operators. Indeed, if we denote

$$S(u) := \frac{1}{2} \int_\Omega (|\nabla_x u|^2 + |x|^{2\gamma} |\nabla_y u|^2) dxdy + \frac{1}{q} \int_\Omega |u|^q dxdy$$

and

$$T(u) := \frac{1}{r} \int_\Omega |u|^r uv dxdy$$

then λ is an eigenvalue for the pair (S, T) of nonlinear operators (in the sense of [10]) if and only if there is a corresponding eigenfunction that is a solution of problem (3).

Taking $v = u$ in relation (4), we deduce that any solution u of problem (3) verifies

$$\|u\|^2 + \|u\|_q^q = \lambda \|u\|_r^r. \tag{5}$$

This implies that problem (3) does not have any solution if $\lambda \le 0$. So, we shall be concerned only with positive values of the bifurcation parameter λ.

We first consider the case $2 < r < q < Q^*$.

We state in what follows the first result of this paper.

Theorem 2.1. *Assume that $2 < r < q < Q^*$. Then there exists $\lambda^* > 0$ such that problem (3) has solutions if and only if $\lambda \ge \lambda^*$.*

Next, we assume that $q < r$. The following result shows that, in this case, problem (3) has at least one solution for all positive values of the parameter λ.

Theorem 2.2. *Assume that $2 < q < r < Q^*$. Then problem (3) has at least one positive solution for all $\lambda > 0$.*

Problem (3) has a variational structure and the associated energy functional should be well defined. For this purpose, we need Sobolev-type embeddings. More details will be given in Sec. 3.

The proof of Theorem 2.1 is developed in Sec. 3, while Sec. 4 will include the details of the proof of Theorem 2.2. We conclude this paper with some open problems and perspectives.

3. Proof of Theorem 2.1

The energy functional associated to problem (3) is $\mathcal{J} : \mathcal{X} \to \mathbb{R}$ defined by

$$\mathcal{J}(u) = \frac{1}{2}\int_\Omega \left(|\nabla_x u|^2 + |x|^{2\gamma}|\nabla_y u|^2\right)dxdy + \frac{1}{q}\int_\Omega |u|^q dxdy - \frac{\lambda}{r}\int_\Omega |u|^r dxdy.$$

We prove that $\mathcal{J}$ is well defined. For this purpose, we need suitable Sobolev-type embeddings. For Sobolev–Poincaré inequalities for weighted gradients we refer to Franchi *et al.* [9], Garofalo and Nhieu [12], Laptev *et al.* [14], Lu [15], and Song and Li [30]. Partial results are also given by Bahrouni *et al.* [2, Theorem 3.2] in the anisotropic setting.

In this paper, we refer to Monti [20] who obtained the following Sobolev–Poincaré inequality. We recall that in (2) we defined the homogeneous dimension Q as $Q = m + (1+\gamma)n = N + \gamma n > N \geq 2$.

Theorem 3.1. *Assume that $\gamma > 0$. Let $D^1(\mathbb{R}^N)$ be the Sobolev space obtained as completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm*

$$\|u\|_{D^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N} [|\nabla_x u|^2 + |x|^{2\gamma}|\nabla_y u|^2]dxdy\right)^{1/2}.$$

Then there exists a positive constant $C = C(m, n, \gamma)$ such that for all $u \in D^1(\mathbb{R}^{m+n})$,

$$\left(\int_{\mathbb{R}^{m+n}} |u|^{2Q/(Q-2)}dxdy\right)^{(Q-2)/Q} \leq C\int_{\mathbb{R}^{m+n}} (|\nabla_x u|^2 + |\nabla_y u|^2)dxdy.$$

The space $D^1(\mathbb{R}^N)$ can be equivalently defined as the space of functions $u \in L^{2Q/(Q-2)}(\mathbb{R}^N)$ with weak derivatives satisfying $\|u\|_{D^1(\mathbb{R}^N)} < \infty$.

Theorem 3.1 implies the following compact embedding theorem for function spaces with weighted gradients.

Corollary 3.1. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$ $(N \geq 2)$. Then the space $\mathcal{X}$ is continuously embedded into $L^{2Q/(Q-2)}(\Omega)$ and is compactly embedded into $L^s(\Omega)$ for all $1 < s < 2Q/(Q-2)$.*

Using Corollary 3.1, we deduce that the energy functional $\mathcal{J}$ is well defined.

We first observe that $\mathcal{J}$ is coercive. Indeed, since $r < q$, then the Hölder inequality yields for all $u \in \mathcal{X}$

$$\int_\Omega |u|^r dxdy \le \left(\int_\Omega |u|^q dxdy\right)^{r/q} |\Omega|^{(q-r)/q}.$$

Thus, for all $u \in \mathcal{X}$

$$\mathcal{J}(u) \ge \frac{1}{2}\|u\|^2 + \frac{1}{q}\|u\|_q^q - \frac{\lambda|\Omega|^{(q-r)/r}}{r}\|u\|_q^r.$$

Fix positive numbers a and b. Since $0 < r < q$ then there exists $C(a,b)$ such that

$$ax^r - bx^q \le C(a,b) \quad \text{for all } x \ge 0.$$

We conclude that for all $u \in \mathcal{X}$

$$\mathcal{J}(u) \ge \frac{1}{2}\|u\|^2 - C(q,r,\Omega) \to +\infty \quad \text{as } \|u\| \to \infty,$$

hence $\mathcal{J}$ is coercive. This means that any minimizing sequence $(u_n) \subset \mathcal{X}$ of $\mathcal{J}$ is bounded. Thus, by the reflexivity of $\mathcal{X}$ and Corollary 3.1, we can assume that (u_n) converges weakly in $\mathcal{X}$ and strongly in $L^q(\Omega)$ and $L^r(\Omega)$ to u. Since $\mathcal{J}(|u|) \le \mathcal{J}(u)$, we can assume that $u \ge 0$. Finally, by lower semicontinuity, we deduce that u is a global minimizer of the energy functional $\mathcal{J}$.

Next, we prove that solutions of problem (3) could exist only in the case of high perturbations of the reaction, that is, for large values of the positive parameter λ.

By Young's inequality and using the basic hypothesis $2 < r < q$, we deduce that for all $u \in \mathcal{X}\backslash\{0\}$

$$\lambda|u|^r \le \frac{r}{q}|u|^q + \frac{q-r}{q}\lambda^{q/(q-r)},$$

hence

$$\lambda \int_\Omega |u|^r dxdy \le \frac{r}{q}\int_\Omega |u|^q dxdy + \frac{q-r}{q}|\Omega|\,\lambda^{q/(q-r)}. \tag{6}$$

Assume that u is a solution of problem (3). It follows that

$$\begin{aligned} \|u\|^2 &= \int_\Omega |\nabla_x u|^2 dxdy + \int_\Omega |x|^{2\gamma}|\nabla_y u|^2 dxdy \\ &= \lambda\int_\Omega |u|^r dxdy - \int_\Omega |u|^q dxdy. \end{aligned} \tag{7}$$

Combining (6) and (7), we obtain that any solution u of (3) satisfies

$$0 < \|u\|^2 \le \frac{r-q}{q}|\Omega|\,\lambda^{q/(q-r)} = C_1(q,r,\Omega)\lambda^{q/(q-r)}. \tag{8}$$

By Corollary 3.1, relation (8) yields

$$\|u\|_r^2 \le C_2(q,r,\Omega)\lambda^{q/(q-r)}. \tag{9}$$

But by (7), we have

$$\lambda\|u\|_r^r \geq \|u\|^2.$$

By the Sobolev embedding theorem, this relation yields

$$\lambda\|u\|^r \geq C_3\|u\|^2. \tag{10}$$

Combining relations (8) and (10), we obtain

$$\lambda \geq C_3\|u\|^{2-r} \geq C_4(q,r,\Omega)\lambda^{\frac{q(2-r)}{2(q-r)}},$$

hence

$$\lambda^{1+\frac{q(r-2)}{2(q-r)}} \geq C_4(q,r,\Omega). \tag{11}$$

Recall that $2 < r < q$. Relation (11) implies that there exists

$$\lambda_0 := C_4(q,r,\Omega)^{-1-\frac{q(r-2)}{2(q-r)}} > 0 \tag{12}$$

such that if problem (3) has solutions, then $\lambda \geq \lambda_0$.

We now prove that the global minimizer $u \geq 0$ of $\mathcal{J}$ is nontrivial. For this purpose, it is sufficient to show that the corresponding energy level is negative. The idea is to construct a natural constrained minimization problem and to show that the corresponding value is negative, provided that λ is large enough.

Consider the problem

$$0 < m := \inf\left\{\frac{1}{2}\int_\Omega(|\nabla_x v|^2 + |x|^{2\gamma}|\nabla_y v|^2)dxdy + \frac{1}{q}\int_\Omega |v|^q dxdy;\right.$$
$$\left. v \in \mathcal{X},\ \int_\Omega |v|^r dxdy = 1\right\}.$$

If $(v_n) \subset \mathcal{X}$ is a corresponding minimizing sequence, then (v_n) is bounded. Thus, up to a subsequence,

$$v_n \rightharpoonup v_0 \quad \text{in } \mathcal{X}$$

and

$$v_n \to v_0 \quad \text{in } L^r(\Omega).$$

We conclude that $\|v_0\|_r^r = 1$ and

$$\frac{1}{2}\int_\Omega(|\nabla_x v_0|^2 + |x|^{2\gamma}|\nabla_y v_0|^2)dxdy + \frac{1}{q}\int_\Omega |v_0|^q dxdy = m.$$

Therefore

$$\mathcal{J}(v_0) = m - \frac{\lambda}{r},$$

hence $\mathcal{J}(v_0) < 0$ if $\lambda > mr$. It follows that the global minimizer u of $\mathcal{J}$ is nontrivial, hence problem (3) has a nonnegative solution.

Let

$$\lambda^* = \inf\{\lambda > 0;\ \text{problem (3) has at least one nonnegative solution}\}.$$

Then $\lambda^* \geq \lambda_0$, where λ_0 is defined in (12).

We prove in what follows that problem (3) admits solutions for all $\lambda > \lambda^*$. The basic abstract tool is the comparison principle for degenerate operators, see Pucci and Serrin [28, Theorem 3.6.1].

Fix $\bar{\lambda} > \lambda^*$. By the definition of λ^*, there exists $\nu \in (\lambda^*, \bar{\lambda})$ such that problem (3) has a nonnegative solution u_ν for $\lambda = \nu$ in (3). Then u_ν is a nonnegative lower solution of problem (3) for $\lambda = \bar{\lambda}$.

We claim that problem (3) for $\lambda = \bar{\lambda}$ has a nonnegative upper solution U such that $U \geq u_\nu$ in Ω. For this purpose, we define the following minimization problem:

$$\inf\left\{\frac{1}{2}\int_\Omega (|\nabla_x v|^2 + |x|^{2\gamma}|\nabla_y v|^2)dxdy + \frac{1}{q}\int_\Omega |v|^q dxdy \right.$$
$$\left. - \frac{\bar{\lambda}}{r}\int_\Omega |v|^r dxdy;\ v \in \mathcal{X},\ v \geq u_\nu\right\}.$$

With the same arguments as for the previous constrained minimization problem, we obtain that this problem has a solution $U \geq u_\nu$, which proves our claim. We conclude that problem (3) admits at least one nonnegative solution for all $\lambda > \lambda^*$.

Now, assuming that there exists $0 < \mu < \lambda^*$ such that (3) for $\lambda = \mu$ has a solution, then the previous argument shows that problem (3) has solutions for all $\lambda \geq \mu$, which contradicts the definition of λ^*. It follows that (3) does not have any solution for all $\lambda \in (0, \lambda^*)$.

We now establish that problem (3) has a solution for $\lambda = \lambda^*$. The idea is to consider a sequence of real numbers (λ_n) such that $\lambda_n \searrow \lambda^*$ as $n \to \infty$. Let $u_n \geq 0$ be a solution of (3) for $\lambda = \lambda_n$. It follows that for all $v \in \mathcal{X}$

$$\int_\Omega \nabla_x u_n \nabla_x v dz + \int_\Omega |x|^{2\gamma}\nabla_y u_n \nabla_y v dz + \int_\Omega |u_n|^{q-2}u_n v dz = \lambda_n \int_\Omega |u_n|^{r-2}u_n v dz. \tag{13}$$

We have already remarked that the sequence $(u_n) \subset \mathcal{X}$ is bounded. Thus, up to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup u^* \quad \text{in } \mathcal{X} \text{ as } n \to \infty, \\ u_n &\to u^* \quad \text{in } L^q(\Omega) \text{ as } n \to \infty. \end{aligned} \tag{14}$$

So, taking $n \to \infty$ in relation (13), we deduce that $u^* \geq 0$ is a solution of problem (3) for $\lambda = \lambda^*$.

To conclude the proof, it remains to argue that u^* is nontrivial. Returning to the sequence of solutions (u_n) corresponding to $\lambda_n \searrow \lambda^*$, we have

$$\lambda_n \|u_n\|_r^r = \|u_n\|^2 + \|u_n\|_q^q \quad \text{for all } n \geq 1.$$

If C is the best constant of the continuous embedding of $\mathcal{X}$ into $L^r(\Omega)$, it follows that

$$\lambda_n C \|u_n\|^r \geq \|u_n\|^2 \quad \text{for all } n \geq 1,$$

hence $\liminf_{n\to\infty} \|u_n\| > 0$. We obtain

$$\liminf_{n\to\infty} \|u_n\|_r > 0.$$

Thus, by (14), we conclude that $u^* \geq 0$ is nontrivial solution.

We have proved that problem (3) has a nontrivial solution u if and only if $\lambda \geq \lambda^*$. This solution is nonnegative, since the above arguments show that we can replace u by $|u|$. By symmetry reasons, problem (3) admits also nonpositive solutions for all $\lambda \geq \lambda^*$. □

4. Proof of Theorem 2.2

The proof relies on the mountain pass theorem, which we shall use in the following form (see Brezis and Nirenberg [6, p. 943]).

Theorem 4.1. *Let X be a Banach space and assume that $F : X \to \mathbb{R}$ is a C^1 function satisfying the following condition: there are an open neighborhood U of 0 and some point $v_0 \notin \overline{U}$ such that*

$$F(0), \quad F(v_0) < c_0 \leq F(u) \quad \textit{for all } u \in \partial U.$$

Consider the family $\mathcal{P}$ of all continuous paths p joining 0 and v_0 and set

$$c = \inf_{A\in\mathcal{P}} \max_{u\in A} F(u). \tag{15}$$

Then there exists a sequence (u_n) in X such that

$$F(u_n) \to c \quad \textit{and} \quad \|F'(u_n)\| \to 0 \quad \textit{as } n \to \infty.$$

If, in addition, we assume $(PS)_c$ with c given in (15), then c is a critical value of F.

In our case, we observe that for all $u \in \mathcal{X}$

$$\begin{aligned}\mathcal{J}(u) &= \frac{1}{2}\|u\|^2 + \frac{1}{q}\|u\|_q^q - \frac{\lambda}{r}\|u\|_r^r \\ &\geq \frac{1}{2}\|u\|^2 - \frac{\lambda}{r}C^r\|u\|_r^r,\end{aligned}$$

where C denotes the best Sobolev constant of the embedding $\mathcal{X} \subset L^r(\Omega)$.

Since $r > 2$, there are positive numbers ρ and c_0 such that $\mathcal{J}(u) \geq c_0$ for all $u \in \mathcal{X}$ with $\|u\| = \rho$.

Next, let $v \in \mathcal{X}\setminus\{0\}$. Then for all $t > 0$

$$\mathcal{J}(tv) = At^2 + Bt^q - Ct^r,$$

where

$$A = \frac{1}{2}\int_\Omega |\nabla_x v|^2 dxdy + \frac{1}{2}\int_\Omega |x|^{2\gamma}|\nabla_y v|^2 dxdy,$$

$$B = \frac{1}{q}\int_\Omega |v|^q dxdy \quad \text{and} \quad C = \frac{\lambda}{r}\int_\Omega |v|^r dxdy.$$

Since $2 \le q < r$ we have $\mathcal{J}(tv) < 0$ if $t > 0$ is large enough.

Fix $v_0 \in \mathcal{X}$ with $\mathcal{J}(v_0) < 0$ and $\|v_0\| > \rho$. Set

$$\mathcal{P} = \{p \in C([0,1];\mathcal{X});\ p(0) = 0, p(1) = v_0\} \tag{16}$$

and

$$c = \inf_{p\in\mathcal{P}} \max_{t\in[0,1]} \mathcal{J}(p(t)).$$

Let $(u_n) \subset \mathcal{X}$ be a $(PS)_c$ sequence of $\mathcal{J}$, that is,

$$\mathcal{J}(u_n) = c + O(1) \quad \text{as } n \to \infty$$

and for all $v \in \mathcal{X}$

$$\mathcal{J}'(u_n)(v) = o(1)\|v\| \quad \text{as } n \to \infty.$$

Using the expression of the energy functional $\mathcal{J}$, these conditions yield

$$\frac{1}{2}\|u_n\|^2 + \frac{1}{q}\|u_n\|_q^q - \frac{\lambda}{r}\|u_n\|_r^r = O(1) \quad \text{as } n \to \infty$$

and

$$\|u_n\|^2 + \|u_n\|_q^q - \lambda\|u_n\|_r^r = o(1)\|u_n\| \quad \text{as } n \to \infty.$$

Combining these relations, we obtain

$$\left(\frac{q}{2} - 1\right)\|u_n\|^2 + \lambda\left(1 - \frac{q}{r}\right)\|u_n\|_r^r = O(1) + o(1)\|u_n\|.$$

Since $2 < q < r$, this estimate shows that the arbitrary $(PS)_c$ sequence $(u_n) \subset \mathcal{X}$ is convergent.

We now claim that the $(PS)_c$ sequence $(u_n) \subset \mathcal{X}$ is relatively compact. Here, we use a method developed by Brezis and Nirenberg [6], which is based on the qualitative properties of the differential operator that describes problem (3).

We first observe that for all $v \in \mathcal{X}$

$$\int_\Omega \nabla_x u_n \nabla_x v dxdy + \int_\Omega |x|^{2\gamma}\nabla_y u_n \nabla v dxdy = \int_\Omega \phi(u_n) v dxdy, \tag{17}$$

where

$$\phi(v) = \lambda|v|^{r-2}v - |v|^{q-2}v \tag{18}$$

satisfies

$$|\phi(v)| \le C(1 + |v|^{r-1}) \quad \text{for all } v \in \mathcal{X}.$$

By (17), our claim follows if we prove that the sequence $(\phi(u_n))$ is relatively compact in $\mathcal{X}^*$. By Theorem 3.1, this is done if we prove that $(\phi(u_n))$ converges, up to a subsequence, in $(L^{2Q/(Q-2)}(\Omega))^* = L^{2Q/(Q+2)}(\Omega)$.

We can assume, going again to a subsequence, that

$$u_n \to u \quad \text{a.e. in } \Omega,$$

where $u \in L^{2Q/(Q-2)}(\Omega)$.

Fix $\eta > 0$. Thus, by Egorov's theorem, there exists a set $\omega \subset \Omega$ such that $|\omega| < \eta$ and

$$u_n \to u \quad \text{uniformly in } \Omega \backslash \omega.$$

To conclude the proof, it is enough to show that

$$\int_\omega |\phi(u_n) - \phi(u)|^{2Q/(Q+2)} dz$$

can be made arbitrarily small.

We first observe that

$$\int_\omega |\phi(u)|^{2Q/(Q+2)} dz \leq C(1 + |u|^{2Q/(Q-2)}) dz.$$

So, choosing $\eta > 0$ small enough, the right-hand side of this inequality can be made as small as we wish.

Next, we observe that for all $\varepsilon > 0$ small enough, there exists $C_\varepsilon > 0$ such that for all $n \geq 1$

$$\int_\omega |\phi(u_n)|^{2Q/(Q+2)} dz \leq \varepsilon \int_\omega |u_n|^{2Q/(Q-2)} dz + C_\varepsilon\, |\omega|. \tag{19}$$

By Theorem 3.1 and since $(u_n) \subset \mathcal{X}$ is bounded, this relation implies that

$$\int_\omega |\phi(u_n)|^{2Q/(Q+2)} dz \leq C\,\varepsilon + C_\varepsilon\, |\omega| \leq 2C\,\varepsilon,$$

provided that $|\omega|$ is sufficiently small. The proof of the claim is complete.

So, by Theorem 4.1, problem (3) admits a solution for all $\lambda > 0$.

Finally, we argue that this solution is positive. This follows by a standard truncation argument, which consists in replacing the reaction $|u|^{r-2}u$ of problem (3) by

$$g(u) = \begin{cases} u^{r-1} & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases}$$

Then, with the same arguments as above, one can prove that the problem

$$\begin{cases} -\Delta_x u - |x|^{2\gamma}\Delta_y u + |u|^{q-2}u = \lambda\, g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u \not\equiv 0 & \text{in } \Omega \end{cases} \tag{20}$$

admits a solution U. Multiplying by U^- in (20) and integrating, we find

$$\int_\Omega \nabla_x U \nabla_x U^- dxdy + \int_\Omega |x|^{2\gamma} \nabla_y U \nabla_y U^- dxdy + \int_\Omega |U|^{q-2} U U^- dxdy = 0,$$

hence

$$\int_\Omega |\nabla_x U^-|^2 dxdy + \int_\Omega |x|^{2\gamma} |\nabla_y U^-|^2 dxdy + \int_\Omega |U|^{q-2} (U^-)^2 dxdy = 0.$$

It follows that $U^- = 0$, that is, U is a positive solution of (20). It follows that $g(U) = U^{r-1}$, hence U is also a solution of problem (3). $\square$

In the final part of this section, we emphasize two key facts concerning the proof of Theorem 2.2.

We first observe that the proof of the Palais–Smale condition also holds if the reaction of problem (3) has an "almost" critical growth. Indeed, in the estimate (19) we only need that

$$\phi(u) = o(|u|^{(Q+2)/(Q-2)}) \quad \text{as } |u| \to +\infty.$$

Next, we point out that the conclusion of Theorem 2.2 remains true if we replace the reaction $|u|^{r-2}u$ with a smooth function $g(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$|g(x,u)| \le C(1 + |u|^{r-1}) \quad \text{for all } (x,u) \in \Omega \times \mathbb{R}$$

$$g(x,0) = g_u(x,0) = 0 \quad \text{for all } x \in \Omega$$

and there exists $\mu > 2$ such that

$$0 < \mu G(x,u) \le u g(x,u) \quad \text{for large } u > 0,$$

where $G(x,u) = \int_0^u g(x,t)dt$.

4.1. *Case of small perturbations*

In a seminal paper, Baras and Pierre [4] established a striking result concerning the existence of solutions for perturbed nonlinear elliptic equations. They studied the problem

$$\begin{cases} -\Delta u = u^\gamma + \mu g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \ge 0 & \text{in } \Omega, \end{cases} \tag{21}$$

where $g \not\equiv 0$, $\mu \ge 0$ and $\gamma > 1$.

Baras and Pierre [4] proved that problem (21) does not have any solution if μ is bigger than a certain critical value. More precisely, they established that problem (21) has a solution for all $\mu \le 1$ if and only if

$$\int_\Omega \varphi g dx \le \frac{\gamma - 1}{\gamma^{\gamma'}} \int_\Omega \frac{(-\Delta\varphi)^{\gamma'}}{\varphi^{\gamma'-1}} dx$$

for all $\varphi \in W_0^{1,\infty}(\Omega) \cap W^{2,\infty}(\Omega)$ nonnegative and superharmonic function with compact support. In terms of duality, this condition expresses that a certain norm of g should be sufficiently small. Baras and Pierre [4] also proved that the optimal size is given by the constant $(\gamma - 1)/\gamma^{\gamma'}$

Inspired by the above mentioned result of Baras and Pierre, we study in what follows the following perturbed problem:

$$\begin{cases} -\Delta_x u - |x|^{2\gamma}\Delta_y u + |u|^{q-2}u = \lambda |u|^{r-2}u + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \tag{22}$$

Multiply by $\varphi > 0$ such that $\varphi = 0$ on $\partial\Omega$. Thus, if $f \geq 0$ and u is a solution of problem (22), then

$$\begin{aligned} \int_\Omega f\varphi dxdy = \int_\Omega u[-\Delta_x\varphi - |x|^{2\gamma}\Delta_y\varphi]dsdy + \int_\Omega |u|^{q-2}u\varphi dxdy \\ - \lambda \int_\Omega |u|^{r-2}u\varphi dxdy. \end{aligned} \tag{23}$$

As in Theorem 2.2, assume that $2 < q < r$. So, by the Cauchy–Schwarz inequality, relation (23) yields the following necessary condition for the existence of solutions to problem (22):

$$0 \leq \int_\Omega f\varphi dz \leq c\int_\Omega \frac{(-\Delta_x\varphi - |x|^{2\gamma}\Delta_y\varphi)^2}{\varphi}dz,$$

for every $\varphi > 0$ smooth such that $\varphi = 0$ on Ω, where c is positive constant not depending on φ. This means that if problem (22) admits solutions, then f should be small in a suitable topology. The following result gives a description of this property.

Theorem 4.2. *Assume that hypotheses of Theorem* 2.2 *are fulfilled. Then there exists a positive number δ such that problem* (22) *has at least one solution for all $f \in L^\infty(\Omega)$ with $\|f\|_\infty \leq \delta$.*

Proof. By Theorem 2.2, problem (22) has nontrivial solutions for all $\lambda > 0$, provided that $f = 0$. That is why, we assume from now on that $\lambda = 1$.

Arguing by contradiction, there exists a sequence $(f_n) \subset L^\infty(\Omega)$ such that $\|f_n\|_\infty \to 0$ and problem (22) (with $f = f_n$) has no solution for all $n \geq 1$. Consider the functional $\mathcal{J}_n : \mathcal{X} \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{J}_n(u) = \frac{1}{2}\int_\Omega (|\nabla_x u|^2 + |x|^{2\gamma}|\nabla_y u|^2)dxdy + \frac{1}{q}\int_\Omega |u|^q dxdy - \frac{1}{r}\int_\Omega (u^+)^r dxdy \\ - \int_\Omega f_n u dxdy. \end{aligned}$$

We observe that $\mathcal{J}_n$ satisfies the geometric hypotheses of the mountain pass theorem. Thus, there exists $v_0 \in \mathcal{X}$ such that

$$\mathcal{J}_n(v_0) \leq 0 \quad \text{for all } n \geq 1.$$

Moreover, there exist positive constants c_0 and ρ such that for all $n \geq 1$

$$\mathcal{J}_n(u) \geq c_0 \quad \text{for all } u \in \mathcal{X} \text{ with } \|u\| = \rho.$$

These properties follow by combining the arguments used in the proof of Theorem 2.2 with the fact that $f_n \to 0$ in $L^\infty(\Omega)$.

Let c_n be the corresponding minimax value of $\mathcal{J}_n$, that is,

$$c_n = \inf_{p \in \mathcal{P}} \max_{t \in [0,1]} \mathcal{J}_n(p(t)),$$

where $\mathcal{P}$ is defined in (16). It follows that $c_n \geq c_0 > 0$ for all $n \geq 1$. Moreover, since $\|f_n\|_\infty \to 0$, we deduce that (c_n) is also bounded from above.

By the mountain pass theorem, there exists $u_n \in \mathcal{X}$ such that $\mathcal{J}_n(u_n) = c_n$ and $\mathcal{J}'(u_n) = 0$ for all $n \geq 1$. Therefore

$$\begin{cases} -\Delta_x u_n - |x|^{2\gamma}\Delta_y u_n + |u_n|^{q-2}u_n = (u_n^+)^{r-1} + f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Next, with the same arguments as in the proof of Theorem 2.2, we prove that the sequence $(u_n) \subset \mathcal{X}$ is bounded. We remember that the basic argument for this conclusion is the subcritical growth of the nonlinear terms combined with the Egorov theorem and the monotonicity property of the differential operator. It follows that, up to a subsequence, we can assume that $u_n \rightharpoonup u_0$ in $\mathcal{X}$, $u_n \to u_0$ in $L^r(\Omega)$, and $u_n \to u_0$ a.e. in Ω as $n \to \infty$. It follows that u_0 verifies

$$\begin{cases} -\Delta_x u_0 - |x|^{2\gamma}\Delta_y u_0 + |u_0|^{q-2}u_0 = (u_0^+)^{r-1} & \text{in } \Omega \\ u_0 = 0 & \text{on } \partial\Omega \end{cases} \tag{24}$$

and

$$\begin{aligned} \mathcal{J}_0(u_0) = \frac{1}{2}\int_\Omega \left(|\nabla_x u_0|^2 + |x|^{2\gamma}|\nabla_y u_0|^2\right) dxdy + \frac{1}{q}\int_\Omega |u_0|^q dxdy \\ - \frac{1}{r}\int_\Omega (u_0^+)^r dxdy \geq c_0 > 0. \end{aligned}$$

It follows that $u_0 \neq 0$. Moreover, multiplying by u_0^- in (24) and integrating over Ω we obtain $u_0^- = 0$, hence $u_0 > 0$ in Ω. Since $u_n \to u_0$ a.e. in Ω, we deduce that $u_n > 0$ in Ω for n large is a solution of problem (22) for $f = f_n$. This contradicts our initial assumption on the sequence (f_n). We conclude that problem (22) admits at least one solution, provided that $\|f\|_\infty$ is sufficiently small. $\square$

Final remarks and open problems

(i) We can replace that the potential $|x|^{2\gamma}$ in problem (3) with a weight $\lambda(x) > 0$ (except for at most a finite number of points) belonging to the class A_∞ introduced by David and Semmes [7] and satisfying the RH_∞ condition (cf. [9]), that is

$$\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\lambda(x)dx \sim \max\{\lambda(x);\ x\in B(x_0,r)\}.$$

We do not know any results on PDEs involving the Baouendi–Grushin operator if the weight λ does not belong to the A_∞ class.

(ii) An interesting research direction concerns the study of the nonlinear eigenvalue problem (3) if the differential operator is replaced by the magnetic Baouendi–Grushin operator, see Laptev *et al.* [1, 14]. This operator is

$$G_{\mathcal{A}} := -(\mathrm{grad}_G + i\beta\mathcal{A}_0)^2 \quad \text{for } -\frac{1}{2}\le\beta\le\frac{1}{2},$$

where

$$\mathcal{A}_0 = (\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4) = \left(-\frac{\partial_y d}{d},\frac{\partial_x d}{d},-2y\frac{\partial_t d}{d},2x\frac{\partial_t d}{d}\right),$$

$$\mathrm{grad}_G = (\partial_x,\partial_y,2x\partial_t,2y\partial_t),$$

where $|z| = |(x,y)| = \sqrt{x^2+y^2}$ and $d(z,t) = (|z|^4+t^2)^{1/4}$ is the Kaplan distance.

(iii) We now refer to a very innovative field with large perspectives, intimately connected with the seminal contributions of Marcellini and Mingione [5, 17, 18] and inspired by patterns in nonlinear elasticity. This new research direction concerns the study of double-phase problems for nonlinear problems with mixed regime. In this case, the differential operator that describes problem (3) should be of the type

$$\begin{aligned}&-\mathrm{div}((|\nabla_x u|^{p_1-2}+|\nabla_x u|^{p_2-2})\nabla u)\\&\quad-\mathrm{div}((|x|^{\alpha_1}|\nabla_y u|^{q_1-2}+|x|^{\alpha_2}|\nabla_y u|^{q_2-2})\nabla u),\end{aligned}$$

for related values of $1<p_1\neq p_2$, $1<q_1\neq q_2$, and $0<\alpha_1\neq\alpha_2$.

For recent contributions to isotropic or anisotropic problems we refer to [2, 24, 25, 29, 33].

(iv) Another valuable possible research direction concerns degenerate problems with weighted gradients with variable exponent, that is, nonlinear problems associated with variational integrals of the type

$$\int_\Omega (|\nabla_x u|^{G(x,y)} + |x|^{\gamma(x,y)}|\nabla_y u|^{G(x,y)})dxdy,$$

with appropriate hypotheses on the variable potentials $G(x,y)$ and $\gamma(x,y)$.

References

[1] L. Aermark and A. Laptev, Hardy's inequality for the Grushin operator with a magnetic field of Aharanov–Bohm type, *Algebra i Analiz* **23**(2) (2011) 1–8 (Russian); English translation in *St. Petersburg Math. J.* **23**(2) (2012) 203–208.

[2] A. Bahrouni, V. D. Rădulescu and D. D. Repovš, Double phase transonic flow problems with variable growth: Nonlinear patterns and stationary waves, *Nonlinearity* **32**(7) (2019) 2481–2495.

[3] M. S. Baouendi, Sur une classe d'opérateurs elliptiques dégénérés, *Bull. Soc. Math. France* **95** (1967) 45–87.

[4] P. Baras and M. Pierre, Critère d'existence de solutions positives pour des équations semi-linéaires non monotones. (French) [Criteria for the existence of positive solutions of nonmonotonic semilinear equations], *Ann. Inst. H. Poincaré Anal. Non Linéaire* **2**(3) (1985) 185–212.

[5] P. Baroni, M. Colombo and G. Mingione, Regularity for general functionals with double phase, *Calc. Var. Partial Differential Equations* **57**(2) (2018) 62.

[6] H. Brezis and L. Nirenberg, Remarks on finding critical points, *Comm. Pure Appl. Math.* **44** (1991) 939–963.

[7] G. David and S. Semmes, Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings, in *Analysis and Partial Differential Equations*, Lecture Notes in Pure and Applied Mathematics, Vol. 122 (Marcel Dekker, New York, 1990), pp. 101–111.

[8] R. Filippucci, P. Pucci and V. D. Rădulescu, Existence and non-existence results for quasilinear elliptic exterior problems with nonlinear boundary conditions, *Comm. Partial Differential Equations* **33** (2008) 706–717.

[9] B. Franchi, C. Gutierrez and R. Wheeden, Weighted Sobolev-Poincaré inequalities for Grushin type operators, *Comm. Partial Differential Equations* **19**(3–4) (1994) 523–604.

[10] S. Fučik, J. Nečas, J. Souček and V. Souček, *Spectral Analysis of Nonlinear Operators*, Lecture Notes in Mathematics, Vol. 346 (Springer-Verlag, Berlin, New York, 1973).

[11] N. Garofalo and E. Lanconelli, Frequency functions on Heisenberg group, the uncertainty principle and unique continuation, *Ann. Inst. Fourier* **40** (1990) 313–356.

[12] N. Garofalo and D. M. Nhieu, Isoperimetric and Sobolev inequalities in Carnot-Carathéodory spaces and the existence of minimal surfaces, *Comm. Pure Appl. Math.* **49** (1996) 1081–1144.

[13] V. V. Grushin, On a class of hypoelliptic operators, *Math. USSR-Sb.* **12** (1970) 458–476.

[14] A. Laptev, M. Ruzhansky and N. Yessirkegenov, Hardy inequalities for Landau Hamiltonian and for Baouendi–Grushin operator with Aharonov–Bohm type magnetic field. Part I, *Math. Scand.* **125**(2) (2019) 239–269.

[15] G. Lu, Existence and size estimates for the Green's functions of differential operators constructed from degenerate vector fields, *Comm. Partial Differential Equations* **17**(7–8) (1992) 1213–1251.

[16] A. R. Manwell, *The Tricomi Equation with Applications to the Theory of Plane Transonic Flow*, Research Notes in Mathematics, Vol. 35 (Pitman, London, 1979).

[17] P. Marcellini, On the definition and the lower semicontinuity of certain quasiconvex integrals, *Ann. Inst. H. Poincaré, Anal. Non Linéaire* **3** (1986) 391–409.

[18] P. Marcellini, Regularity and existence of solutions of elliptic equations with p, q–growth conditions, *J. Differential Equations* **90** (1991) 1–30.

[19] E. Mitidieri and S. I. Pohozaev, *A priori* estimates of solutions to nonlinear partial differential equations and inequalities and applications, *Proc. Steklov Inst. Math.* **234** (2001) 1–375.

[20] R. Monti, Sobolev inequalities for weighted gradients, *Comm. Partial Differential Equations* **31**(10–12) (2006) 1479–1504.

[21] C. Morawetz, On the non-existence of continuous transonic flows past profiles. I, *Comm. Pure Appl. Math.* **9** (1956) 45–68.

[22] C. Morawetz, On the non-existence of continuous transonic flows past profiles. II, *Comm. Pure Appl. Math.* **10** (1957) 107–131.

[23] C. Morawetz, On the non-existence of continuous transonic flows past profiles. III, *Comm. Pure Appl. Math.* **11** (1958) 129–144.

[24] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, Double-phase problems with reaction of arbitrary growth, *Zeitschrift Angew. Math. Phys.* (*ZAMP*) **69** (2018) 69–108.

[25] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, Double-phase problems and a discontinuity property of the spectrum, *Proc. Amer. Math. Soc.* **147** (2019) 2899–2910.

[26] N. S. Papageorgiou, V. D. Rădulescu and D. D. Repovš, *Nonlinear Analysis–Theory and Methods*, Springer Monographs in Mathematics (Springer, Cham, 2019).

[27] P. Pucci and J. Serrin, On the strong maximum and compact support principles and some applications, in *Handbook of Differential Equations — Stationary Partial Differential Equations*, ed. M. Chipot, Vol. 4 (Elsevier 2007), pp. 355–483.

[28] P. Pucci and J. Serrin, *The Maximum Principle*, Progress in Nonlinear Differential Equations and their Applications, Vol. 73 (Birkhäuser Verlag, Basel, 2007).

[29] M. A. Ragusa and A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, *Adv. Nonlinear Anal.* **9**(1) (2020) 710–728.

[30] M. Song and W. Li, Weighted Caffarelli-Kohn-Nirenberg type inequalities related to Grushin type operators, *Adv. Nonlinear Anal.* **8**(1) (2019) 130–143.

[31] F. G. Tricomi, Sulle equazioni lineari alle derivate parziali di secondo ordine di tipo misto, *Mem. Lincei* **14** (1923) 133–247.

[32] L. S. Yu, Nonlinear p-Laplacian problems on unbounded domains, *Proc. Amer. Math. Soc.* **115** (1992) 1037–1045.

[33] Q. Zhang and V. D. Rădulescu, Double phase anisotropic variational problems and combined effects of reaction and absorption terms, *J. Math. Pures Appl. C* **118** (2018) 159–203.