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Spectrum in an unbounded interval for a class of nonhomogeneous differential operators

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Abstract

The present paper deals with the spectrum of a nonhomogeneous problem involving variable exponents on an exterior domain in $\mathbb{R}^N$. The existence of two positive real numbers λ_0 and λ_1, is established satisfying the condition $\lambda_0 \leqslant \lambda_1$, such that the problem has no eigenvalue in the interval $(0, \lambda_0)$ while any number in the interval $[\lambda_1, \infty)$ is an eigenvalue.

1. *Introduction*

Let Ω be a smooth domain (bounded or unbounded) in $\mathbb{R}^N$ $(N \geqslant 1)$. Consider the eigenvalue problem

$$\begin{aligned} &-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda\, g(x)\, |u|^{p(x)-2}u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \\ &u \not\equiv 0 \quad \text{in } \Omega. \end{aligned} \tag{1}$$

If $g \equiv 1$ and $p \equiv 2$, problem 1 is related to the Riesz–Fredholm theory of self-adjoint and compact operators. The linear case corresponding to $p \equiv 2$ but for a non-constant potential g has been first considered in the pioneering papers of Bocher [**4**], Hess and Kato [**18**], Minakshisundaran and Pleijel [**26**] and Pleijel [**28**]. For instance, Minakshisundaran and Pleijel [**26**, **28**] studied the case where Ω is bounded, $g \in L^\infty(\Omega)$, $g \geqslant 0$ in Ω and $g > 0$ in $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$. An important contribution in the study of 1 if Ω is not necessarily bounded has been given by Szulkin and Willem [**33**].

Let Λ denote the set of eigenvalues of 1, that is,

$$\Lambda = \Lambda_{p(x)} = \{\lambda \in \mathbb{R};\ \lambda \text{ is an eigenvalue of problem 1}\}.$$

In [**16**] Garcia and Peral established that if $p(x) \equiv p > 1$, then problem 1 has a sequence of eigenvalues, $\sup \Lambda = +\infty$ and $\inf \Lambda = \lambda_1 = \lambda_{1,p} > 0$, where $\lambda_{1,p}$ is the first eigenvalue of $(-\Delta_p)$ in $W_0^{1,p}(\Omega)$ and

$$\lambda_1 = \lambda_{1,p} = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^p\, dx}{\displaystyle\int_\Omega |u|^p\, dx}.$$

In Fan, Zhang and Zhao [**13**] it is shown that for general functions $p(\cdot)$ the set Λ is infinite and $\sup \Lambda = +\infty$. Moreover, it may arise that $\inf \Lambda = 0$. Set

$$\lambda^* = \lambda^*_{p(x)} = \inf \Lambda.$$

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In Fan, Zhang and Zhao [**13**] it is argued that if $N = 1$ then $\lambda^* > 0$ if and only if the function $p(x)$ is monotone. In arbitrary dimension, $\lambda^* = 0$ provided that there exist an open set $U \subset \Omega$ and a point $x_0 \in U$ such that $p(x_0) <$ (or $>$) $p(x)$ for all $x \in \partial U$.

The existence of a principal positive eigenvalue and of a corresponding unique (up to a multiplicative constant) positive eigenfunction both for the *linear* Laplace operator and for the *nonlinear* p–Laplace operator (see [**3**]) follows by the Krein–Rutman theorem [**20**]. Using variational techniques, Sreenadh [**31**] has extended this study to the Hardy–Sobolev operator $-\Delta_p u - \mu w(x)|u|^{p-2}u$ (assuming that $0 \in \Omega$), where μ is smaller than the best Hardy–Sobolev constant $(N-p)^p p^{-p}$ and w denotes the weight function

$$w(x) = \begin{cases} |x|^{-p} & \text{if } 1 < p < N, \\ \left(|x| \log \dfrac{1}{|x|}\right)^{-N} & \text{if } p = N. \end{cases}$$

The case of fully nonlinear elliptic operators has been considered by Felmer and Quaas [**14**], in the framework of the Pucci maximal operators. In all these cases, the existence part is guaranteed as long as the differential operator is positively homogeneous and is monotone with respect to a convex cone. The uniqueness part in such arguments based on the Krein–Rutman theorem requires that the operator is strictly increasing and strongly positive. In some recent papers we have considered nonlinear eigenvalue problems with a *nonhomogeneous* structure. In such nonstandard cases we have pointed out that some strange phenomena may occur. For instance, in [**22**, **24**] we have proved that, under some appropriate conditions, some classes of such eigenvalue problems admit solutions for any $\lambda \in (0, \lambda_0)$, where λ_0 is a positive real number. Taking into account the 'competition' between several variable exponents, we establish in the present work that the eigenvalues may concentrate in a neighborhood of infinity and that this phenomenon strongly depends on the decay rate of exponents in relationship with the variable potential g.

In this paper we are concerned with the study of the eigenvalue problem

$$\begin{aligned} &-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u + |u|^{q(x)-2}u = \lambda g(x)|u|^{r(x)-2}u \quad \text{for } x \in \Omega, \\ &u = 0 \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{2}$$

where Ω is a smooth exterior domain in $\mathbb{R}^N$ $(N \geqslant 3)$, that is, Ω is the complement of a bounded domain with Lipschitz boundary. We notice that this smoothness assumption is needed only to ensure the existence of Sobolev embeddings; see Edmunds and Rákosník [**10**]. The mappings $p,\ q,\ r : \overline{\Omega} \to [2, \infty)$ are Lipschitz continuous functions, while $g : \overline{\Omega} \to [0, \infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g(x) > 0$ for any $x \in \Omega_0$, and $\lambda \geqslant 0$ is a real number.

The study of problems of type 2 has a strong motivation, and important research efforts have been made with the aim of understanding anisotropic phenomena described by nonhomogeneous differential operators. We remember that equations of this type can be regarded as models for phenomena arising in the study of electrorheological fluids [**1**, **2**, **6**, **17**, **29**], elasticity [**35**] or image processing and restoration [**5**, **11**]. A survey of the history of this research field with a comprehensive bibliography is provided by Diening, Hästö and Nekvinda [**7**].

Next, we point out that this paper extends to the case of unbounded domains some recent results on eigenvalue problems involving variable exponent conditions (see [**24**, **25**]).

Finally, the present paper carries on some recent results obtained on some problems related with 2 but studied in the case when p, q and r are positive constants (see [**15**, **34**]).

2. *A brief review on variable exponent Lebesgue-Sobolev spaces*

In order to study problem 2 we introduce a variable exponent Lebesgue–Sobolev setting. For more details we refer to the book by Musielak [27] and the papers by Edmunds and co-workers [8–10], Kovacik and Rákosník [19], Mihăilescu and Rădulescu [23] and Samko and Vakulov [30].

Throughout this paper, for any Lipschitz continuous function $h : \overline{\Omega} \to (1, \infty)$ we denote

$$h^- = \operatorname{ess}\inf_{x\in\Omega} h(x) \quad \text{and} \quad h^+ = \operatorname{ess}\sup_{x\in\Omega} h(x).$$

Usually it is assumed that $h^+ < +\infty$, since this condition is known to imply many desirable features for the associated variable exponent Lebesgue space $L^{h(x)}(\Omega)$. This function space is defined by

$$L^{h(x)}(\Omega) = \left\{ u;\ u \text{ is a measurable real-valued function such that} \int_\Omega |u(x)|^{h(x)}\, dx < \infty \right\}.$$

On this space we define a norm, the so-called *Luxemburg norm*, by the formula

$$|u|_{h(x)} = \inf \left\{ \mu > 0;\ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{h(x)} dx \leqslant 1 \right\}.$$

The variable exponent Lebesgue space is a special case of an Orlicz–Musielak space. For a constant function h the variable exponent Lebesgue space coincides with the standard Lebesgue space.

We recall that the variable exponent Lebesgue spaces are separable and reflexive Banach spaces. If $0 < |\Omega| < \infty$ and h_1, h_2 are variable exponents so that $h_1(x) \leqslant h_2(x)$ almost everywhere in Ω, then there exists the continuous embedding $L^{h_2(x)}(\Omega) \hookrightarrow L^{h_1(x)}(\Omega)$.

We denote by $L^{h'(x)}(\Omega)$ the conjugate space of $L^{h(x)}(\Omega)$, where $1/h(x) + 1/h'(x) = 1$. For any $u \in L^{h(x)}(\Omega)$ and $v \in L^{h'(x)}(\Omega)$ the Hölder-type inequality

$$\left| \int_\Omega uv\, dx \right| \leqslant \left(\frac{1}{h^-} + \frac{1}{h'^-} \right) |u|_{h(x)} |v|_{h'(x)} \tag{3}$$

holds true.

Moreover, if h_1, h_2, $h_3 : \overline{\Omega} \to (1, \infty)$ are three Lipschitz continuous functions such that $1/h_1(x) + 1/h_2(x) + 1/h_3(x) = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$ and $w \in L^{h_3(x)}(\Omega)$ the following inequality holds (see [12, Proposition 2.5]):

$$\left| \int_\Omega uvw\, dx \right| \leqslant \left(\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-} \right) |u|_{h_1(x)}\, |v|_{h_2(x)}\, |w|_{h_3(x)}\,. \tag{4}$$

An important role in manipulating the generalized Lebesgue–Sobolev spaces is played by the *modular* of the $L^{h(x)}(\Omega)$ space, which is the mapping $\rho_{h(x)} : L^{h(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{h(x)}(u) = \int_\Omega |u(x)|^{h(x)}\, dx.$$

Lebesgue–Sobolev spaces with $h^+ = +\infty$ have been investigated in [8, 19]. In such a case we denote $\Omega_\infty = \{x \in \Omega;\ h(x) = +\infty\}$ and define the modular by setting

$$\rho_{h(x)}(u) = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{h(x)}\, dx + \operatorname{ess}\sup_{x\in\Omega_\infty} |h(x)|.$$

If (u_n), $u \in L^{h(x)}(\Omega)$ then the following relations hold true:

$$|u|_{h(x)} > 1 \quad \Longrightarrow \quad |u|_{h(x)}^{h^-} \leqslant \rho_{h(x)}(u) \leqslant |u|_{h(x)}^{h^+}, \tag{5}$$

$$|u|_{h(x)} < 1 \quad \Longrightarrow \quad |u|_{h(x)}^{h^+} \leqslant \rho_{h(x)}(u) \leqslant |u|_{h(x)}^{h^-}, \tag{6}$$

$$|u_n - u|_{h(x)} \to 0 \quad \Longleftrightarrow \quad \rho_{h(x)}(u_n - u) \to 0. \tag{7}$$

Next, we define the variable exponent Sobolev space

$$W^{1,h(x)}(\Omega) = \{u \in L^{h(x)}(\Omega) : \ |\nabla u| \in L^{h(x)}(\Omega)\}.$$

On $W^{1,h(x)}(\Omega)$ we may consider one of the following equivalent norms:

$$\|u\|_{h(x)} = |u|_{h(x)} + |\nabla u|_{h(x)}$$

or

$$\|u\| = \inf \left\{ \mu > 0; \ \int_\Omega \left(\left| \frac{\nabla u(x)}{\mu} \right|^{h(x)} + \left| \frac{u(x)}{\mu} \right|^{h(x)} \right) \, dx \leqslant 1 \right\}.$$

We also define $W_0^{1,h(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,h(x)}(\Omega)$. Assuming that $h^- > 1$, the function spaces $W^{1,h(x)}(\Omega)$ and $W_0^{1,h(x)}(\Omega)$ are separable and reflexive Banach spaces. Set

$$\varrho_{h(x)}(u) = \int_\Omega \left(|\nabla u(x)|^{h(x)} + |u(x)|^{h(x)} \right) \, dx.$$

For all (u_n), $u \in W_0^{1,h(x)}(\Omega)$ the following relations hold:

$$\|u\| > 1 \quad \Longrightarrow \quad \|u\|^{h^-} \leqslant \varrho_{h(x)}(u) \leqslant \|u\|^{h^+}, \tag{8}$$

$$\|u\| < 1 \quad \Longrightarrow \quad \|u\|^{h^+} \leqslant \varrho_{h(x)}(u) \leqslant \|u\|^{h^-}, \tag{9}$$

$$\|u_n - u\| \to 0 \quad \Longleftrightarrow \quad \varrho_{h(x)}(u_n - u) \to 0. \tag{10}$$

Finally, we remember some embedding results regarding variable exponent Lebesgue–Sobolev spaces. If h, $\theta : \Omega \to (1, \infty)$ are Lipschitz continuous and $h^+ < N$ and $h(x) \leqslant \theta(x) \leqslant h^\star(x)$ for any $x \in \Omega$ where $h^\star(x) = Nh(x)/(N - h(x))$, then there exists a continuous embedding $W_0^{1,h(x)}(\Omega) \hookrightarrow L^{\theta(x)}(\Omega)$. Furthermore, assuming that Ω_0 is a bounded subset of Ω, the embedding $W_0^{1,h(x)}(\Omega_0) \hookrightarrow L^{\theta(x)}(\Omega_0)$ is continuous and compact.

3. *The main result*

In this paper we study problem 2 assuming that the functions p, q and r satisfy the hypotheses

$$2 \leqslant p^- \leqslant p^+ < N, \tag{11}$$

$$p^+ < r^- \leqslant r^+ < q^- \leqslant q^+ < \frac{Np^-}{N - p^-}. \tag{12}$$

Furthermore, we assume that the function $g(x)$ satisfies the hypothesis

$$g \in L^\infty(\Omega) \cap L^{p_0(x)}(\Omega)\,, \tag{13}$$

where $p_0(x) = p^\star(x)/(p^\star(x) - r^-)$ for any $x \in \overline{\Omega}$.

Obviously, the natural space where we should seek solutions for problem 2 is the space $W_0^{1,p(x)}(\Omega)$.

We say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem 2 if there exists a $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + |u|^{p(x)-2} uv + |u|^{q(x)-2} uv) \, dx - \lambda \int_\Omega g(x) |u|^{r(x)-2} uv \, dx = 0,$$

for all $v \in W_0^{1,p(x)}(\Omega)$. We point out that if λ is an eigenvalue of the problem 2 then the corresponding $u \in W_0^{1,p(x)}(\Omega) \setminus \{0\}$ is a weak solution of 2.

Define

$$\lambda_1 := \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (1/p(x))(|\nabla u|^{p(x)} + |u|^{p(x)})\, dx + \int_\Omega (1/q(x))|u|^{q(x)}\, dx}{\displaystyle\int_\Omega (g(x)/r(x))|u|^{r(x)}\, dx}$$

and

$$\lambda_0 := \inf_{u \in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx + \int_\Omega |u|^{q(x)}\, dx}{\displaystyle\int_\Omega g(x)|u|^{r(x)}\, dx}.$$

Our main result is given by the following theorem.

Theorem 1. *Let Ω be an exterior domain with Lipschitz boundary in $\mathbb{R}^N$, where $N \geqslant 3$. Suppose that p, q, $r : \overline{\Omega} \to [2,\infty)$ are Lipschitz continuous functions and that $g : \overline{\Omega} \to [0,\infty)$ is a measurable function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g > 0$ in Ω_0. Assume that conditions* 11–13 *are fulfilled.*

Then

$$0 < \lambda_0 \leqslant \lambda_1.$$

Furthermore, each $\lambda \in [\lambda_1, \infty)$ is an eigenvalue of problem 2 *while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem* 2.

At this stage we are not able to deduce whether $\lambda_0 = \lambda_1$ or $\lambda_0 < \lambda_1$. In the latter case an interesting open problem concerns the existence of eigenvalues of problem 2 in the interval $[\lambda_0, \lambda_1)$.

Remark. We notice that a similar result as Theorem 1 can be proved (with similar arguments as those used in the case of problem 2) for the problem

$$\begin{aligned} &-\Delta u + u + |u|^{q(x)-2}u = \lambda g(x)|u|^{r(x)-2}u \quad \text{for } x \in \Omega,\\ &u = 0 \quad \text{for } x \in \partial\Omega, \end{aligned}$$

where Ω is a smooth exterior domain in $\mathbb{R}^2$. The mappings q and $r : \overline{\Omega} \to [2,\infty)$ are still Lipschitz continuous functions, while $g : \overline{\Omega} \to [0,\infty)$ is a function for which there exists a nonempty set $\Omega_0 \subset \Omega$ such that $g(x) > 0$ for any $x \in \Omega_0$, and $\lambda \geqslant 0$ is a real number. This time, conditions 11–13 should be replaced by the following conditions:

$$2 < r^- \leqslant r^+ < q^- \leqslant q^+ < \infty$$

and

$$g \in L^\infty(\Omega) \cap L^1(\Omega).$$

4. *Proof of the main result*

Let E denote the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$.

Define the functionals $J_1,\ I_1,\ J_0,\ I_0 : E \to \mathbb{R}$ by

$$J_1(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\ dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ dx,$$

$$I_1(u) = \int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}\ dx,$$

$$J_0(u) = \int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\ dx + \int_\Omega |u|^{q(x)}\ dx,$$

$$I_0(u) = \int_\Omega g(x)|u|^{r(x)}\ dx.$$

Standard arguments imply that $J_1,\ I_1 \in C^1(E, \mathbb{R})$ and for all $u,\ v \in E$,

$$\langle J_1'(u), v\rangle = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u \nabla v + |u|^{p(x)-2}uv + |u|^{q(x)-2}uv)\ dx,$$

$$\langle I_1'(u), v\rangle = \int_\Omega g(x)|u|^{r(x)-2}uv\ dx.$$

For any $\lambda > 0$ we also define the functional $T_\lambda : E \to \mathbb{R}$ by

$$T_\lambda(u) = J_1(u) - \lambda \cdot I_1(u) \quad \forall\, u \in E.$$

It is clear that λ is an eigenvalue for problem 2 if and only if there exists a $u_\lambda \in E \setminus \{0\}$, a critical point of the functional T_λ.

We split the proof of Theorem 1 into four steps.

Step 1. We show that $\lambda_0,\ \lambda_1 > 0$.

Indeed, since by relation 12 we have $p(x) < r(x) < q(x)$ for any $x \in \overline{\Omega}$, we deduce that

$$|u(x)|^{p(x)} + |u(x)|^{q(x)} \geqslant |u(x)|^{r(x)} \quad \forall\, u \in E \text{ and } \forall\, x \in \overline{\Omega}.$$

The above relation shows that

$$\int_\Omega (|u|^{p(x)} + |u|^{q(x)})\ dx \geqslant \frac{1}{|g|_\infty} \cdot \int_\Omega g(x)|u|^{r(x)}\ dx \quad \forall\, u \in E, \tag{14}$$

or

$$J_0(u) \geqslant \frac{1}{|g|_\infty} \cdot I_0(u) \quad \forall\, u \in E.$$

We deduce that $\lambda_0 > 0$.

On the other hand, by 14 we have

$$\int_\Omega \frac{1}{p(x)}|u|^{p(x)}\ dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\ dx \geqslant \frac{r^-}{q^+ \cdot |g|_\infty} \cdot \int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}\ dx, \quad \forall\, u \in E,$$

and thus

$$J_1(u) \geqslant \frac{r^-}{q^+ \cdot |g|_\infty} \cdot I_1(u) \quad \forall\, u \in E.$$

Consequently, $\lambda_1 > 0$ and Step 1 is verified.

Step 2. We show that any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem 2.

Indeed, assuming by contradiction that there exists a $\lambda \in (0, \lambda_0)$ that is an eigenvalue of problem 2, we deduce that there exists a $u_\lambda \in E \setminus \{0\}$ such that

$$\langle J_1'(u_\lambda), v\rangle = \lambda \cdot \langle I_1'(u_\lambda), v\rangle \quad \forall\, v \in E.$$

Taking $v = u_\lambda$ in the above equality we find that

$$\langle J_1'(u_\lambda), u_\lambda\rangle = \lambda \cdot \langle I_1'(u_\lambda), u_\lambda\rangle$$

or

$$J_0(u_\lambda) = \lambda \cdot I_0(u_\lambda).$$

Since $u_\lambda \in E \setminus \{0\}$ it follows that $J_0(u_\lambda) > 0$ and $I_0(u_\lambda) > 0$. This information combined with the above inequality, the definition of λ_0 and the fact that $\lambda \in (0, \lambda_0)$ yield

$$J_0(u_\lambda) \geqslant \lambda_0 \cdot I_0(u_\lambda) > \lambda \cdot I_0(u_\lambda) = J_0(u_\lambda),$$

and this is a contradiction. Thus Step 2 is verified.

Step 3. We show that any $\lambda \in (\lambda_1, \infty)$ is an eigenvalue for problem 2.

In order to verify that the conclusion of Step 3 holds true, we first prove two auxiliary results.

LEMMA 1. *Assume that the hypotheses of Theorem* 1 *are satisfied and* s *is a real number such that*

$$r^+ < s < (p^-)^\star,$$

where $(p^-)^\star = Np^-/(N-p^-)$*. Then* $g \in L^{s/(s-r^-)}(\Omega) \cap L^{s/(s-r^+)}(\Omega)$ *and*

$$\int_\Omega g(x)|u|^{r(x)} \leqslant |g|_{s/(s-r^-)}|u|_s^{r^-} + |g|_{s/(s-r^+)}|u|_s^{r^+} \quad \forall\, u \in E. \tag{15}$$

Proof. First we remark that the following inequalities hold true:

$$\frac{s}{s-r^+} \geqslant \frac{s}{s-r^-} > \frac{(p^-)^\star}{(p^-)^\star - r^-} \geqslant \frac{p^\star(x)}{p^\star(x) - r^-} = p_0(x), \quad \forall\, x \in \overline{\Omega}$$

and

$$p_0^+ = \frac{(p^-)^\star}{(p^-)^\star - r^-}.$$

On the other hand, taking into account the above information and the fact that relation 13 holds true, we have

$$\begin{aligned} \int_\Omega [g(x)]^{s/(s-r^-)}\, dx &= \int_\Omega [g(x)]^{p_0(x)} \cdot [g(x)]^{s/(s-r^-)-p_0(x)}\, dx \\ &\leqslant \int_\Omega [g(x)]^{p_0(x)} \cdot |g|_\infty^{s/(s-r^-)-p_0(x)}\, dx \\ &\leqslant [|g|_\infty^{s/(s-r^-)-p_0^+} + |g|_\infty^{s/(s-r^-)-p_0^-}] \cdot \int_\Omega [g(x)]^{p_0(x)}\, dx < \infty. \end{aligned}$$

Thus we have found that $g \in L^{s/(s-r^-)}(\Omega)$. Similar arguments show that $g \in L^{s/(s-r^+)}(\Omega)$.

Inequality 15 follows from the remark that

$$|u(x)|^{r(x)} \leqslant |u(x)|^{r^-} + |u(x)|^{r^+} \quad \forall\, u \in E \,\text{and}\, \forall\, x \in \overline{\Omega}$$

or

$$\begin{aligned} \int_\Omega g(x)|u|^{r(x)}\, dx &\leqslant \int_\Omega g(x)|u|^{r^-}\, dx + \int_\Omega g(x)|u|^{r^+}\, dx \\ &\leqslant |g|_{s/(s-r^-)}|u|_s^{r^-} + |g|_{s/(s-r^+)}|u|_s^{r^+} \quad \forall\, u \in E. \end{aligned}$$

The proof of Lemma 1 is complete. □

LEMMA 2. *For any* $\lambda > 0$ *we have*

$$\lim_{\|u\| \to \infty} T_\lambda(u) = \infty.$$

Proof. We fix $\lambda > 0$ and s such that

$$r^+ < s < q^- < (p^-)^\star.$$

Then using 15 we deduce that the following inequalities hold true for any $u \in E$ with $\|u\| > 1$:

$$\begin{aligned}
T_\lambda(u) &= \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx - \lambda \int_\Omega \frac{g(x)}{r(x)}|u|^{r(x)}\,dx \\
&\geqslant \frac{1}{2p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \frac{1}{\max\{2p^+, q^+\}}\int_\Omega (|u|^{p(x)} + |u|^{q(x)})\,dx \\
&\quad - \frac{\lambda}{r^-}\int_\Omega g(x)(|u|^{r^+} + |u|^{r^-})\,dx \\
&\geqslant \frac{1}{2p^+}\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \frac{1}{\max\{2p^+, q^+\}}\int_\Omega |u|^s\,dx - C_1 \cdot |u|_s^{r^-} - C_2 \cdot |u|_s^{r^+} \\
&\geqslant C_0 \cdot \|u\|^{p^-} + \left(\frac{1}{2\max\{2p^+, q^+\}}\int_\Omega |u|^s\,dx - C_1 \cdot |u|_s^{r^-}\right) \\
&\quad + \left(\frac{1}{2\max\{2p^+, q^+\}}\int_\Omega |u|^s\,dx - C_2 \cdot |u|_s^{r^+}\right),
\end{aligned}$$

where C_0, C_1 and C_2 are positive constants.

Actually, by the above inequality we found that there exist four positive constants C_0, C_1, C_2 and C_3 (with $C_3 = 1/2\max\{2p^+, q^+\}$) such that

$$T_\lambda(u) \geqslant C_0 \cdot \|u\|^{p^-} + \left(C_3 \cdot |u|_s^s - C_1 \cdot |u|_s^{r^-}\right) + \left(C_3 \cdot |u|_s^s - C_2 \cdot |u|_s^{r^+}\right), \tag{16}$$

for any $u \in E$ with $\|u\| > 1$.

Next, we show that for any $u \in E$ there exist two positive constants $M_1 = M_1(r^-, s, C_1, C_3)$ and $M_2 = M_2(r^+, s, C_2, C_3)$ such that

$$C_3 \cdot |u|_s^s - C_1 \cdot |u|_s^{r^-} \geqslant -M_1 \tag{17}$$

and

$$C_3 \cdot |u|_s^s - C_2 \cdot |u|_s^{r^+} \geqslant -M_2. \tag{18}$$

In order to prove this, we point out that the functional $\Phi : (0, \infty) \to \infty$ defined by

$$\Phi(t) = \alpha \cdot t^a - \beta \cdot t^b,$$

where α, β, a, b are positive constants with $a < b$, achieves its positive global maximum

$$\Phi(t_0) = \frac{b-a}{a} \cdot \left(\frac{a}{b}\right)^{a/(b-a)} \cdot \alpha^{b/(b-a)} \cdot \beta^{a/(a-b)} > 0,$$

where $t_0 = ((\alpha \cdot a)/(\beta \cdot b))^{1/(b-a)} > 0$. Thus we deduce that

$$\alpha \cdot t^a - \beta \cdot t^b \leqslant C(a, b) \cdot \alpha^{b/(b-a)} \cdot \beta^{a/(a-b)} \quad \forall\, t > 0, \tag{19}$$

where $C(a, b) = (b-a) \cdot (a^a/b^b)^{1/(b-a)}$.

Finally, we notice that taking in 19 $a = r^-$, $b = s$, $\alpha = C_1$ and $\beta = C_3$ we deduce that 17 holds true with $M_1 = C(r^-, s) \cdot C_1^{s/(s-r^-)} \cdot C_3^{r^-/(r^- - s)}$. Similarly, taking in 19 $a = r^+$, $b = s$, $\alpha = C_2$ and $\beta = C_3$ we deduce that 18 is valid with $M_2 = C(r^+, s) \cdot C_2^{s/(s-r^+)} \cdot C_3^{r^+/(r^+ - s)}$.

Combining relations 16–18 we deduce that Lemma 2 holds true. □

Now, we return to the proof of Step 3. First, we fix $\lambda \in (\lambda_1, \infty)$. By Lemma 2 we deduce that $\lim_{\|u\| \to \infty} T_\lambda(u) = \infty$, that is, T_λ is coercive. On the other hand, similar arguments as those used in the proof of [**23**, Lemma 3.4] show that the functional T_λ is weakly lower semi-continuous. These two facts enable us to apply [**32**, Theorem 1.2] in order to prove that there exists a $u_\lambda \in E$, a global minimum point of T_λ and thus a critical point of T_λ. In order to conclude that Step 3 holds true it is enough to show that u_λ is not trivial. Indeed, since $\lambda_1 = \inf_{u\in E\setminus\{0\}} J_1(u)/I_1(u)$ and $\lambda > \lambda_1$ it follows that there exists a $v_\lambda \in E$ such that

$$J_1(v_\lambda) < \lambda I_1(v_\lambda)$$

or

$$T_\lambda(v_\lambda) < 0.$$

Thus

$$\inf_E T_\lambda < 0,$$

and we conclude that u_λ is a nontrivial critical point of T_λ or λ is an eigenvalue of problem 2. Thus Step 3 is verified.

Step 4. We show that λ_1 is an eigenvalue of problem 2.
We begin by proving two auxiliary results.

LEMMA 3. *The following relation holds true:*

$$\lim_{\|u\|\to 0} \frac{J_0(u)}{I_0(u)} = +\infty.$$

Proof. Let s be a real number satisfying the following inequality:

$$r^+ < s < q^- < (p^-)^\star.$$

Thus we deduce that E is continuously embedded in $L^s(\Omega)$. It follows that there exists a positive constant C such that

$$|u|_s \leqslant C \cdot \|u\| \quad \forall\, u \in E.$$

Using the above inequality and relation 15 from Lemma 1 we find that for any $u \in E$ with $\|u\| < 1$, we have

$$\begin{aligned}
\frac{J_0(u)}{I_0(u)} &= \frac{\displaystyle\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx + \int_\Omega |u|^{q(x)}\,dx}{\displaystyle\int_\Omega g(x)|u|^{r(x)}\,dx} \\
&\geqslant \frac{\displaystyle\int_\Omega (|\nabla u|^{p(x)} + |u|^{p(x)})\,dx}{\left(\displaystyle\int_\Omega g(x)|u|^{r^-}\,dx + \int_\Omega g(x)|u|^{r^+}\,dx\right)} \\
&\geqslant \frac{\|u\|^{p^+}}{|g|_{s/(s-r^-)}|u|_s^{r^-} + |g|_{s/(s-r^+)}|u|_s^{r^+}} \\
&\geqslant \frac{\|u\|^{p^+}}{|g|_{s/(s-r^-)}\cdot C^{r^-}\cdot\|u\|^{r^-} + |g|_{s/(s-r^+)}\cdot C^{r^+}\cdot\|u\|^{r^+}}.
\end{aligned}$$

Since $r^+ \geqslant r^- > p^+$, passing to the limit in the above inequality we deduce that $\lim_{\|u\|\to 0} J_0(u)/I_0(u) = +\infty$, and thus Lemma 3 holds true. $\square$

LEMMA 4. *Assume that $\{u_n\}$ converges weakly to u in E. Then the following relations hold true:*

$$\lim_{n\to\infty} I_0(u_n) = I_0(u), \tag{20}$$

$$\lim_{n\to\infty} \langle I_1'(u_n), u_n - u\rangle = 0. \tag{21}$$

Proof. We shall prove just relation 20. Relation 21 can be obtained by using similar arguments.

Since $\{u_n\}$ converges weakly to u in E and since E is continuously embedded in $L^{p^\star(x)}(\Omega)$, it follows that the sequence $\{|u_n|_{p^\star(x)}\}$ is bounded. Using this fact we deduce that there exists a positive constant M such that

$$\max_n \left\{ \left||u_n|^{r^+}\right|_{p^\star(x)/\,r^+}, \left||u|^{r^+}\right|_{p^\star(x)/r^+}, \left||u_n|^{r^-}\right|_{p^\star(x)/r^-}, \left||u|^{r^-}\right|_{p^\star(x)/r^-} \right\} \leqslant M. \tag{22}$$

On the other hand, let k be a positive integer such that

$$\mathbb{R}^N \setminus \Omega \subset B_k(0),$$

where $B_k(0) = \{x \in \mathbb{R}^N; |x| < k\}$ is the ball centered in the origin with the radius k.

Let $\epsilon > 0$ be fixed. Since by relation 13 we have $g \in L^{p_0(x)}(\Omega)$, we can consider k defined above to be sufficiently large such that

$$|g|_{L^{p_0(x)}(\Omega\setminus B_k(0))} < \frac{\epsilon}{8M}. \tag{23}$$

Relations 3, 22 and 23 imply that

$$\begin{aligned}
&\int_{\Omega\setminus B_k(0)} g(x)||u_n|^{r(x)} - |u|^{r(x)}|\,dx \\
&\leqslant \int_{\Omega\setminus B_k(0)} g(x)(|u_n|^{r^+} + |u_n|^{r^-} + |u|^{r^+} + |u|^{r^-})\,dx \\
&\leqslant |g|_{L^{p_0(x)}(\Omega\setminus B_k(0))} \left(\left||u_n|^{r^+}\right|_{p^\star(x)/r^+}, \left||u|^{r^+}\right|_{p^\star(x)/r^+}, \left||u_n|^{r^-}\right|_{p^\star(x)/r^-}, \left||u|^{r^-}\right|_{p^\star(x)/r^-} \right) \\
&\leqslant \frac{\epsilon}{2}.
\end{aligned}$$

On the other hand, since $W_0^{1,p(x)}(B_k(0)\cap\Omega)$ is compactly embedded in $L^{r(x)}(B_k(0)\cap\Omega)$ and $g \in L^\infty(\Omega)$, we find that

$$\lim_{n\to\infty} \int_{B_k(0)\cap\Omega} g(x)|u_n|^{r(x)}\,dx = \int_{B_k(0)\cap\Omega} g(x)|u|^{r(x)}\,dx$$

or

$$\left| \int_{B_k(0)\cap\Omega} g(x)|u_n|^{r(x)}\,dx - \int_{B_k(0)\cap\Omega} g(x)|u|^{r(x)}\,dx \right| < \frac{\epsilon}{2},$$

for n large enough.

The above piece of information assures that relation 20 holds true. Thus the proof of Lemma 4 is complete. □

We return now to the proof of Step 4. Let $\lambda_n \searrow \lambda_1$. By Step 3 we deduce that for each n there exists a $u_n \in E \setminus \{0\}$ such that

$$\langle J_1'(u_n), v\rangle = \lambda_n \cdot \langle I_1'(u_n), v\rangle \quad \forall\, v \in E. \tag{24}$$

Taking $v = u_n$ we find that

$$J_0(u_n) = \lambda_n \cdot I_0(u_n). \tag{25}$$

Passing to the limit as $n \to \infty$ in relation 25 and we deduce that

$$\lim_{n\to\infty} (J_0(u_n) - \lambda_n I_0(u_n)) = 0\,.$$

The above equality and a similar argument as those used in the proof of Lemma 2 show that the sequence $\{u_n\}$ *is bounded in* E. Since E is a reflexive Banach space it follows that there exists a $u \in E$ such that, up to a subsequence, $\{u_n\}$ converges weakly to u in E. Then by relations 20 and 21 it follows that

$$\lim_{n\to\infty} I_0(u_n) = I_0(u)$$

and

$$\lim_{n\to\infty} \langle I_1^{'}(u_n), u_n - u\rangle = 0.$$

On the other hand, by [**21**, Lemma 4.2] we find that for any $\theta \geqslant 2$ and any ξ, $\eta \in \mathbb{R}^N$

$$\frac{2}{2^{\theta-1}-1}|\xi-\eta|^\theta \leqslant \theta(|\xi|^{\theta-2}\xi - |\eta|^{\theta-2}\eta)\cdot(\xi-\eta)\,. \tag{26}$$

Using inequality 26 and the above relations we deduce that there exist two positive constants L_1 and L_2 such that

$$\begin{aligned} L_1 &\int_\Omega (|\nabla(u_n-u)|^{p(x)} + |u_n-u|^{p(x)})\,dx \\ &\leqslant \int_\Omega (|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u)\cdot(\nabla u_n - \nabla u)\,dx \\ &\quad + \int_\Omega (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)\cdot(u_n-u)\,dx \end{aligned}$$

and

$$L_2\int_\Omega |u_n-u|^{q(x)}\,dx \leqslant \int_\Omega (|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u)\cdot(u_n-u)\,dx.$$

Adding the two relations above, using relations 21 and 24 and the fact that $\{u_n\}$ converges weakly to u in E we deduce that

$$\begin{aligned} L_1\int_\Omega (|\nabla(u_n-u)|^{p(x)} + |u_n-u|^{p(x)})\,dx &\leqslant \langle J_1^{'}(u_n) - J_1^{'}(u), u_n-u\rangle \\ &= |\langle J_1^{'}(u_n), u_n-u\rangle| + |\langle J_1^{'}(u), u_n-u\rangle| \\ &= |\lambda_n\cdot\langle I_1^{'}(u_n), u_n-u\rangle| + |\langle J_1^{'}(u), u_n-u\rangle| \to 0, \end{aligned}$$

as $n \to \infty$.

The above inequalities and relations 7 and 10 show that u_n converges strongly to u in E. Then passing to the limit as $n \to \infty$ in 24, it follows that

$$\langle J_1^{'}(u), v\rangle = \lambda_1 \cdot \langle I_1^{'}(u), v\rangle \quad \forall\, v \in E.$$

Thus u is a critical point for T_{λ_1}. In order to prove that λ_1 is an eigenvalue for problem 2 it remains to show that $u \neq 0$. Indeed, passing to the limit as $n \to \infty$ in 25 we find that

$$\lim_{n\to\infty}\frac{J_0(u_n)}{I_0(u_n)} = \lambda_1.$$

On the other hand, if we assume by contradiction that $u = 0$ then we have $u_n \to 0$ in E or $\|u_n\| \to 0$. However, by Lemma 3 we deduce that

$$\lim_{n\to\infty}\frac{J_0(u_n)}{I_0(u_n)} = \infty,$$

which represents a contradiction. Consequently, $u \neq 0$ and thus λ_1 is an eigenvalue for problem 2.

By Steps 2–4 we deduce that $\lambda_0 \leqslant \lambda_1$. The proof of Theorem 1 is now complete. □

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