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Non-autonomous double phase eigenvalue problems with indefinite weight and lack of compactness

Tianxiang Gou[1] | **Vicenţiu D. Rădulescu**[2,3,4,5,6]

[1]School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, Shaanxi, China

[2]Faculty of Applied Mathematics, AGH University of Science and Technology, Krakow, Poland

[3]Faculty of Electrical Engineering and Communication, Brno University of Technology, Brno, Czech Republic

[4]Department of Mathematics, University of Craiova, Craiova, Romania

[5]Simion Stoilow Institute of Mathematics of the Romanian Academy, Bucharest, Romania

[6]School of Mathematics, Zhejiang Normal University, Jinhua, China

Correspondence
Vicenţiu D. Rădulescu, Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Krakow, Poland.
Email: vicentiu.radulescu@imar.ro

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Abstract

In this paper, we consider eigenvalues to the following double phase problem with unbalanced growth and indefinite weight,

$$-\Delta_p^a u - \Delta_q u = \lambda m(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N,$$

where $N \geqslant 2$, $1 < p, q < N$, $p \neq q$, $a \in C^{0,1}(\mathbb{R}^N, [0, +\infty))$, $a \not\equiv 0$ and $m : \mathbb{R}^N \to \mathbb{R}$ is an indefinite sign weight which may admit non-trivial positive and negative parts. Here, Δ_q is the q-Laplacian operator and Δ_p^a is the weighted p-Laplace operator defined by $\Delta_p^a u := \mathrm{div}(a(x)|\nabla u|^{p-2}\nabla u)$. The problem can be degenerate, in the sense that the infimum of a in $\mathbb{R}^N$ may be zero. Our main results distinguish between the cases $p < q$ and $q < p$. In the first case, we establish the existence of a *continuous* family of eigenvalues, starting from the principal frequency of a suitable single phase eigenvalue problem. In the latter case, we prove the existence of a *discrete* family of positive eigenvalues, which diverges to infinity.

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1 | INTRODUCTION

In this paper, we investigate eigenvalues to the following double phase problem with unbalanced growth and indefinite weight,

$$-\Delta_p^a u - \Delta_q u = \lambda m(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $N \geqslant 2, 1 < p, q < N, p \neq q, a \in C^{0,1}(\mathbb{R}^N, [0, +\infty)), a \not\equiv 0$ and $m : \mathbb{R}^N \to \mathbb{R}$ is an indefinite sign weight which may admit non-trivial positive and negative parts. Here, Δ_q is the q-Laplacian operator and Δ_p^a is the weighted p-Laplace operator defined by $\Delta_p^a u := \operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u)$. Throughout of this paper, we shall always assume that the weight function $m : \mathbb{R}^N \to \mathbb{R}$ satisfies the following assumption,

(H) $m = m_1 - m_2$, where $m_1, m_2 \geqslant 0$, $m_1 \not\equiv 0$, $m_1 \in L^{\frac{N}{q}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $m_2 \in L^\infty(\mathbb{R}^N)$.

Remark 1.1. In our case, $m_2 = 0$ is allowable.

Problems like (1.1) arise when one looks for the stationary solutions of reaction–diffusion systems of the form

$$u_t = \operatorname{div}[D(x, \nabla u)\nabla u] + g(x, u) \quad (x, t) \in \mathbb{R}^N \times (0, \infty),$$

where $D(x, \nabla u) = a(x)|\nabla u|^{p-2} + |\nabla u|^{q-2}$. This system has a wide range of applications in physics and related fields, such as biophysics, plasma physics and chemical reaction design (see [7, 26]). In such applications, the function u is a state variable and describes density or concentration of multi-component substances, $\operatorname{div}[D(x, \nabla u)\nabla u]$ corresponds to the diffusion with a diffusion coefficient $D(x, \nabla u)$ and $g(x, u)$ is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $g(x, u)$ has a polynomial form with respect to the unknown concentration denoted by u.

The analysis of the double phase eigenvalue problem (1.1) is closely associated with the following single phase quasilinear eigenvalue problem,

$$-\Delta_r^a u = \mu m(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The first part of the paper is devoted to the study of (1.2). The main results we establish regarding (1.2) are upcoming Theorem 3.1 and Proposition 3.1, which reveal that there exist a sequence of eigenvalues to (1.2) and the first eigenvalue is simple. In the case of bounded domains and $r = 2$, this problem is related to the Riesz–Fredholm theory of self-adjoint and compact operators. The anisotropic linear case (if $r = 2$ and $m(\cdot)$ is non-constant) was first considered in the pioneering papers of Bocher [6], Hess and Kato [17] and Pleijel [25]. An important contribution in the case of unbounded domains is due to Allegretto and Huang [1] and Szulkin and Willem [27]. In [27], the authors assumed that weight function may have singular points.

Equation (1.1) contains the contribution of two differential operators in the left-hand side, so this problem is not homogeneous. In fact, the differential operator $u \mapsto -\Delta_p^a u - \Delta_q u$ is related to the ‘double-phase variational functional defined by

$$u \mapsto \int_{\mathbb{R}^N} a(x)|\nabla u|^p + |\nabla u|^q \, dx.$$

The integrand of this functional is the function

$$\xi(x,t) = a(x)t^p + t^q \quad \text{for all } x \in \mathbb{R}^N \text{and } t \geqslant 0.$$

When $a \equiv 1$, then (1.1) becomes the so-called p & q Laplacian problem, which was investigated by Benouhiba and Belyacine [4, 5]. A feature of this paper is that we do not assume that the function $a(\cdot)$ is bounded away from zero, that is, we do not require that $\operatorname{essinf}_{x\in\mathbb{R}^N} a(x) > 0$. This implies that the integrand $\xi(x,t)$ exhibits unbalanced growth, namely there holds that

$$t^q \leqslant \xi(x,t) \leqslant C_0(t^p + t^q) \text{ for all } x \in \mathbb{R}^N \text{ and } t \geqslant 0, \tag{1.3}$$

where $C_0 > 0$ is a constant. In this scenario, the study is carried out in the framework of Musielak–Orlicz–Sobolev spaces. Such functionals were first investigated by Marcellini [18–20] in the context of problems of the calculus of variations and of non-linear elasticity for strongly anisotropic materials. For such problems, there is no global (that is, up to the boundary) regularity theory. There are only interior regularity results, which are primarily due to Baroni et al. [3] and Marcellini [10, 20, 21]. In fact, most of works dealt with double phase problems having unbalanced growth in bounded domains of $\mathbb{R}^N$, we refer the readers to [12–15, 22–24] and references therein. However, there exist relatively few ones treating the problems in $\mathbb{R}^N$. The study of eigenvalue problems like (1.1) is open until now. Since (1.1) is set in the whole space $\mathbb{R}^N$, lack of compactness is one of major difficulties we encounter to discuss the eigenvalue problem (1.1) in Musielak–Orlicz–Sobolev spaces and more careful analysis is needed in suitable weighted functions spaces. Indeed, this is mainly because the embedding $W^{1,\xi}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is only continuous for any $q \leqslant r \leqslant q^*$ (see Lemma 2.3) and the weight function $m : \mathbb{R}^N \to \mathbb{R}$ is indefinite, which cause that the verification of the compactness of the underlying (minimizing and Palasi–Smale) sequences becomes difficult. Consequently, we manage to study the problem (1.1) in a new weighted Sobolev space E defined by the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_E := \|\nabla u\|_\xi + \left(\int_{\mathbb{R}^N} |u|^q \max\{m_2, \omega\} \, dx\right)^{\frac{1}{q}}, \quad \omega(x) := \frac{1}{(1+|x|)^q}, \quad x \in \mathbb{R}^N,$$

where $\|\cdot\|_\xi$ denotes the standard norm in $D^{1,\xi}(\mathbb{R}^N)$. Here, $W^{1,\xi}(\mathbb{R}^N)$ and $D^{1,\xi}(\mathbb{R}^N)$ are Musielak–Orlicz–Sobolev spaces defined in Section 2. In this paper, when $p < q$, we establish the existence of a continuous family of eigenvalues to (1.1), starting from the principal frequency to (1.2), see Theorems 3.2 and 3.3. While $q < p$, we prove the existence of a discrete family of positive eigenvalues to (1.1), which diverges to infinity, see Theorem 3.4 and Proposition 3.2. The results we derive reveal new facts of eigenvalues to double phase problems in $\mathbb{R}^N$. In both cases, we actually need to assume $q < q^* := \frac{Nq}{N-q}$, because of the unbalanced growth property (1.3) with respect to the double phase operator and the dominance is the q-Laplacian term. Thus, the problem under consideration is Sobolev subcritical and the energy functional J corresponding to (1.1) is well-defined in the Sobolev space E by Theorem 2.3, where

$$J(u) := \frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx.$$

Observe that $\frac{p}{q} < 1 + \frac{1}{N}$ implies $p < q^*$. When double phase problems are set in bounded domains in $\mathbb{R}^N$, then the condition $\frac{p}{q} < 1 + \frac{1}{N}$ can be applied to prove the desired compact embedding results, for example [22, Proposition 4]. While double phase problems are set in $\mathbb{R}^N$, the condition $\frac{p}{q} < 1 + \frac{1}{N}$ can no longer be applicable to derive the compact embedding results, which leads to lack of compactness for the study. In this paper, such a condition is actually used to guarantee the regularity of solutions to (1.1) (see [8, 9]), which along with the maximum principle developed in [23, 24] can lead to the simplicity of eigenvalues, see Proposition 3.2.

2 | PRELIMINARIES

In the section, we are going to present some preliminary results used to establish our main theorems. To deal with the eigenvalue problem (1.1), we shall work in the corresponding Musielak–Orlicz–Sobolev space. For the convenience of the readers, let us first present a few definitions from [11, Section 2] concerning the main notions and function spaces used in this paper.

Definition 2.1. A function $\varphi : [0, +\infty] \to [0, +\infty)$ is called a Φ-function if φ is convex and left-continuous on $[0, +\infty)$. In addition, φ satisfies that

$$\varphi(0) = 0, \quad \lim_{t\to 0^+} \varphi(t) = 0, \quad \lim_{t\to +\infty} \varphi(t) = +\infty.$$

Definition 2.2. A function $\xi : \mathbb{R}^N \times [0, +\infty] \to [0, +\infty)$ is called a generalized Φ-function if it satisfies the following conditions:

(i) for almost every $x \in \mathbb{R}^N$, $\xi(x, \cdot)$ is a Φ-function;
(ii) for almost every $t \in [0, +\infty)$, $\xi(\cdot, t)$ is measurable.

Definition 2.3. A generalized Φ-function $\xi : \mathbb{R}^N \times [0, +\infty] \to [0, +\infty)$ satisfies Δ_2-condition if there exists $K \geqslant 2$ such that, for almost every $x \in \mathbb{R}^N$ and $t \geqslant 0$,

$$\xi(x, 2t) \geqslant K\xi(x, t).$$

Definition 2.4. A Φ-function $\varphi : [0, +\infty] \to [0, +\infty)$ is said to be an N-function if it is continuous and positive on $[0, +\infty)$. In addition, it satisfies that

$$\lim_{t\to 0^+} \frac{\varphi(t)}{t} = 0, \quad \lim_{t\to +\infty} \frac{\varphi(t)}{t} = +\infty.$$

A generalized Φ-function $\xi : \mathbb{R}^N \times [0, +\infty] \to [0, +\infty)$ is said to be a generalized N-function if, for almost every $x \in \mathbb{R}^N$, $\xi(x, \cdot)$ is an N-function.

Definition 2.5. A generalized N-function $\xi : \mathbb{R}^N \times [0, +\infty] \to [0, +\infty)$ is called uniformly convex if, for any $\epsilon > 0$, there exists $\delta > 0$ such that, for almost every $x \in \mathbb{R}^N$,

$$\xi\left(x, \frac{s+t}{2}\right) \leqslant (1-\delta)\frac{\xi(x,s)+\xi(x,t)}{2},$$

whenever $s, t \geqslant 0$ and $|x - t| \geqslant \epsilon \max\{|s|, |t|\}$.

With these definitions in hand, we are now ready to introduce the double phase function $\xi : \mathbb{R}^N \times [0, +\infty) \to [0, +\infty)$ corresponding to (1.1) as

$$\xi(x,t) := a(x)t^p + t^q, \quad x \in \mathbb{R}^N,\ t \geqslant 0. \tag{2.1}$$

It is simple to check that ξ is a generalized N-function. Moreover, ξ is uniformly convex and it satisfies the Δ_2-condition. Let us denote by $M(\mathbb{R}^N)$ the space consisting of all Lebesgue measurable function $u : \mathbb{R}^N \to \mathbb{R}$. The Musielak–Orlicz space $L^\xi(\mathbb{R}^N)$ is defined by

$$L^\xi(\mathbb{R}^N) := \left\{u \in M(\mathbb{R}^N) : \rho_\xi(u) < +\infty\right\},$$

where ρ_ξ is the modular function given by

$$\rho_\xi(u) := \int_{\mathbb{R}^N} \xi(x, |u|)\, dx = \int_{\mathbb{R}^N} a(x)|u|^p + |u|^q\, dx. \tag{2.2}$$

Here, the space $L^\xi(\mathbb{R}^N)$ is equipped with the Luxemburg norm given by

$$\|u\|_\xi := \inf\left\{\lambda > 0 : \rho_\xi\left(\frac{u}{\lambda}\right) \leqslant 1\right\}. \tag{2.3}$$

Using the above properties satisfied by ξ, we can easily check that $L^\xi(\mathbb{R}^N)$ is a Banach space, which is also separable and reflexive. The Musielak–Orlicz–Sobolev space $W^{1,\xi}(\mathbb{R}^N)$ is defined by

$$W^{1,\xi}(\mathbb{R}^N) := \left\{u \in L^\xi(\mathbb{R}^N) : |\nabla u| \in L^\xi(\mathbb{R}^N)\right\}.$$

Here, the space $W^{1,\xi}(\mathbb{R}^N)$ is equipped with the norm

$$\|u\|_{1,\xi} := \|u\|_\xi + \|\nabla u\|_\xi,$$

where $\|\nabla u\|_\xi := \||\nabla u|\|_\xi$. Clearly, $W^{1,\xi}(\mathbb{R}^N)$ is a separable, reflexive Banach space. Let us introduce the associated homogeneous Musielak–Orlicz–Sobolev $D^{1,\xi}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $\|\nabla u\|_\xi$.

Next, we are going to show some relations between the norm in $L^\xi(\mathbb{R}^N)$ and the modular function ρ_ξ given by (2.2) and (2.3), respectively, proofs of which can be completed by using the ingredients presented in [16, Section 3.2].

Lemma 2.1. *Let $\xi : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$ be defined by (2.1). Then, the following assertions hold.*

(i) $\|u\|_\xi = \lambda$ *if and only if* $\rho_\xi(\frac{u}{\lambda}) = 1$.
(ii) $\|u\|_\xi < 1 (= 1, > 1, respectively)$ *if and only if* $\rho_\xi(u) < 1 (= 1, > 1, respectively)$.
(iii) *If* $\|u\|_\xi < 1$, *then* $\|u\|_\xi^{\max\{p,q\}} \leqslant \rho_\xi(u) \leqslant \|u\|_\xi^{\min\{p,q\}}$.
(iv) *If* $\|u\|_\xi > 1$, *then* $\|u\|_\xi^{\min\{p,q\}} \leqslant \rho_\xi(u) \leqslant \|u\|_\xi^{\max\{p,q\}}$.
(v) $\lim_{n\to+\infty} \|u_n\|_\xi = 0(+\infty, respectively)$ *if and only if* $\lim_{n\to+\infty} \rho_\xi(u_n) = 0(+\infty, respectively)$.

Note that $t^q \leqslant \xi(x,t)$ for any $x \in \mathbb{R}^N$ and $t \in \mathbb{R}$, by assertion (ii) of Lemma 2.1, then there holds the following embedding result.

Lemma 2.2. *Let $\xi : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$ be defined by (2.1). Then, the embedding $L^\xi(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous.*

As a consequence of Lemma 2.2 and Sobolev's embeddings in $W^{1,q}(\mathbb{R}^N)$ and $D^{1,q}(\mathbb{R}^N)$ for $1 < q < N$, we have the following embedding result.

Lemma 2.3. *Let $\xi : \mathbb{R}^N \times [0,+\infty) \to [0,+\infty)$ be defined by (2.1). Then, the embedding $W^{1,\xi}(\mathbb{R}^N) \hookrightarrow W^{1,q}(\mathbb{R}^N) \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for any $q \leqslant r \leqslant q^*$. Moreover, the embedding $D^{1,\xi}(\mathbb{R}^N) \hookrightarrow D^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$ is continuous.*

3 | MAIN RESULTS

In this section, we shall consider the eigenvalue problem (1.1) under the assumption (H). The hypothesis (H) is always assumed to hold in what follows. First, we shall present some results related to the following eigenvalue problem,

$$-\Delta_r^a u = \mu m(x)|u|^{r-2}u \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Theorem 3.1. *Assume (H) holds, $N \geqslant 2$, $1 < r < N$, $a \in C^{0,1}(\mathbb{R}^N, [0,+\infty))$ and $a \not\equiv 0$. Then, there exists a sequence of solutions $(\mu_{a,r,k}, u_{a,r,k}) \in \mathbb{R} \times D^{1,\eta}(\mathbb{R}^N)$ to (3.1) with $u_{a,r,k} \in \mathcal{M}$ and*

$$0 < \mu_{a,r,1} < \mu_{a,r,2} \leqslant \cdots \leqslant \mu_{a,r,k} \leqslant \cdots, \quad \lim_{k\to\infty} \mu_{a,r,k} \to +\infty \quad as\ k \to +\infty,$$

where $\eta(x,t) = a(x)t^r$ for $x \in \mathbb{R}^N$ and $t \geqslant 0$,

$$\mathcal{M}_r := \left\{ u \in D^{1,\eta}(\mathbb{R}^N) : \int_{\mathbb{R}^N} m(x)|u|^r \, dx = 1 \right\}.$$

Proof. Define

$$\Psi(u) := \int_{\mathbb{R}^N} a(x)|\nabla u|^r \, dx, \quad M_r := \mathcal{M}_r \cap \mathcal{V},$$

where the Sobolev space $\mathcal{V}$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|\nabla u\|_\eta + \left(\int_{\mathbb{R}^N} |u|^r \max\{m_2, \omega\}\, dx\right)^{\frac{1}{r}}, \quad \omega(x) = \frac{1}{(1+|x|)^r}, \quad x \in \mathbb{R}^N.$$

Reasoning as the proof of [1, Lemma 1], we are able to show $\Psi(u)$ restricted on M_r satisfies the Palais–Smale condition. Then, by adapting Ljusternik–Schnirelman theory as the proof of forthcoming Theorem 3.4, we can derive the desired conclusion. Thus, the proof is completed. □

Proposition 3.1. *Assume* (H) *holds,* $N \geqslant 2$, $1 < r < N$, $a \in C^{0,1}(\mathbb{R}^N, [0,+\infty))$ *and* $a \not\equiv 0$. *Then, the first eigenvalue* $\mu_{a,r,1}$ *obtained in Theorem 3.1 is simple and the eigenfunction* $u_{a,r,1}$ *has constant sign. Moreover, if* $u \in D^{1,\eta}(\mathbb{R}^N)$ *is a non-trivial solution to (3.1) corresponding to* $\mu > \mu_{a,r,1}$, *then* u *is sign-changing.*

Since the function m is an indefinite sign weight, then proof of Proposition 3.1 is not straightforward. To prove this, we need the following auxiliary result.

Lemma 3.1. *Define*

$$I(u,v) := -\int_{\mathbb{R}^N} (\Delta_r^a u)\frac{u^r - v^r}{u^{r-1}}\, dx - \int_{\mathbb{R}^N} (\Delta_r^a v)\frac{v^r - u^r}{v^{r-1}}\, dx, \quad u, v \in D^{1,\eta}(\mathbb{R}^N), u, v > 0.$$

Then, $I(u,v) \geqslant 0$. *Moreover,* $I(u,v) = 0$ *if and only if* $u = kv$ *for some* $k \in \mathbb{R}$.

Proof. Observe that

$$\nabla\left(\frac{u^r - v^r}{u^{r-1}}\right) = \left(1 + (r-1)\left(\frac{v}{u}\right)^r\right)\nabla u - r\left(\frac{v}{u}\right)^{r-1}\nabla v,$$
$$\nabla\left(\frac{v^r - u^r}{v^{r-1}}\right) = \left(1 + (r-1)\left(\frac{u}{v}\right)^r\right)\nabla v - r\left(\frac{u}{v}\right)^{r-1}\nabla u.$$

Then, by the divergence theorem, we see

$$\begin{aligned} I(u,v) = &\int_{\mathbb{R}^N} a(x)\left(\left(1 + (r-1)\left(\frac{v}{u}\right)^r\right)|\nabla u|^r - r\left(\frac{v}{u}\right)^{r-1}|\nabla u|^{r-2}(\nabla v\cdot\nabla u)\right) dx \\ &+ \int_{\mathbb{R}^N} a(x)\left(\left(1 + (r-1)\left(\frac{u}{v}\right)^r\right)|\nabla v|^r - r\left(\frac{u}{v}\right)^{r-1}|\nabla v|^{r-2}(\nabla u\cdot\nabla v)\right) dx. \end{aligned} \tag{3.2}$$

Using Young's inequality, we have

$$r\left(\frac{v}{u}\right)^{r-1}|\nabla u|^{r-2}(\nabla v\cdot\nabla u) \leqslant r\left(\frac{v}{u}\right)^{r-1}|\nabla u|^{r-1}|\nabla v| \leqslant (r-1)\left(\frac{v}{u}\right)^r|\nabla u|^r + |\nabla v|^r,$$
$$r\left(\frac{u}{v}\right)^{r-1}|\nabla v|^{r-2}(\nabla u\cdot\nabla v) \leqslant r\left(\frac{u}{v}\right)^{r-1}|\nabla v|^{r-1}|\nabla u| \leqslant (r-1)\left(\frac{u}{v}\right)^r|\nabla v|^r + |\nabla u|^r.$$

As a consequence, coming back to (3.2), we can conclude $I(u,v) \geqslant 0$. If $I(u,v) = 0$, then

$$\nabla u\cdot\nabla v = |\nabla u||\nabla v|, \quad \left(\frac{v}{u}\right)^r|\nabla u|^r = |\nabla v|^r, \quad \left(\frac{u}{v}\right)^r|\nabla v|^r = |\nabla u|^r.$$

It then follows that

$$|u\nabla v - v\nabla u| = 0.$$

This implies that there exists $k \in \mathbb{R}$ such that $u = kv$ and the proof is completed. □

Proof of Proposition 3.1. Note first that

$$\mu_{a,r,1} = \inf_{u\in\mathcal{M}_r} \Psi(u).$$

If $u \in \mathcal{M}_r$ satisfies $\Psi(u) = u_{a,r,1}$, then $|u| \in \mathcal{M}_r$ and $\Psi(|u|) = u_{a,r,1}$. Therefore, without restriction, we may assume $u_{a,r,1}$ is non-negative. Observe that $u_{a,r,1} \in D^{1,\eta}(\mathbb{R}^N)$ satisfies the equation

$$-\Delta_r^a u_{a,r,1} + \mu_{a,r,1} m_2(x)|u_{a,r,1}|^{r-2}u_{a,r,1} = \mu_{a,r,1} m_1(x)|u_{a,r,1}|^{r-2}u_{a,r,1} \geqslant 0 \quad \text{in } \mathbb{R}^N.$$

By maximum principle, $u_{a,r,1} > 0$. Let $u_{a,r,1} \in \mathcal{M}_r$ and $v_{a,r,1} \in \mathcal{M}_r$ be two positive eigenfunctions corresponding to $\mu_{a,r,1}$, then

$$-\Delta_r^a u_{a,r,1} = \mu_{a,r,1} m(x) u_{a,r,1}^{r-1}, \quad -\Delta_r^a v_{a,r,1} = \mu_{a,r,1} m(x) v_{a,r,1}^{r-1} \quad \text{in } \mathbb{R}^N.$$

It is simple to calculate $I(u_{a,r,1}, v_{a,r,1}) = 0$. As a result of Lemma 3.1, we have $u_{a,r,1} = kv_{a,r,1}$ for some $k \in \mathbb{R}$. This indicates that $\mu_{a,r,1}$ is simple.

Arguing by contradiction, we suppose $u \in D^{1,\eta}(\mathbb{R}^N)$ is a non-negative solution to (3.1) corresponding to $\mu > \mu_{a,r,1}$. By the maximum principle, $u > 0$. Notice

$$\int_{\mathbb{R}^N} a(x)|\nabla u|^r\,dx = \mu\int_{\mathbb{R}^N} m(x)|u|^r\,dx > 0.$$

In addition, we know that if $u \in D^{1,\eta}(\mathbb{R}^N)$ is a solution to (3.1), then $ku \in D^{1,\eta}(\mathbb{R}^N)$ is also a solution to (3.1) for any $k \in \mathbb{R}\backslash\{0\}$. Then, by scaling, we may assume

$$0 < \int_{\mathbb{R}^N} m(x)|u|^r\,dx < 1. \tag{3.3}$$

Let $u_{a,r,1} \in \mathcal{M}$ and $u_{a,r,1} > 0$ be an eigenfunction to (3.1) corresponding to $\mu_{a,r,1}$. Then, $u_{a,r,1}$ solves the equation

$$-\Delta_r^a u_{a,r,1} = \mu_{a,r,1} m(x)|u_{a,r,1}|^{r-2}u_{a,r,1} \quad \text{in } \mathbb{R}^N.$$

As a consequence of Lemma 3.1 and (3.3), we have

$$\begin{aligned} 0 \leqslant I(u, u_{a,r,1}) &= \mu\int_{\mathbb{R}^N} m(x)\left(u^r - u_{a,r,1}^r\right)dx + \mu_{a,r,1}\int_{\mathbb{R}^N} m(x)\left(u_{a,r,1}^r - u^r\right)dx \\ &= (\mu - \mu_{a,r,1})\int_{\mathbb{R}^N} m(x)|u|^r\,dx - (\mu - \mu_{a,r,1}) < 0. \end{aligned}$$

This is impossible, hence u is sign-changing and the proof is completed. □

Theorem 3.2. *Assume* (H) *holds,* $N \geqslant 2$, $1 < p, q < N$, $p \neq q$, $a \in C^{0,1}(\mathbb{R}^N, [0, +\infty))$ *and* $a \not\equiv 0$. *Then, (1.1) has no non-trivial solutions in* $D^{1,\xi}(\mathbb{R}^N)$ *for any* $0 \leqslant \lambda \leqslant \mu_{1,q,1}$, *where* $\mu_{1,q,1} > 0$ *is the first eigenvalue to (3.1) with* $a \equiv 1$ *and* $r = q$.

Proof. Let $u \in D^{1,\xi}(\mathbb{R}^N)$ be a solution to (1.1) for some $0 \leqslant \lambda \leqslant \mu_{1,q,1}$. Observe first that

$$\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx + \int_{\mathbb{R}^N} |\nabla u|^q\,dx = \lambda \int_{\mathbb{R}^N} m(x)|u|^q\,dx. \tag{3.4}$$

This implies $u = 0$ if $\lambda = 0$. Let us assume $0 < \lambda < \mu_{1,q,1}$. Assume $u \neq 0$, it then follows from (3.4) that

$$\int_{\mathbb{R}^N} m(x)|u|^q\,dx > 0. \tag{3.5}$$

In addition, since $\mu_{1,q,1} > 0$ is the first eigenvalue to (3.1), then

$$\int_{\mathbb{R}^N} |\nabla u|^q\,dx \geqslant \mu_{1,q,1} \int_{\mathbb{R}^N} m(x)|u|^q\,dx. \tag{3.6}$$

This along with (3.4) leads to

$$\mu_{1,q,1} \int_{\mathbb{R}^N} m(x)|u|^q\,dx \leqslant \lambda \int_{\mathbb{R}^N} m(x)|u|^q\,dx.$$

Using (3.5), we then get $u = 0$. This is a contradiction. Next we assume $\lambda = \mu_1$. In this case, by combining (3.4) and (3.6), we obtain

$$\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx \leqslant 0,$$

hence $u = 0$. Thus, the proof is completed. □

3.1 | Case $p < q$

In this case, to establish the existence of solutions to (1.1), we shall adapt some ideas from [1]. Let us first introduce the weight function

$$\omega(x) = \frac{1}{(1+|x|)^q}, \quad x \in \mathbb{R}^N.$$

Let E be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_E := \|\nabla u\|_\xi + \left(\int_{\mathbb{R}^N} |u|^q \max\{m_2, \omega\}\,dx\right)^{\frac{1}{q}}.$$

It is standard to conclude that E is a separable and reflexive Banach space. In order to prove the existence of solutions to (1.1), we shall define the associated energy functional $J : E \to \mathbb{R}$ by

$$J(u) := \frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx - \frac{\lambda}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx.$$

Theorem 3.3. *Assume* (H) *holds,* $N \geqslant 2$, $1 < p < q < N$, $a \in C^{0,1}(\mathbb{R}^N, [0,+\infty))$ *and* $a \not\equiv 0$. *Then, there exist positive solutions to (1.1) for any* $\lambda > \mu_{1,q,1}$.

In this case, we find that J is unbounded from below in E. Indeed, let $u_{1,q,1} \in D^{1,q}(\mathbb{R}^N)$ be an eigenfunction of (3.1) corresponding to $\mu_{1,q,1}$. We observe

$$J(tu_1) = \frac{t^p}{p} \int_{\mathbb{R}^N} a(x)|\nabla u_{1,q,1}|^p\, dx + \frac{t^q}{q}\left(1 - \frac{\lambda}{\mu_{1,q,1}}\right) \int_{\mathbb{R}^N} |\nabla u_{1,q,1}|^q\, dx.$$

Since $p < q$ and $\lambda > \mu_1$, $J(tu_{1,q,1}) \to -\infty$ as $t \to +\infty$. In this situation, to seek solutions to (1.1), we introduce the Nehari manifold

$$\mathcal{N} := \{u \in E\backslash\{0\} : I(u) = 0\},$$

where

$$I(u) := \int_{\mathbb{R}^N} a(x)|\nabla u|^p\, dx + \int_{\mathbb{R}^N} |\nabla u|^q\, dx - \lambda \int_{\mathbb{R}^N} m(x)|u|^q\, dx.$$

Then, we are able to define the minimization problem

$$m := \inf_{u \in \mathcal{N}} J(u). \tag{3.7}$$

Obviously, any minimizer of (3.7) is a solution to (1.1).

Proof of Theorem 3.3. Let $\{u_n\} \subset \mathcal{N}$ be a minimizing sequence to (3.7). Then, $m = J(u_n) + o_n(1)$ and $I(u_n) = 0$. Since $I(|u|) \leqslant I(u) = 0$ for any $u \in \mathcal{N}$, then there exists a unique $0 < t_{|u|} \leqslant 1$ such that $I(t_{|u|}|u|) = 0$, where

$$t_{|u|} = \left(\frac{\int_{\mathbb{R}^N} a(x)|\nabla |u||^p\, dx}{\lambda \int_{\mathbb{R}^N} m(x)|u|^q\, dx - \int_{\mathbb{R}^N} |\nabla |u||^q\, dx}\right)^{\frac{1}{q-p}}.$$

Moreover, for any $u \in \mathcal{N}$, we see

$$J(u) = J(u) - \frac{1}{q} I(u) = \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} a(x)|\nabla u|^p\, dx. \tag{3.8}$$

Therefore, for any $u \in \mathcal{N}$,

$$J(t_{|u|}|u|) = t_{|u|}^p \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} a(x)|\nabla |u||^p\, dx \leqslant \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} a(x)|\nabla u|^p\, dx = J(u).$$

As a consequence, we shall assume $\{u_n\} \subset \mathcal{N}$ is a non-negative minimizing sequence to (3.7). Otherwise, we can replace $\{u_n\}$ by $\{t_{|u_n|}|u_n|\}$ as a new minimizing sequence to (3.7).

First we are going to prove $m > 0$. It follows from (3.8) that $m \geqslant 0$. Let us argue by contradiction that $m = 0$. Then, by (3.8), we have

$$\int_{\mathbb{R}^N} a(x)|\nabla u_n|^p\, dx = o_n(1).$$

Let us first assume

$$\int_{\mathbb{R}^N} m(x)|u_n|^q\, dx = o_n(1). \tag{3.9}$$

Since $I(u_n) = 0$, there holds

$$\int_{\mathbb{R}^N} |\nabla u_n|^q \, dx = o_n(1).$$

In this case, we set

$$v_n := \frac{u_n}{\left(\int_{\mathbb{R}^N} m(x)|u_n|^q \, dx\right)^{\frac{1}{q}}} \geqslant 0, \quad \forall \, n \in \mathbb{N}^+. \tag{3.10}$$

It is easy to see $\{v_n\} \subset \mathcal{M}_q$. Since $I(u_n) = 0$,

$$\int_{\mathbb{R}^N} a(x)|\nabla v_n|^p \, dx = \frac{\int_{\mathbb{R}^N} a(x)|\nabla u_n|^p \, dx}{\left(\int_{\mathbb{R}^N} m(x)|u_n|^q \, dx\right)^{\frac{p}{q}}} = \frac{1}{\left(\int_{\mathbb{R}^N} m(x)|u_n|^q \, dx\right)^{\frac{p}{q}-1}} \left(\lambda - \int_{\mathbb{R}^N} |\nabla v_n|^q \, dx\right). \tag{3.11}$$

In view of (3.9) and (3.11),

$$\int_{\mathbb{R}^N} a(x)|\nabla v_n|^p \, dx = o_n(1).$$

It then yields that

$$\int_{\mathbb{R}^N} m(x)|v_n|^p \, dx \leqslant \frac{1}{\mu_{a,p,1}} \int_{\mathbb{R}^N} a(x)|\nabla v_n|^p \, dx = o_n(1). \tag{3.12}$$

Invoking Hölder's inequality, Sobolev's inequality and (3.12), we then get

$$\begin{aligned}
\int_{\mathbb{R}^N} m(x)|v_n|^q \, dx &\leqslant \left(\int_{\mathbb{R}^N} m(x)|v_n|^p \, dx\right)^{\theta} \left(\int_{\mathbb{R}^N} m(x)|v_n|^{q^*} \, dx\right)^{1-\theta} \\
&\leqslant \|m\|_\infty^{1-\theta} \left(\int_{\mathbb{R}^N} m(x)|v_n|^p \, dx\right)^{\theta} \left(\int_{\mathbb{R}^N} |v_n|^{q^*} \, dx\right)^{1-\theta} \\
&\leqslant C\|m\|_\infty^{1-\theta} \left(\int_{\mathbb{R}^N} m(x)|v_n|^p \, dx\right)^{\theta} \left(\int_{\mathbb{R}^N} |\nabla v_n|^q \, dx\right)^{\frac{q^*(1-\theta)}{q}} = o_n(1),
\end{aligned}$$

where $0 < \theta < 1$ and $q = \theta p + (1-\theta)q^*$. This is a contradiction, because of $v_n \in \mathcal{M}_q$. Let us next assume that there exists some $\lambda_0 > 0$ such that

$$\int_{\mathbb{R}^N} m(x)|u_n|^q \, dx = \lambda_0 + o_n(1).$$

Since $I(u_n) = 0$,

$$\int_{\mathbb{R}^N} |\nabla v_n|^q \, dx = \lambda - \frac{\int_{\mathbb{R}^N} a(x)|\nabla u_n|^p \, dx}{\int_{\mathbb{R}^N} m(x)|u_n|^q \, dx} = \lambda + o_n(1).$$

Therefore, there holds $\|\nabla v_n\|_p^p = \lambda + o_n(1)$. In virtue of [1, Lemma 1], we then get that $\{v_n\}$ is compact in V, where the Sobolev space V is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|\nabla u\|_q + \left(\int_{\mathbb{R}^N} |u|^q \max\{m_2, \omega\}\, dx\right)^{\frac{1}{q}}.$$

Let $v \in V$ be such that $v_n \to v$ in V as $n \to \infty$, then $v \neq 0$ and $v \geqslant 0$. It then infers that $v \in V$ is a non-negative eigenfunction to (3.1) corresponding to λ. By Lemma 3.1, we reach a contradiction, because $\lambda > \mu_{1,q,1}$. As a consequence, we have $m > 0$.

It is standard to show that $\mathcal{N}$ is a natural constraint. By the fact that there exists a non-negative minimizing sequence to (3.7) and applying Ekeland's variational principle, there exists a Palais–Smale sequence $\{u_n\} \subset E$ with $u_n^- = o_n(1)$ and $I(u_n) = o_n(1)$ for J at the level $m > 0$. Let us now prove that $\{u_n\}$ is bounded in E. Observe

$$m + o_n(1) = J(u_n) - \frac{1}{q} I(u_n) + o_n(1) = \left(\frac{1}{p} - \frac{1}{q}\right) \int_{\mathbb{R}^N} a(x)|u_n|^p\, dx. \tag{3.13}$$

Let us verify that $\{\|\nabla u_n\|_q\}$ is bounded. On the contrary, we may assume $\|\nabla u_n\|_q \to +\infty$ as $n \to \infty$. Define v_n by (3.10), use the fact $I(u_n) = o_n(1)$ and (3.13), then there holds that $\|\nabla v_n\|_p^p = \lambda + o_n(1)$. With the help of [1, Lemma 1], we can also reach a contradiction. This implies $\{\|\nabla u_n\|_q\}$ is bounded. By Hardy's inequality, it then follows that

$$\int_{\mathbb{R}^N} \frac{|u_n|^q}{(1+|x|)^q}\, dx \leqslant \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u_n|^q\, dx \leqslant C.$$

Notice that $I(u_n) = o_n(1)$, then

$$\begin{aligned}\int_{\mathbb{R}^N} m_2(x)|u_n|^q\, dx &= \int_{\mathbb{R}^N} m(x)|u_n|^q\, dx - \int_{\mathbb{R}^N} m_1(x)|u_n|^q\, dx \\ &\leqslant \int_{\mathbb{R}^N} m(x)|u_n|^q\, dx = \int_{\mathbb{R}^N} a(x)|\nabla u_n|^p\, dx + \int_{\mathbb{R}^N} |\nabla u_n|^q\, dx + o_n(1) \leqslant C.\end{aligned}$$

As a result, we get that $\{u_n\}$ is bounded in E. Then, there exists $u \in E$ such that $u_n \rightharpoonup u$ in E as $n \to \infty$. Since $\{u_n\} \subset E$ is a bounded Palais–Smale sequence for J,

$$-\Delta_p^a u_n - \Delta_q u_n = \lambda m(x)|u_n|^{q-2}u_n + o_n(1) \quad \text{in } \mathbb{R}^N. \tag{3.14}$$

Therefore, we are able to derive that $u \in E$ satisfies the following equation:

$$-\Delta_p^a u - \Delta_q u = \lambda m(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{3.15}$$

Since the embedding $E \hookrightarrow D^{1,\xi}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$ is continuous by Lemma 2.3, $\{u_n\}$ is bounded in $L^{q^*}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $L^{q^*}(\mathbb{R}^N)$ as $n \to \infty$. It follows that $\{|u_n|^q\}$ is bounded in $L^{\frac{N}{N-q}}(\mathbb{R}^N)$ and $|u_n|^q \rightharpoonup |u|^q$ in $L^{\frac{N}{N-q}}(\mathbb{R}^N)$ as $n \to \infty$. Due to $m_1 \in L^{\frac{N}{q}}(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} m_1(x)|u_n|^q\, dx = \int_{\mathbb{R}^N} m_1(x)|u|^q\, dx + o_n(1). \tag{3.16}$$

This readily indicates $u \neq 0$. Otherwise, there holds

$$\int_{\mathbb{R}^N} m(x)|u_n|^q\,dx = \int_{\mathbb{R}^N} m_1(x)|u_n|^q\,dx - \int_{\mathbb{R}^N} m_2(x)|u_n|^q\,dx \leqslant o_n(1). \tag{3.17}$$

Since $I(u_n) = o_n(1)$,

$$\int_{\mathbb{R}^N} a(x)|\nabla u_n|^p\,dx + \int_{\mathbb{R}^N} |\nabla u_n|^q\,dx \leqslant o_n(1).$$

This in turn gives $J(u_n) = o_n(1)$, which is impossible, because of $m > 0$. Therefore, u is a non-trivial solution to (1.1). Moreover, as a consequence of maximum principle, see [24, Proposition 2.3], we have $u > 0$. Thus, the proof is completed. □

3.2 | Case $q < p$

Next, we are going to deal with the case $q < p$. In this case, we define

$$\Phi(u) := \frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx,$$

$$\mathcal{S} := \left\{u \in E : \frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx = 1\right\}.$$

Lemma 3.2. *Assume (H) holds, $N \geqslant 2$, $1 < q < p < N$, $a \in C^{0,1}(\mathbb{R}^N, [0, +\infty))$ and $a \not\equiv 0$. Then, Φ restricted on $\mathcal{S}$ satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$.*

Proof. Let $\{u_n\} \subset E$ be a Palais–Smale sequence for Φ restricted on $\mathcal{S}$ at the level $c \in \mathbb{R}$. Then,

$$\Phi(u_n) = c + o_n(1), \quad (\Phi_{|_\mathcal{S}})'(u_n) = o_n(1).$$

The aim is to prove that $\{u_n\}$ is compact in E. It is straightforward to see that $\{u_n\}$ is bounded in $D^{1,\xi}(\mathbb{R}^N)$, because of $\{u_n\} \subset \mathcal{S}$. In virtue of Hardy's inequality, we obtain

$$\int_{\mathbb{R}^N} \frac{|u_n|^q}{(1+|x|)^q}\,dx \leqslant \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u_n|^q\,dx \leqslant C.$$

In addition, note that

$$\int_{\mathbb{R}^N} m_2(x)|u_n|^q\,dx = \int_{\mathbb{R}^N} m_1(x)|u_n|^q\,dx - \int_{\mathbb{R}^N} m(x)|u_n|^q\,dx.$$

By Hölder's inequality and Sobolev's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} m_1(x)|u_n|^q\,dx &\leqslant \left(\int_{\mathbb{R}^N} |m_1|^{\frac{N}{q}}\,dx\right)^{\frac{q}{N}} \left(\int_{\mathbb{R}^N} |u_n|^{\frac{qN}{N-q}}\,dx\right)^{\frac{N-q}{N}} \\ &\leqslant C\left(\int_{\mathbb{R}^N} |m_1|^{\frac{N}{q}}\,dx\right)^{\frac{q}{N}} \int_{\mathbb{R}^N} |\nabla u_n|^q\,dx \leqslant C.\end{aligned}$$

Accordingly, we obtain that $\{u_n\}$ is bounded in E. It then yields that there exists $u \in E$ such that $u_n \rightharpoonup u$ in E as $n \to \infty$. Since the embedding $E \hookrightarrow L^{q^*}(\mathbb{R}^N)$ is continuous, $\{u_n\}$ is bounded in $L^{q^*}(\mathbb{R}^N)$ and $u_n \rightharpoonup u$ in $L^{q^*}(\mathbb{R}^N)$ as $n \to \infty$. It then follows that $\{|u_n|^q\}$ is bounded in $L^{\frac{N}{N-q}}(\mathbb{R}^N)$ and $|u_n|^q \rightharpoonup |u|^q$ in $L^{\frac{N}{N-q}}(\mathbb{R}^N)$ as $n \to \infty$. Due to $m_1 \in L^{\frac{N}{q}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} m_1(x)|u_n|^q\,dx = \int_{\mathbb{R}^N} m_1(x)|u|^q\,dx + o_n(1). \tag{3.18}$$

It readily indicates that $u \neq 0$. Otherwise, by (3.18), there holds that

$$q = \int_{\mathbb{R}^N} m(x)|u_n|^q\,dx = \int_{\mathbb{R}^N} m_1(x)|u_n|^q\,dx - \int_{\mathbb{R}^N} m_2(x)|u_n|^q\,dx \leqslant o_n(1). \tag{3.19}$$

This is impossible. Since $\{u_n\} \subset E$ is a bounded Palais–Smale sequence for Φ restricted on $\mathcal{S}$, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that $u_n \in E$ satisfies the equation

$$-\Delta_p^a u_n - \Delta_q u_n = \lambda_n m(x)|u_n|^{q-2}u_n + o_n(1) \quad \text{in } \mathbb{R}^N, \tag{3.20}$$

where

$$\lambda_n = \frac{1}{q}\int_{\mathbb{R}^N} a(x)|\nabla u_n|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u_n|^q\,dx + o_n(1).$$

Notice that $\{u_n\}$ is bounded in $D^{1,\xi}(\mathbb{R}^N)$, then $\{\lambda_n\}$ is bounded in $\mathbb{R}$ and there exists $\lambda \in \mathbb{R}$ such that $\lambda_n \to \lambda$ in $\mathbb{R}$ as $n \to \infty$. Furthermore, $u \in E$ and it satisfies the equation

$$-\Delta_p^a u - \Delta_q u = \lambda m(x)|u|^{q-2}u \quad \text{in } \mathbb{R}^N. \tag{3.21}$$

Thanks to $u \neq 0$, we then have $\lambda > 0$. Taking into account (3.20) and (3.21), we conclude

$$\begin{aligned}
&\int_{\mathbb{R}^N} \left(a(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right) + \left(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\right)\right)\cdot(\nabla u_n - \nabla u)\,dx \\
&= \lambda_n \int_{\mathbb{R}^N} m(x)|u_n|^{q-2}u_n(u_n-u)\,dx - \lambda\int_{\mathbb{R}^N} m(x)|u|^{q-2}u(u_n-u)\,dx + o_n(1) \\
&= (\lambda_n - \lambda)\int_{\mathbb{R}^N} m(x)|u_n|^{q-2}u_n(u_n-u)\,dx \\
&\quad + \lambda\int_{\mathbb{R}^N} m(x)\left(|u_n|^{q-2}u_n - |u|^{q-2}u\right)(u_n-u)\,dx + o_n(1).
\end{aligned} \tag{3.22}$$

Observe first that

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} m(x)|u_n|^{q-2}u_n(u_n-u)\,dx\right| &\leqslant \left|\int_{\mathbb{R}^N} m_1(x)|u_n|^{q-2}u_n(u_n-u)\,dx\right| \\
&\quad + \left|\int_{\mathbb{R}^N} m_2(x)|u_n|^{q-2}u_n(u_n-u)\,dx\right|.
\end{aligned}$$

In addition, we see

$$\left|\int_{\mathbb{R}^N} m_1(x)|u_n|^{q-2}u_n(u_n-u)\,dx\right|$$
$$\leqslant \left(\int_{\mathbb{R}^N} |m_1|^{\frac{N}{q}}\,dx\right)^{\frac{q}{N}} \left(\int_{\mathbb{R}^N} |u_n|^{q^*}\,dx\right)^{\frac{q-1}{q^*}} \left(\int_{\mathbb{R}^N} |u_n-u|^{q^*}\,dx\right)^{\frac{1}{q^*}},$$
$$\left|\int_{\mathbb{R}^N} m_2(x)|u_n|^{q-2}u_n(u_n-u)\,dx\right|$$
$$\leqslant \left(\int_{\mathbb{R}^N} m_2(x)|u_n|^{q}\,dx\right)^{\frac{q-1}{q}} \left(\int_{\mathbb{R}^N} m_2(x)|u|^{q}\,dx\right)^{\frac{1}{q}} + \int_{\mathbb{R}^N} m_2(x)|u_n|^q\,dx.$$

Therefore, utilizing the fact that $\{u_n\}$ is bonded in E, we get

$$\left|\int_{\mathbb{R}^N} m(x)|u_n|^{q-2}u_n(u_n-u)\,dx\right| \leqslant C.$$

It necessarily follows that

$$(\lambda_n-\lambda)\int_{\mathbb{R}^N} m(x)|u_n|^{q-2}u_n(u_n-u)\,dx = o_n(1). \tag{3.23}$$

Note that $u_n \rightharpoonup u$ in E as $n\to\infty$, then $u_n \rightharpoonup u$ in $D^{1,q}(\mathbb{R}^N)$ as $n\to\infty$. We then deduce that $u_n \to u$ in $L^q_{loc}(\mathbb{R}^N)$ as $n\to\infty$. As a consequence, we have

$$\int_{B(0,R)} m(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n-u)\,dx = o_n(1). \tag{3.24}$$

On the other hand, by Hölder's inequality and Sobolev's inequality, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus B(0,R)} m(x)\big(|u_n|^{q-2}u_n - |u|^{q-2}u\big)(u_n-u)\,dx\\
&\leqslant \left(\int_{\mathbb{R}^N\setminus B(0,R)} |m_1|^{\frac{N}{q}}\,dx\right)^{\frac{q}{N}} \left(\|u_n\|_{q^*}^{q-1} + \|u\|_{q^*}^{q-1}\right)\|u_n-u\|_{q^*} \qquad (3.25)\\
&\leqslant C\left(\int_{\mathbb{R}^N\setminus B(0,R)} |m_1|^{\frac{N}{q}}\,dx\right)^{\frac{q}{N}} \left(\|\nabla u_n\|_{q}^{q-1} + \|\nabla u\|_{q}^{q-1}\right)\|\nabla u_n-\nabla u\|_{q} = o_R(1),
\end{aligned}$$

where we also used the facts

$$\big(|s|^{q-2}s - |t|^{q-2}t\big)(s-t) \geqslant 0, \quad \forall\ s,t\in\mathbb{R}, q>1,$$
$$\int_{\mathbb{R}^N\setminus B(0,R)} |m_1|^{\frac{N}{q}}\,dx = o_R(1),$$

where the second fact holds because of $m_1 \in L^{\frac{N}{q}}(\mathbb{R}^N)$ from the assumption (H). Combining (3.23), (3.24) and (3.25), by (3.22), we then obtain

$$\int_{\mathbb{R}^N} \left(a(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right) + \left(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\right)\right) \cdot (\nabla u_n - \nabla u)\, dx = o_n(1).$$

Observe that

$$|z_1 - z_2|^r \leqslant C\left(\left(|z_1|^{r-2}z_1 - |z_2|^{r-2}z_2\right) \cdot (z_1 - z_2)\right)^{\frac{\theta}{2}} (|z_1|^r + |z_2|^r)^{1-\frac{\theta}{2}}, \quad \forall\, z_1, z_2 \in \mathbb{R}^N, \tag{3.26}$$

where $\theta = r$ if $1 < r < 2$ and $\theta = 2$ if $r \geqslant 2$. Then, we see

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(x)(|\nabla u_n - \nabla u|^p)\, dx + \int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^q\, dx \\
&\leqslant C\left(\int_{\mathbb{R}^N} a(x)\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right) \cdot (\nabla u_n - \nabla u)\, dx\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} a(x)(|\nabla u_n|^p + |\nabla u|^p)\, dx\right)^{1-\frac{\theta}{2}} \\
&\quad + C\left(\int_{\mathbb{R}^N} \left(|\nabla u_n|^{q-2}\nabla u_n - |\nabla u|^{q-2}\nabla u\right) \cdot (\nabla u_n - \nabla u)\, dx\right)^{\frac{\theta}{2}} \left(\int_{\mathbb{R}^N} |\nabla u_n|^q + |\nabla u|^q\, dx\right)^{1-\frac{\theta}{2}} = o_n(1).
\end{aligned}$$

This immediately indicates that $u_n \to u$ in $D^{1,\xi}(\mathbb{R}^N)$ as $n \to \infty$. Taking advantage of (3.20) and (3.21), we then get

$$\int_{\mathbb{R}^N} m(x)|u_n|^q\, dx = \int_{\mathbb{R}^N} m(x)|u|^q\, dx + o_n(1),$$

because of $\lambda_n = \lambda + o_n(1)$ and $\lambda \neq 0$. In view of (3.18),

$$\int_{\mathbb{R}^N} m_2(x)|u_n|^q\, dx = \int_{\mathbb{R}^N} m_2(x)|u|^q\, dx + o_n(1).$$

Since $u_n \to u$ in $D^{1,q}(\mathbb{R}^N)$ as $n \to \infty$, by Hardy's inequality,

$$\int_{\mathbb{R}^N} \frac{|u_n - u|^q}{(1+|x|)^q}\, dx \leqslant \left(\frac{p}{N-p}\right)^p \int_{\mathbb{R}^N} |\nabla u_n - \nabla u|^q\, dx = o_n(1).$$

Consequently, we derive that $u_n \to u$ in E as $n \to \infty$. Thus, the proof is completed. □

Theorem 3.4. *Assume (H) holds, $N \geqslant 2$, $1 < q < p < N$, $a \in C^{0,1}(\mathbb{R}^N, [0, +\infty))$ and $a \not\equiv 0$. Then, there exists a sequence of solutions $(\lambda_k, u_k) \in \mathbb{R} \times E$ with $u_k \in \mathcal{S}$ and*

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots, \quad \lim_{k\to\infty} \lambda_k \to +\infty \text{ as } k \to +\infty.$$

Proof. To establish the existence of a sequence of eigenvalues to (1.1), we shall take into account Ljusternik–Schnirelman theory in [2]. Define

$$\Sigma := \{A \subset \mathcal{S} : A \text{ is compact and } A = -A\}.$$

For a set $A \in \Sigma$, the genus of A is defined by

$$\gamma(A) := \min\{n \in \mathbb{N} : \text{exists a function } \varphi \in C(A, \mathbb{R}^n \backslash \{0\}) \text{ satisfying } \varphi(-x) = -\varphi(x)\}.$$

If such a minimum does not exist, we set $\gamma(A) = +\infty$.

Let us now define

$$\Sigma_k := \{A \in \Sigma : \gamma(A) \geqslant k\}, \quad \forall\, k \in \mathbb{N}^+.$$

First we see that, for any $k \in \mathbb{N}^+$, $\Sigma_k \neq \emptyset$. Indeed, let X_k be a k-dimensional subspace of E, by Borsuk–Ulam's theorem, then $\gamma(S \cap X_k) \geqslant k$. Define

$$\tilde{\lambda}_k := \inf_{A \in \Sigma_k} \sup_{u \in A} \Phi(u).$$

Since $\Sigma_{k+1} \subset \Sigma_k$, $\tilde{\lambda}_k \leqslant \tilde{\lambda}_{k+1}$ for any $k \in \mathbb{N}^+$. From Lemma 3.2, $\tilde{\lambda}_k$ is a critical point of J restricted on S for any $k \in \mathbb{N}^+$. Then, we derive that

$$0 < \tilde{\lambda}_1 < \tilde{\lambda}_2 \leqslant \cdots \leqslant \tilde{\lambda}_k \leqslant \tilde{\lambda}_{k+1} \leqslant \cdots.$$

Next we prove $\tilde{\lambda}_k \to +\infty$ as $k \to +\infty$. Let $\{e_i\} \subset E$ be such that $E = \overline{\operatorname{span}\{e_1, e_2, \dots, e_i, \dots\}}$. Let $\{e_i'\} \subset E$ be such that $E' = \overline{\operatorname{span}\{e_1', e_2', \dots, e_i', \dots\}}$, where E' denotes the dual space of E. Define $X_i := \operatorname{span}\{e_i\}$ and

$$Y_k := \bigoplus_{i=1}^{k} X_i, \quad Z_k := \overline{\bigoplus_{i=k}^{\infty} X_i}, \quad \forall\, k \in \mathbb{N}^+.$$

Let $A \in \Sigma_k$ satisfy $\gamma(A) \geqslant k$. By basic properties of the genus, we have $A \cap Z_k \neq \emptyset$. Define

$$\beta_k := \inf_{A \in \Sigma_k} \sup_{u \in A \cap Z_k} J(u), \quad \forall\, k \in \mathbb{N}^+.$$

Then, $\beta_k \to +\infty$ as $k \to \infty$. Otherwise, we may assume $\{\beta_k\} \subset \mathbb{R}$ is bounded. Thus, there exists a sequence $\{u_k\} \subset A \cap Z_k$ such that $\{\Phi(u_k)\} \subset \mathbb{R}$ is bounded. It then follows that $\{u_k\}$ is bounded in E. Further, there exists $u \in E$ such that $u_k \rightharpoonup u$ in E as $n \to \infty$. Observe that $\langle e_i', u\rangle = \langle e_i', u_k\rangle + o_k(1) = o_k(1)$, because of $u_k \in Z_k$. Therefore, we have $u = 0$ and $u_k \rightharpoonup 0$ in E as $k \to \infty$. This along with the assumption that $m_1 \in L^{\frac{N}{q}}(\mathbb{R}^N)$ from the assumption (H) leads to

$$\int_{\mathbb{R}^N} m_1(x)|u_k|^q \, dx = o_k(1).$$

Since $m_2 \geqslant 0$ from the assumption (H),

$$\int_{\mathbb{R}^N} m(x)|u_k|^q \, dx = \int_{\mathbb{R}^N} m_1(x)|u_k|^q \, dx - \int_{\mathbb{R}^N} m_2(x)|u_k|^q \, dx \leqslant o_k(1),$$

which is impossible due to $u_k \in S$. Consequently, we get that $\beta_k \to +\infty$ as $k \to \infty$. Thanks to $\tilde{\lambda}_k \geqslant \beta_k$ for any $k \in \mathbb{N}^+$, $\tilde{\lambda}_k \to +\infty$ as $k \to \infty$. Since $u_k \in E$ is a critical point for E restricted on S, there exists $\lambda_k \in \mathbb{R}$ such that

$$-\Delta_p^a u_k - \Delta_q u_k = \lambda_k m(x)|u_k|^{q-2} u_k \quad \text{in } \mathbb{R}^N,$$

where

$$\lambda_k = \frac{1}{q}\int_{\mathbb{R}^N} a(x)|\nabla u_k|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u_k|^q\,dx > \Phi(u_k) = \tilde{\lambda}_k, \quad \forall\, k \in \mathbb{N}^+.$$

Thus, the proof is completed. □

Lemma 3.3. *Define*

$$\begin{aligned} I(u,v) :&= -\int_{\mathbb{R}^N} \left(\Delta_p^a u\right)\frac{u^q - v^q}{u^{q-1}}\,dx - \int_{\mathbb{R}^N} \left(\Delta_q u\right)\frac{u^q - v^q}{u^{q-1}}\,dx \\ &\quad - \int_{\mathbb{R}^N} \left(\Delta_p^a v\right)\frac{v^q - u^q}{v^{q-1}}\,dx - \int_{\mathbb{R}^N} \left(\Delta_q v\right)\frac{v^q - u^q}{v^{q-1}}\,dx, \end{aligned} \tag{3.27}$$

where $u, v \in D^{1,\xi}(\mathbb{R}^N)$, $u, v > 0$ and $1 < q < p$. Then, $I(u,v) \geqslant 0$. Moreover, $I(u,v) = 0$ if and only if $u = kv$ for some $k \in \mathbb{R}$.

Proof. Let us first show

$$I_1(u,v) := -\int_{\mathbb{R}^N} \left(\Delta_p^a u\right)\frac{u^q - v^q}{u^{q-1}}\,dx - \int_{\mathbb{R}^N} \left(\Delta_p^a v\right)\frac{v^q - u^q}{v^{q-1}}\,dx \geqslant 0, \quad u, v \in D^{1,\xi}(\mathbb{R}^N), u, v > 0.$$

It is straightforward to compute

$$\nabla\left(\frac{u^q - v^q}{u^{q-1}}\right) = \left(1 + (q-1)\left(\frac{v}{u}\right)^q\right)\nabla u - q\left(\frac{v}{u}\right)^{q-1}\nabla v, \tag{3.28}$$

$$\nabla\left(\frac{v^q - u^q}{v^{q-1}}\right) = \left(1 + (q-1)\left(\frac{u}{v}\right)^q\right)\nabla v - q\left(\frac{u}{v}\right)^{q-1}\nabla u. \tag{3.29}$$

Therefore, by the divergence theorem, we derive that

$$\begin{aligned} I_1(u,v) &= \int_{\mathbb{R}^N} a(x)\left(\left(1 + (q-1)\left(\frac{v}{u}\right)^q\right)|\nabla u|^p - q\left(\frac{v}{u}\right)^{q-1}|\nabla u|^{p-2}\nabla u\cdot\nabla v\right)dx \\ &\quad + \int_{\mathbb{R}^N} a(x)\left(\left(1 + (q-1)\left(\frac{u}{v}\right)^q\right)|\nabla v|^p - q\left(\frac{u}{v}\right)^{q-1}|\nabla v|^{p-2}\nabla v\cdot\nabla u\right)dx. \end{aligned}$$

Using Young's inequality, we know that

$$\begin{aligned} q\left(\frac{v}{u}\right)^{q-1}|\nabla u|^{p-2}|\nabla u\cdot\nabla v| &\leqslant q\left(\frac{v}{u}\right)^{q-1}|\nabla u|^{p-1}|\nabla v| \\ &\leqslant \frac{q(p-1)}{p}\left(\frac{v}{u}\right)^{\frac{p(q-1)}{p-1}}|\nabla u|^p + \frac{q}{p}|\nabla v|^p \\ &= \frac{q(p-1)}{p}\left(\frac{v}{u}\right)^{\frac{p(q-1)}{p-1}}|\nabla u|^{\frac{p^2(q-1)}{q(p-1)}}|\nabla u|^{\frac{p(p-q)}{q(p-1)}} + \frac{q}{p}|\nabla v|^p \\ &\leqslant (q-1)\left(\frac{v}{u}\right)^q|\nabla u|^p + \frac{p-q}{p}|\nabla u|^p + \frac{q}{p}|\nabla v|^p. \end{aligned}$$

Similarly, we can get

$$q\Big(\frac{u}{v}\Big)^{q-1}|\nabla v|^{p-2}|\nabla v\cdot\nabla u| \leqslant q\Big(\frac{u}{v}\Big)^{q-1}|\nabla v|^{p-1}|\nabla u| \leqslant (q-1)\Big(\frac{u}{v}\Big)^{q}|\nabla v|^{p} + \frac{p-q}{p}|\nabla v|^{p} + \frac{q}{p}|\nabla u|^{p}.$$

It then follows that $I_1(u,v)\geqslant 0$. Next, we prove that

$$I_2(u,v) := -\int_{\mathbb{R}^N}(\Delta_q u)\frac{u^q-v^q}{u^{q-1}}\,dx - \int_{\mathbb{R}^N}(\Delta_q v)\frac{v^q-u^q}{v^{q-1}}\,dx \geqslant 0, \quad u,v\in D^{1,\xi}(\mathbb{R}^N), u,v>0.$$

In view of (3.28) and (3.29), by the divergence theorem,

$$\begin{aligned} I_2(u,v) = &\int_{\mathbb{R}^N}\Big(1+(q-1)\Big(\frac{v}{u}\Big)^{q}\Big)|\nabla u|^q - q\Big(\frac{v}{u}\Big)^{q-1}|\nabla u|^{q-2}\nabla u\cdot\nabla v\,dx \\ &+ \int_{\mathbb{R}^N}\Big(1+(q-1)\Big(\frac{u}{v}\Big)^{q}\Big)|\nabla v|^q - q\Big(\frac{u}{v}\Big)^{q-1}|\nabla v|^{q-2}\nabla v\cdot\nabla u\,dx. \end{aligned}$$

Using again Young's inequality, we obtain

$$q\Big(\frac{v}{u}\Big)^{q-1}|\nabla u|^{q-2}|\nabla u\cdot\nabla v| \leqslant q\Big(\frac{v}{u}\Big)^{q-1}|\nabla u|^{q-1}|\nabla v| \leqslant (q-1)\Big(\frac{v}{u}\Big)^{q}|\nabla u|^{q} + |\nabla v|^q,$$

$$q\Big(\frac{u}{v}\Big)^{q-1}|\nabla v|^{q-2}|\nabla v\cdot\nabla u| \leqslant q\Big(\frac{u}{v}\Big)^{q-1}|\nabla v|^{q-1}|\nabla u| \leqslant (q-1)\Big(\frac{u}{v}\Big)^{q}|\nabla v|^{q} + |\nabla u|^q.$$

Therefore, we have $I_2(u,v)=0$. Accordingly, there holds that $I(u,v)\geqslant 0$ for any $u,v\in D^{1,\xi}(\mathbb{R}^N)$ and $u,v>0$. If $I(u,v)=0$, then $I_2(u,v)=0$. This leads to

$$\nabla u\cdot\nabla v = |\nabla u||\nabla v|, \quad \Big(\frac{v}{u}\Big)^{q}|\nabla u|^q = |\nabla v|^q, \quad \Big(\frac{u}{v}\Big)^{q}|\nabla v|^q = |\nabla u|^q,$$

As a consequence, we see that

$$|u\nabla v - v\nabla u| = 0.$$

This implies that there exists $k\in\mathbb{R}$ such that $u=kv$ and the proof is completed. □

Remark 3.1. In fact, Lemma 3.3 is established for the double phase operator under the assumption $q<p$, which is not a direct consequence of Lemma 3.1. It is unknown to us if Lemma 3.3 remains valid for the case $p<q$. From the proof of Lemma 3.3, one can see that the assumption $q<p$ is crucial, which is the premise of the use of Young's inequality.

Proposition 3.2. *Assume* (H) *holds,* $N\geqslant 2$, $1<q<p<N$, $\frac{p}{q}<1+\frac{1}{N}$, $a\in C^{0,1}(\mathbb{R}^N,[0,+\infty))$ *and* $a\not\equiv 0$. *Assume that any eigenfunction to (1.1) corresponding to* λ *is non-negative. Then,* λ *is simple.*

Proof. Let $u\in E$ be a non-negative eigenfunction to (1.1) corresponding to λ. It follows from [8] and [23, Proposition 3] or [24, Proposition 2.3] that $u>0$. Let $u>0$ and $v>0$ be two eigenfunctions to (1.1) corresponding to λ. Then, we see that

$$-\Delta_p^a u - \Delta_q u = \lambda m(x)u^{q-1}, \quad -\Delta_p^a v - \Delta_q v = \lambda m(x)v^{q-1} \quad \text{in } \mathbb{R}^N.$$

As a result, there holds that

$$I(u,v)=\lambda\int_{\mathbb{R}^N} m(x)(u^q-v^q)\,dx+\lambda\int_{\mathbb{R}^N} m(x)(v^q-u^q)\,dx=0.$$

It then follows from Lemma 3.3 that the desired conclusion holds. This completes the proof. □

Proposition 3.3. *Assume (H) holds, $N\geqslant 2$, $1<p,q<N$, $p\neq q$, $a\in C^{0,1}(\mathbb{R}^N,[0,+\infty))$ and $a\not\equiv 0$. Then,*

$$\mu_{1,q,1}=\inf\left\{\frac{\frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,dx}{\frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx}:u\in E\backslash\{0\},\ \int_{\mathbb{R}^N} m(x)|u|^q\,dx>0\right\}.$$

Proof. Since $\mu_{1,q,1}$ is the first eigenvalue to (3.1) and $E\subset D^{1,q}(\mathbb{R}^N)$,

$$\begin{aligned}\mu_{1,q,1}&=\inf\left\{\frac{\int_{\mathbb{R}^N}|\nabla u|^q\,dx}{\int_{\mathbb{R}^N} m(x)|u|^q\,dx}:u\in D^{1,q}(\mathbb{R}^N)\backslash\{0\},\ \int_{\mathbb{R}^N} m(x)|u|^q\,dx>0\right\}\\&\leqslant\inf\left\{\frac{\frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u_{1,q,1}|^p\,dx+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,dx}{\frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx}:u\in E\backslash\{0\},\ \int_{\mathbb{R}^N} m(x)|u|^q\,dx>0\right\}.\end{aligned}$$

Let $u_{1,q,1}\in E$ be an eigenfunction to (3.1) corresponding to $\mu_{1,q,1}$ and $p<q$, then

$$\begin{aligned}&\inf\left\{\frac{\frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,dx}{\frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx}:u\in E\backslash\{0\},\ \int_{\mathbb{R}^N} m(x)|u|^q\,dx>0\right\}\\&\leqslant\frac{\frac{n^p}{p}\int_{\mathbb{R}^N} a(x)|\nabla u_{1,q,1}|^p\,dx+\frac{n^q}{q}\int_{\mathbb{R}^N}|\nabla u_{1,q,1}|^q\,dx}{\frac{n^q}{q}\int_{\mathbb{R}^N} m(x)|u_{1,q,1}|^q\,dx}=\mu_{1,q,1}+o_n(1)\quad\text{as } n\to\infty.\end{aligned}$$

Similarly, if $q<p$, then

$$\begin{aligned}&\inf\left\{\frac{\frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx+\frac{1}{q}\int_{\mathbb{R}^N}|\nabla u|^q\,dx}{\frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx}:u\in E\backslash\{0\},\ \int_{\mathbb{R}^N} m(x)|u|^q\,dx>0\right\}\\&\leqslant\frac{\frac{1}{pn^p}\int_{\mathbb{R}^N} a(x)|\nabla u_{1,q,1}|^p\,dx+\frac{1}{qn^q}\int_{\mathbb{R}^N}|\nabla u_{1,q,1}|^q\,dx}{\frac{1}{qn^q}\int_{\mathbb{R}^N} m(x)|u_{1,q,1}|^q\,dx}=\mu_{1,q,1}+o_n(1)\quad\text{as } n\to\infty.\end{aligned}$$

Thus, the desired result follows and the proof is completed. □

Remark 3.2. Under the assumptions of Theorem 3.4, by Theorem 3.2 and Proposition 3.3, we have

$$\lambda_1 > \inf\left\{ \frac{\frac{1}{p}\int_{\mathbb{R}^N} a(x)|\nabla u|^p\,dx + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx}{\frac{1}{q}\int_{\mathbb{R}^N} m(x)|u|^q\,dx} : u \in E\backslash\{0\}, \int_{\mathbb{R}^N} m(x)|u|^q\,dx > 0 \right\}.$$

Remark 3.3. The arguments developed in this paper allow to obtain similar results if the hypothesis (H) is replaced by the following condition introduced by Szulkin and Willem [27],

$(\mathcal{H})$ $m \in L^1_{loc}(\mathbb{R}^N)$, $m^+ = m_1 + m_2 \neq 0$, $m_1 \in L^{\frac{N}{q}}(\mathbb{R}^N)$, for every $y \in \mathbb{R}^N$, $\lim_{x\to y} |x - y|^q m_2(x) = 0$ and $\lim_{|x|\to\infty} |x|^q m_2(x) = 0$, where $m^+ := \max\{m(x), 0\}$.

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ORCID

Vicenţiu D. Rădulescu https://orcid.org/0000-0003-4615-5537

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