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A SADDLE POINT THEOREM FOR NON-SMOOTH FUNCTIONALS AND PROBLEMS AT RESONANCE

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Abstract. We prove a saddle point theorem for locally Lipschitz functionals with arguments based on a version of the mountain pass theorem for such kind of functionals. This abstract result is applied to solve two different types of multivalued semilinear elliptic boundary value problems with a Laplace–Beltrami operator on a smooth compact Riemannian manifold.

The mountain pass theorem of Ambrosetti and Rabinowitz (see [1]) and the saddle point theorem of Rabinowitz (see [18]) are very important tools in the critical point theory of C^1-functionals. That is why it is natural to ask what happens if the functional fails to be differentiable. The first who considered such a case were Aubin and Clarke (see [4]) and Chang (see [9]), who gave suitable variants of the mountain pass theorem for locally Lipschitz functionals. For this aim they replaced the usual gradient with a generalized one, which was firstly defined by Clarke (see [10], [11]). In their arguments, the fundamental approach was a "Lipschitz version" of the deformation lemma in reflexive Banach spaces.

In the first part of our paper, after recalling the main properties of the Clarke generalized gradient, we give a variant of the saddle point theorem for locally Lipschitz functionals. As a compactness condition we use the locally Palais–Smale condition, which was introduced for smooth mappings by Brezis, Coron and Nirenberg (see [7]).

We then apply our abstract framework to solve two different types of problems with a Laplace–Beltrami operator on a smooth compact Riemann manifold, possibly with smooth boundary. The first one is related to a multivalued problem with strong resonance at infinity. The literature is very rich in such problems, the first who studied problems at resonance, in the smooth case, being Landesman and Lazer (see [16]). They found sufficient conditions for the existence of solutions for some singlevalued equations with Dirichlet conditions. These problems, which arise frequently in mechanics, were thereafter intensively studied and many applications to concrete situations were given. See e.g. [2], [3], [6], [9], [12], [15], [16], [18], [20], [21].

As a second application we solve another type of multivalued elliptic problem. We assume that the nonlinearity has a subcritical growth and a subresonant

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decay at the origin. Such type of problems also arises frequently in non-smooth mechanics.

1. Critical point theorems for non-smooth functionals

Throughout this paper, X will denote a real Banach space. Let X^* be its dual. For each $x \in X$ and $x^* \in X^*$, we denote by $\langle x^*, x\rangle$ the duality pairing between X^* and X. We say that a function $f\colon X \to \mathbf{R}$ is *locally Lipschitzian* ($f \in \mathrm{Lip}_{\mathrm{loc}}(X, \mathbf{R})$) if, for each $x \in X$, there is a neighbourhood V of x and a constant $k = k(V)$ depending on V such that, for each $y, z \in V$,

$$|f(y) - f(z)| \le k\|y - z\|.$$

We first recall the definition of the Clarke subdifferential and some of its most important properties (see [9], [10], [11] for proofs and further details).

Let $f\colon X \to \mathbf{R}$ be a locally Lipschitzian function. For each $x, v \in X$, we define the generalized directional derivative of f at x in the direction v as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}.$$

It follows by the definition of a locally Lipschitzian function that $f^0(x, v)$ is a finite number and $|f^0(x, v)| \le k\|v\|$. Moreover, the mapping $v \longmapsto f^0(x, v)$ is positively homogeneous and subadditive and so, it is convex continuous. The generalized gradient (the Clarke subdifferential) of f at the point x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^*; f^0(x, v) \ge \langle x^*, v\rangle, \text{ for all } v \in X\}.$$

If f is Fréchet-differentiable at x, then $\partial f(x) = \{f'(x)\}$, and if f is convex, then $\partial f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis.

The fundamental properties of the Clarke subdifferential are:

a) For each $x \in X$, $\partial f(x)$ is a non-empty convex weak-$\star$ compact subset of X^*.

b) For each $x, v \in X$, we have

$$f^0(x, v) = \max\{\langle x^*, v\rangle; x^* \in \partial f(x)\}.$$

c) The set-valued mapping $x \longmapsto \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X$, $\varepsilon > 0$, $v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\|x - x_0\| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $|\langle x^* - x_0^*, v\rangle| < \varepsilon$.

d) The function $f^0(\,\cdot\,, \cdot\,)$ is upper semi-continuous.

e) If f attains a local minimum at x, then $0 \in \partial f(x)$.

f) The function

$$\lambda(x) = \min\{\|x^*\|; x^* \in \partial f(x)\}$$

exists and is lower semi-continuous.

g) Lebourg's mean value theorem: If x and y are distinct points in X, then there is a point z in the open segment between x and y such that

$$f(y) - f(x) \in \langle \partial f(z), y - x\rangle.$$

Definition 1. Let $f\colon X \to \mathbf{R}$ be a locally Lipschitzian function. A point $x \in X$ is said to be a *critical point* of f provided that $0 \in \partial f(x)$, that is, $f^0(x, v) \geq 0$ for every $v \in X$. A real number c is called a *critical value* of f if there is a critical point $x \in X$ such that $f(x) = c$.

Definition 2. Let $f\colon X \to \mathbf{R}$ be a locally Lipschitzian function and let c be a real number. We say that f satisfies the *Palais–Smale condition* at the level c (in short $(\mathrm{PS})_c$) if any sequence $(x_n)_n$ in X with the properties $\lim_{n\to\infty} f(x_n) = c$ and $\lim_{n\to\infty} \lambda(x_n) = 0$ is relatively compact.

Let K be a compact metric space and let K^* be a non-empty closed subset of K. If $p^*\colon K^* \to X$ is a fixed continuous mapping, set

$$\mathscr{P} = \big\{p \in C(K, X); p = p^* \text{ on } K^*\big\}.$$

It follows by a theorem of Dugundji (Theorem 6.1 in [13]) that $\mathscr{P}$ is non-empty.

Define

$$c = \inf_{p\in\mathscr{P}} \max_{t\in K} f\big(p(t)\big). \tag{1}$$

Obviously, $c \geq \max\big\{f\big(p^*(t)\big); t \in K^*\big\}$.

The following result is a generalization of the mountain pass theorem of Ambrosetti–Rabinowitz:

Theorem 1. *Let $f\colon X \to \mathbf{R}$ be a locally Lipschitzian function. Assume that*

$$c > \max\big\{f\big(p^*(t)\big); t \in K^*\big\}. \tag{2}$$

Then there exists a sequence (x_n) in X such that:

i) $\lim_{n\to\infty} f(x_n) = c$;

ii) $\lim_{n\to\infty} \lambda(x_n) = 0$.

Moreover, if f satisfies $(\mathrm{PS})_c$ then c is a critical value of f, corresponding to a critical point which is not in $p^*(K^*)$.

The proof of this theorem can be found in [19]. We only mention that the key facts of the proof are Ekeland's variational principle and the following pseudo-gradient lemma (see [8]) for multivalued mappings:

Lemma 1 (Choulli–Deville–Rhandi). *Let M be a compact metric space and let φ: $M \to 2^{X^*}$ be a set-valued mapping which is upper semi-continuous (in the sense of* c)) *and with weak-$\star$ compact convex values. Let*

$$\gamma = \inf\bigl\{\|x^*\|; x^* \in \varphi(t), t \in M\bigr\}.$$

Then, given $\varepsilon > 0$, there exists a continuous function v: $M \to X$ such that for all $t \in M$ and $x^* \in \varphi(t)$,

$$\|v(t)\| \le 1 \quad \text{and} \quad \langle x^*, v(t)\rangle \ge \gamma - \varepsilon.$$

Let us notice the following two consequences of Theorem 1, the first one being the mountain pass theorem as stated by Ambrosetti and Rabinowitz:

Corollary 1. *Let f: $X \to \mathbf{R}$ be a locally Lipschitzian function. Suppose that $f(0) = 0$ and there is some $v \in X \setminus \{0\}$ such that*

$$f(v) \le 0. \tag{3}$$

Moreover, assume that f satisfies the following geometric hypothesis: there exist $0 < R < \|v\|$ and $\alpha > 0$ such that, for each $u \in X$ with $\|u\| = R$, we have

$$f(u) \ge \alpha. \tag{4}$$

Let $\mathscr{P}$ be the family of all continuous paths p: $[0,1] \to X$ that join 0 to v. Then the conclusion of Theorem 1 holds for c defined as in (1) and K replaced with $[0,1]$.

Corollary 2. *Let f: $X \to \mathbf{R}$ be a locally Lipschitzian function. Suppose there exists $S \subset X$ such that $p(K) \cap S \ne \emptyset$, for each $p \in \mathscr{P}$. If*

$$\inf\{f(x); x \in S\} > \max\bigl\{f\bigl(p^*(t)\bigr); t \in K^*\bigr\}$$

then the conclusion of Theorem 1 holds.

Proof. In order to apply Theorem 1, it is enough to observe that

$$\inf_{p\in\mathscr{P}} \max_{t\in K} f\bigl(p(t)\bigr) \ge \inf_{x\in S} f(x) > \max_{t\in K^*} f\bigl(p^*(t)\bigr). \quad \square$$

The following saddle point type result generalizes Rabinowitz's theorem (see [18]):

Theorem 2. *Let* $f\colon X \to \mathbf{R}$ *be a locally Lipschitzian function. Assume that* $X = Y \oplus Z$, *where* Z *is a finite dimensional subspace of* X *and for some* $z_0 \in Z$ *there exists* $R > \|z_0\|$ *such that*

$$\inf_{y\in Y} f(y+z_0) > \max\bigl\{f(z); z\in Z, \|z\| = R\bigr\}.$$

Let

$$K = \{z \in Z; \|z\| \le R\}$$

and

$$\mathscr{P} = \bigl\{p \in C(K,X); p(x) = x \text{ if } \|x\| = R\bigr\}.$$

If c *is defined as in* (1) *and* f *satisfies* $(\mathrm{PS})_c$, *then* c *is a critical value of* f.

Proof. It suffices to apply Corollary 2 for $S = z_0 + Y$. In this respect we have to prove that for every $p \in \mathscr{P}$,

$$p(K) \cap (z_0 + Y) \ne \emptyset.$$

If $P\colon X \to Z$ is the canonical projection, the above condition is equivalent to the fact that, for each $p \in \mathscr{P}$, there is some $x \in K$ such that

$$P\bigl(p(x) - z_0\bigr) = P(p(x)) - z_0 = 0.$$

This follows easily by a topological degree argument. Indeed, for some fixed $p \in \mathscr{P}$, one has

$$P \circ p = \mathrm{Id} \quad \text{on } K^* = \partial K.$$

Hence

$$d(P\circ p, \mathrm{Int}\, K, 0) = d(P\circ p, \mathrm{Int}\, K, z_0) = d(\mathrm{Id}, \mathrm{Int}\, K, z_0) = 1.$$

By the existence property of the Brouwer degree we get some $x \in \mathrm{Int}\, K$ such that $(P\circ p)(x) - z_0 = 0$, which concludes our proof. □

2. Semilinear elliptic problems with strong resonance at infinity

We shall use the above abstract results to prove the existence of solutions for certain nonlinear problems with strong resonance at infinity. In order to explain what we mean, a few words are necessary about problems at resonance. We shall briefly recall what such problems are in the smooth case.

Let Ω be a smooth bounded open set in $\mathbf{R}^N$ and $f \in C^1(\mathbf{R})$. The aim is to examine the following problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The nature of this problem depends heavily on the asymptotic behaviour of $f(t)$ as $|t| \to \infty$. We shall suppose that f is asymptotically linear, that is, there exists

$$\lim_{|t|\to\infty} \frac{f(t)}{t} = a \in \mathbf{R}.$$

If $g(t) = f(t) - at$, it is obvious that g is "sublinear at infinity", in the sense that

$$\lim_{|t|\to\infty} \frac{g(t)}{t} = 0.$$

The problem (5) is said to be *with resonance at infinity* if the number a defined above is one of the eigenvalues of $-\Delta$ in $H_0^1(\Omega)$. There are different degrees of resonance that depend on the growth of g at infinity: a "smaller" g at infinity generates a "stronger" resonance.

Following Landesman and Lazer one can distinguish the following cases:

(LL1) $$g(\pm\infty) := \lim_{t\to\pm\infty} g(t) \in \mathbf{R} \quad \text{and} \quad \big(g(+\infty), g(-\infty)\big) \neq (0,0);$$

(LL2) $$g(\pm\infty) = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \int_0^t g(s)\,ds = \pm\infty;$$

(LL3) $$g(\pm\infty) = 0 \quad \text{and} \quad \lim_{|t|\to\infty} \int_0^t g(s)\,ds \in \mathbf{R}.$$

The third case is usually refered to as *strong resonance at infinity.*

In what follows M will denote an m-dimensional smooth compact Riemann manifold, possibly with smooth boundary ∂M. Particularly, M can be any open bounded smooth subset of $\mathbf{R}^m$. We shall consider the following multivalued elliptic problem

(P1) $$\begin{cases} -\Delta_M u(x) - \lambda_1 u(x) \in \big[\underline{f}\big(u(x)\big), \overline{f}\big(u(x)\big)\big] & \text{a.e. } x \in M \\ u = 0 & \text{on } \partial M \\ u \not\equiv 0 \end{cases}$$

where:

i) Δ_M is the Laplace–Beltrami operator on M.
ii) λ_1 is the first eigenvalue of $-\Delta_M$ in $H_0^1(M)$.
iii) $f \in L^\infty(\mathbf{R})$.
iv) $\underline{f}(t) = \lim_{\varepsilon\searrow 0} \operatorname{ess\,inf}\{f(s); |t-s| < \varepsilon\}$, $\overline{f}(t) = \lim_{\varepsilon\searrow 0} \operatorname{ess\,sup}\{f(s); |t-s| < \varepsilon\}$.

As proved in [9] (see also [17]), the functions $\underline{f}$ and $\overline{f}$ are measurable on $\mathbf{R}$ and, if

$$F(t) = \int_0^t f(s)\,ds,$$

then the Clarke subdifferential of F satisfies

$$\partial F(t) \subset \left[\underline{f}(t), \overline{f}(t)\right] \qquad \text{a.e. } t \in \mathbf{R}.$$

Let $(g_{ij}(x))_{i,j}$ define the metric on M. We consider on $H_0^1(M)$ the functional

$$\varphi(u) = \varphi_1(u) - \varphi_2(u),$$

where

$$\varphi_1(u) = \frac{1}{2}\int_M \left(\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \lambda_1 u^2\right) dx \quad \text{and} \quad \varphi_2(u) = \int_M F(u)\,dx.$$

Notice that φ is locally Lipschitzian on $H_0^1(M)$. Indeed, it is enough to prove that φ_2 is a Lipschitzian mapping on $H_0^1(M)$, which follows from

$$|\varphi_2(u) - \varphi_2(v)| = \left|\int_M \left(\int_{u(x)}^{v(x)} f(t)\,dt\right)\right|$$
$$\leq \|f\|_{L^\infty} \cdot \|u - v\|_{L^1} \leq C_1\|u - v\|_{L^2} \leq C_2\|u - v\|_{H_0^1}.$$

By a *solution of the problem* (P1) we shall mean any critical point of the energetic functional φ.

Denote

$$f(\pm\infty) = \operatorname{ess\,lim}_{t\to\pm\infty} f(t) \qquad \text{and} \qquad F(\pm\infty) = \lim_{t\to\pm\infty} F(t).$$

Our basic hypothesis on f will be

(f1) $$f(+\infty) = F(+\infty) = 0$$

which makes the problem (P1) a Landesman–Lazer type one with strong resonance at $+\infty$.

As an application of Theorem 2 we shall prove the following sufficient condition for the existence of solutions of our problem:

Theorem 3. *Assume that* f *satisfies* (f1) *and either*

(F1) $$F(-\infty) = -\infty$$

or $-\infty < F(-\infty) \leq 0$ *and there exists* $\eta > 0$ *such that*

(F2) F *is non-negative on* $(0,\eta)$ *or* $(-\eta, 0)$.

Then the problem (P1) *has at least one solution.*

For positive values of $F(-\infty)$ it is necessary to impose additional restrictions on f. Our variant for this case is

Theorem 4. *Assume* (f1) *and* $0 < F(-\infty) < +\infty$. *Then the problem* (P1) *has at least one solution provided the following conditions are satisfied*:

$$f(-\infty) = 0$$

and

$$F(t) \leq \tfrac{1}{2}(\lambda_2 - \lambda_1)t^2 \qquad \text{for each } t \in \mathbf{R}.$$

3. Proof of Theorems 3 and 4

For the proof of Theorem 3 we shall make use of the following non-smooth variants of Lemmas 6 and 7 in [12] which can be obtained in the same manner:

Lemma 2. *Assume* $f \in L^\infty(\mathbf{R})$ *and there exist* $F(\pm\infty) \in \overline{\mathbf{R}}$. *Moreover, suppose that*

(i) $f(+\infty) = 0$ *if* $F(+\infty)$ *is finite*;

and

(ii) $f(-\infty) = 0$ *if* $F(-\infty)$ *is finite.*

Then

$$\mathbf{R} \setminus \left\{a \cdot \operatorname{meas}(M); a = -F(\pm\infty)\right\} \subset \left\{c \in \mathbf{R}; \varphi \textit{ satisfies } (\mathrm{PS})_c\right\}.$$

Lemma 3. *Assume* f *satisfies* (f1). *Then* φ *satisfies* $(\mathrm{PS})_c$, *whenever* $c \neq 0$ *and* $c < -F(-\infty) \cdot \operatorname{meas}(M)$.

Here $\operatorname{meas}(M)$ denotes the Riemannian measure of M.

Proof of Theorem 3. There are two distinct situations:

Case 1. $F(-\infty)$ is finite, that is $-\infty < F(-\infty) \leq 0$. In this case, φ is bounded from below since

$$\varphi(u) = \frac{1}{2}\int_M \left(\sum_{i,j} g_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} - \lambda_1 u^2\right) dx - \int_M F(u)\,dx$$

and, by our hypothesis on $F(-\infty)$,

$$\sup_{u \in H_0^1(M)} \int_M F(u)\,dx < +\infty.$$

Therefore,

$$-\infty < a := \inf_{u \in H_0^1(M)} \varphi(u) \leq 0 = \varphi(0).$$

Choose c small enough in order to have $F(ce_1) < 0$ (note that c may be taken positive if $F > 0$ in $(0, \eta)$ and negative if $F < 0$ in $(-\eta, 0)$). Here $e_1 > 0$ denotes the first eigenfunction of $-\Delta_M$ in $H_0^1(M)$. Hence $\varphi(ce_1) < 0$, so $a < 0$. It now follows from Lemma 3 that φ satisfies $(\mathrm{PS})_a$. The proof ends in this case by applying Corollary 1.

Case 2. $F(-\infty) = -\infty$. Then, by Lemma 2, φ satisfies $(\mathrm{PS})_c$ for each $c \neq 0$.

Let V be the orthogonal complement of the space spanned by e_1 with respect to $H_0^1(M)$, that is

$$H_0^1(M) = \mathrm{Sp}\{e_1\} \oplus V.$$

For fixed $t_0 > 0$, denote

$$V_0 = \{t_0 e_1 + v; v \in V\} \qquad \text{and} \qquad a_0 = \inf_{v \in V_0} \varphi(v).$$

Note that φ is coercive on V. Indeed, if $v \in V$, then

$$\varphi(v) \geq \tfrac{1}{2}(\lambda_2 - \lambda_1)\|v\|^2_{H^1_0} - \int_M F(v) \to +\infty \quad \text{as } \|v\|_{H^1_0} \to +\infty,$$

because the first term has a quadratic growth at infinity (t_0 being fixed), while $\int_M F(v)$ is uniformly bounded (in v), in view of the behaviour of F near $\pm\infty$. Thus, a_0 is attained, because of the coercivity of φ on V. From the boundedness of φ on $H^1_0(M)$ it follows that $-\infty < a \leq 0 = \varphi(0)$ and $a \leq a_0$.

Again, there are two posibilities:

i) $a < 0$. In this case, by Lemma 3, φ satisfies $(\mathrm{PS})_a$. Hence $a < 0$ is a critical value of φ.

ii) $a = 0 \leq a_0$. Then, either $a_0 = 0$ or $a_0 > 0$. In the first case, as we have already remarked, a_0 is attained. Thus, there is some $v \in V$ such that

$$0 = a_0 = \varphi(t_0 e_1 + v).$$

Hence, $u = t_0 e_1 + v \in H^1_0(M) \setminus \{0\}$ is a critical point of φ, that is a solution of (P1).

If $a_0 > 0$, notice that φ satisfies $(\mathrm{PS})_b$ for each $b \neq 0$. Since $\lim_{t\to+\infty} \varphi(te_1) = 0$, we may apply Theorem 2 for $X = H^1_0(M)$, $Y = V$, $Z = \mathrm{Sp}\{e_1\}$, $f = \varphi$, $z = t_0 e_1$. Thus φ has a critical value $c \geq a_0 > 0$. □

Proof of Theorem 4. If V has the same signification as above, let

$$V_+ = \{te_1 + v; t > 0, v \in V\}.$$

It will be sufficient to show that the functional φ has a non-zero critical point. To do this, we shall make use of two different arguments.

If $u = te_1 + v \in V_+$ then

$$\varphi(u) = \tfrac{1}{2}\int_M (|\nabla v|^2 - \lambda_1 v^2) - \int_M F(te_1 + v). \tag{6}$$

In view of the boundedness of F it follows that

$$-\infty < a_+ := \inf_{u \in V_+} \varphi(u) \leq 0.$$

We analyse two distinct situations:

Case 1. $a_+ = 0$. To prove that φ has a critical point, we use the same arguments as in the proof of Theorem 3 (the second case). More precisely, for some fixed $t_0 > 0$ we define in the same way V_0 and a_0. Obviously, $a_0 \geq 0 = a_+$, since $V_0 \subset V_+$. The proof follows from now on the same ideas as in Case 2 of Theorem 3, by considering the two distinct situations $a_0 > 0$ and $a_0 = 0$.

Case 2. $a_+ < 0$. Let $u_n = t_n e_1 + v_n$ be a minimizing sequence of φ in V_+. The proof is divided into three steps:

Step 1. *The sequence* $(v_n)_n$ *is bounded.* Indeed, arguing by contradiction and using the coercivity of φ on V, the boundedness of F and the definion of V_+ we obtain

$$\limsup_{n\to\infty} \varphi(u_n) = +\infty,$$

which is a contradiction since

$$\lim_{n\to\infty} \varphi(u_n) = a_+ < 0.$$

Step 2. *The sequence* $(u_n)_n$ *is bounded.* To prove this, it suffices to show that $(t_n)_n$ is a bounded real sequence. Arguing again by reductio ad absurdum, we apply the Lebesgue dominated convergence theorem to φ_2. We obtain, by using (f1),

$$\lim_{n\to\infty} \varphi_2(u_n) = 0,$$

which leads to

$$\liminf_{n\to\infty} \varphi(u_n) \geq 0,$$

a contradiction.

Step 3. It follows that there exists $w \in H_0^1(M)$, more exactly $w \in \overline{V}_+$, such that, going eventually to a subsequence,

$$\begin{aligned} u_n &\rightharpoonup w \qquad \text{weakly in } H_0^1(M), \\ u_n &\to w \qquad \text{strongly in } L^2(M), \\ u_n &\to w \qquad \text{a.e.} \end{aligned}$$

Applying again the Lebesgue dominated convergence theorem we get

$$\lim_{n\to\infty} \varphi_2(u_n) = \varphi_2(w).$$

On the other hand,

$$\varphi(w) \leq \liminf_{n\to\infty} \varphi_1(u_n) - \lim_{n\to\infty} \varphi_2(u_n) = \liminf_{n\to\infty} \varphi(u_n) = a_+.$$

It follows that, necessarily, $\varphi(w) = a_+ < 0$. Since the boundary of V_+ is V and

$$\inf_{u\in V} \varphi(u) = 0,$$

we conclude that w is a local minimum of φ on V_+ and $w \in V_+$. □

4. Semilinear elliptic problems near resonance

Under the same hypotheses as above about the manifold M, let f be a measurable function defined on $M \times \mathbf{R}$ such that

$$|f(x,t)| \le C(1+|t|^p) \qquad \text{a.e. } (x,t) \in M \times \mathbf{R}, \tag{7}$$

where C is a suitable positive constant and $1 < p < (m+2)/(m-2)$ (if $m > 2$) and $1 < p < \infty$ (if $m = 1, 2$).

Let us consider on the space $L^{p+1}(M)$ the functional

$$\psi(u) = \int_M \int_0^{u(x)} f(x,t)\, dt\, dx.$$

Firstly we observe that ψ is locally Lipschitzian. Indeed, the growth condition (7) and the Hölder inequality lead to

$$|\psi(u) - \psi(v)| \le C' \big\{ [\text{meas}(M)]^{p/(p+1)} + \max_{w \in U} \|w\|_{L^{p+1}}^{p/(p+1)} \cdot \|u - v\|_{L^{p+1}} \big\},$$

where U is an open ball which contains both u and v. Let $F(x,t) = \int_0^t f(x,s)\, ds$ and

$$\underline{f}(x,t) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,inf}\{f(x,s); |t-s| < \varepsilon\}$$
$$\overline{f}(x,t) = \lim_{\varepsilon \searrow 0} \operatorname{ess\,sup}\{f(x,s);\ |t-s| < \varepsilon\}.$$

We make the following assumptions:

$$\lim_{t \to 0} \operatorname{ess\,sup} \Big| \frac{f(x,t)}{t} \Big| < \lambda_1, \qquad \text{uniformly for } x \in M \tag{8}$$

and there exist some $\mu > 2$ and $r > 0$ such that

$$\mu F(x,t) \le \begin{cases} t\underline{f}(x,t), & \text{a.e. } (x,t) \in M \times [r, +\infty), \\ t\overline{f}(x,t), & \text{a.e. } (x,t) \in M \times (-\infty, -r], \end{cases} \tag{9}$$

$$f(x,t) \ge 1 \qquad \text{a.e. } (x,t) \in M \times [r, +\infty). \tag{10}$$

Theorem 5. *Under the hypotheses* (7), (8), (9), (10), *the multivalued elliptic problem*

$$\begin{cases} -\Delta_M u(x) \in \big[\underline{f}\big(x, u(x)\big), \overline{f}\big(x, u(x)\big)\big], & \text{a.e. } x \in M, \\ u = 0, & \text{on } \partial M \end{cases} \tag{P2}$$

has at least a non-trivial solution in $H_0^1(M) \cap W^{2,q}(M)$, *where* q *is the conjugated exponent of* $(p+1)$.

Proof. We consider in the space $H_0^1(M)$ the mapping

$$\varphi(u) = \tfrac{1}{2}\|\Delta_M\|_{L^2}^2 - \psi(u).$$

Taking into account our hypotheses, it follows that φ is locally Lipschitz.

To prove our statement it suffices to show that φ has a critical point $u_0 \in H_0^1(M)$ which corresponds to a positive critical value. Indeed, it is obvious that

$$\partial\varphi(u) = -\Delta_M u - \partial\psi|_{H_0^1(M)}(u) \qquad \text{in } H^{-1}(M).$$

If u_0 would be a critical point of φ then there is some $w \in \partial\psi|_{H_0^1(M)}(u_0)$ such that

$$-\Delta_M u_0 = w \qquad \text{in } H^{-1}(M).$$

But $w \in L^q(M)$. By a classical argument concerning elliptic regularity it follows that $u_0 \in W^{2,q}(M)$ and u_0 is a solution of (P2).

To prove that φ has a critical point we shall apply Corollary 1. To do this, we shall prove that φ satisfies (3), (4) and the Palais–Smale condition.

Verification of (3). Obviously, $\varphi(0) = 0$. On the other hand,

$$\varphi(te_1) = \tfrac{1}{2}\lambda_1 t^2\|e_1\|_{L^2}^2 - \psi(te_1) \le \tfrac{1}{2}\lambda_1 t^2\|e_1\|_{L^2}^2 - \frac{C_1}{\mu}t^\mu \int_M e_1^\mu + C_2 t\int_M e_1 < 0,$$

for t big enough. Thus, for t found above, we can choose $v = te_1$ to ensure the validity of (3).

Verification of (4). By using (8) and the growth condition (7) we get two constants $0 < C_3 < \lambda_1$ and $C_4 > 0$ so that, for almost all $(x,t) \in M \times \mathbf{R}$,

$$|f(x,t)| \le C_3|t| + C_4|t|^p.$$

By the Poincaré inequality and the Sobolev embedding theorem it follows that, for each $u \in H_0^1(M)$,

$$\psi(u) \le \frac{C_3}{2}\int_M u^2 + \frac{C_4}{p+1}\int_M |u|^{p+1} \le \frac{C_3}{2\lambda_1}\ \|\nabla u\|_{L^2}^2 + C'\ \|\nabla u\|_{L^2}^{p+1},$$

where C' is a positive constant. Hence

$$\varphi(u) \ge \Big(\frac{1}{2} - \frac{C_3}{2\lambda_1}\Big)\|\nabla u\|_{L^2}^2 - C'\|\nabla u\|_{L^2}^{p+1} \ge \alpha > 0,$$

for $R > 0$ small enough and each $u \in H_0^1(M)$ with $\|\nabla u\|_{L^2} = R$. Notice that we can choose R so that $R < \|v\|$, for v found above.

Verification of the Palais–Smale condition. Let $(u_n)_n$ be a sequence in $H_0^1(M)$ such that

$$\sup_n |\varphi(u_n)| < +\infty \tag{11}$$

$$\lim_{n\to\infty} \lambda(u_n) = 0. \tag{12}$$

It follows from the definition of λ that there exists $w_n \in \partial\psi_{|H_0^1(M)} \subset L^q(M)$ such that

$$-\Delta_M u_n - w_n \to 0 \qquad \text{in } H^{-1}(M). \tag{13}$$

It follows easily from (7) that the mapping F is locally bounded in the variable t uniformly with respect to x and (9) leads to

$$\mu F\big(x, u(x)\big) \le \begin{cases} u(x)\underline{f}\big(x, u(x)\big) + C & \text{a.e. on } [u \ge 0] \\ u(x)\overline{f}\big(x, u(x)\big) + C & \text{a.e. on } [u \le 0] \end{cases}$$

where u is an arbitrary measurable function defined on M while C is a constant which does not depend on u. So, for each $u \in M$ and $w \in \partial\psi(u)$,

$$\begin{aligned} \psi(u) &= \int_{[u\ge 0]} F\big(x, u(x)\big)\,dx + \int_{[u\le 0]} F\big(x, u(x)\big)\,dx \\ &\le \frac{1}{\mu}\int_{[u\ge 0]} u(x)\underline{f}\big(x, u(x)\big)\,dx + \int_{[u\le 0]} u(x)\overline{f}\big(x, u(x)\big)\,dx + C|\operatorname{meas}(M)|. \end{aligned}$$

Then, by Theorem 5 in [17], one has

$$\psi(u) \le \int_M u(x)w(x)\,dx + C',$$

for each $u \in H_0^1(M)$ and every $w \in \partial\psi(u)$.

We prove in what follows that the sequence $(u_n)_n$ contains a weak convergent subsequence in $H_0^1(M)$. Indeed,

$$\begin{aligned} \varphi(u_n) &= \tfrac{1}{2}\int_M |\nabla u_n|^2 - \psi(u_n) \\ &= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_M |\nabla u_n|^2 + \frac{1}{\mu}\langle -\Delta_M u_n - w_n, u_n\rangle + \frac{1}{\mu}\langle w_n, u_n\rangle - \psi(u_n) \\ &\ge \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\int_M |\nabla u_n|^2 + \frac{1}{\mu}\langle -\Delta_M u_n - w_n, u_n\rangle - C'. \end{aligned}$$

Since $\big(\varphi(u_n)\big)_n$ is a bounded sequence, it follows from the above relation that $(u_n)_n$ is bounded in $H_0^1(M)$. Passing eventually to a subsequence, $(u_n)_n$ is weakly convergent to $u \in H_0^1(M)$. Since the inclusion $H_0^1(M) \subset L^{p+1}(M)$ is compact, passing to another subsequence, we can suppose that (u_n) converges in $L^{p+1}(M)$. Since ψ is a Lipschitz function on the bounded subsets of $L^{p+1}(M)$, it follows that (w_n) is bounded in $L^q(M)$. On the other hand, by using

$$\|\nabla u_n\|_{L^2}^2 = \int_M \nabla u_n \cdot \nabla u + \int_M w_n(u_n - u) + \langle -\Delta_M u_n - w_n, u_n - u\rangle_{H^{-1},H^1}$$

it follows that

$$\|\nabla u_n\|_{L^2} \to \|\nabla u\|_{L^2}.$$

Hence,

$$\|\nabla u_n\|_{H_0^1} \to \|\nabla u\|_{H_0^1}.$$

Our conclusion follows easily by the above relation from

$$u_n \rightharpoonup u \qquad \text{in } H_0^1(M)$$

and the fact that $H_0^1(M)$ is a Hilbert space. So,

$$u_n \to u \qquad \text{in } H_0^1(M). \ \square$$

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