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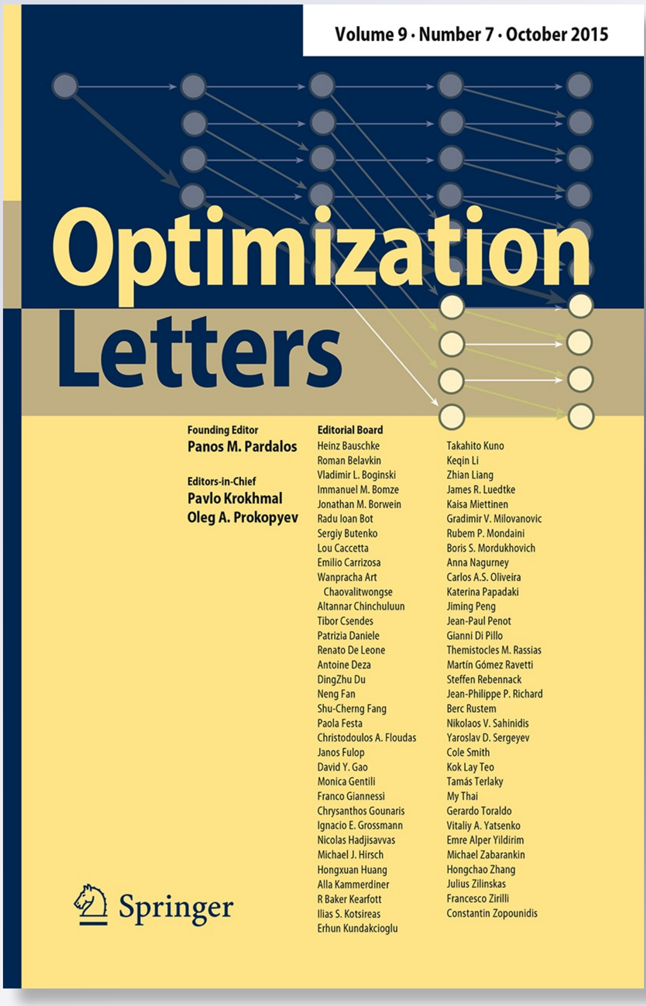


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ORIGINAL PAPER

# Facility location in normed linear spaces

**Sorin Rădulescu · Diana-Olimpia Alexandrescu · Vicenţiu D. Rădulescu**

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**Abstract** We study the Fermat–Torricelli problem in the framework of normed linear spaces by using some ingredients of convex analysis and optimization. Several general formulations of the Fermat–Torricelli problem are presented. Sufficient conditions for the existence and uniqueness of the minimum point are formulated. Existence conditions for the minimum point are related to reflexivity assumptions on the normed space. Uniqueness conditions are related to strict convexity assumptions on the normed space. In the second part of the paper we study the Fermat problem subject to constraints in the plane and on the sphere.

**Keywords** Facility location · Constrained facility location · Minimum point · Convex function · Reflexive Banach space · Fermat problem

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## 1 Introduction

Fermat challenged Torricelli with the following problem: "*given three points in the plane, find the minimizer of the sum of distances to these three points*". Torricelli found several solutions of this problem. The minimizer is a point called the Fermat–Torricelli point.

In 1909, Weber formulated in [24] a generalization of Fermat's problem in economic terms, multiplying the distances by some scalars, which are the costs of the distances. In Weber's problem the costs for transportation net that connects a central point to the branches of the manufacturing facilities were minimized. A great interest for the Fermat–Torricelli problem appeared after Weber's generalization was published. This can be seen in the papers of Haikimi [12], Kuhn [17], Perreur and Thisse [22], Wendel and Hurter [26] and Witzgall [28]. New results were obtained in the optimization problems in the case of different parameters such as natural factors that include the relief, natural resources, human or economic factors were taken into account.

The Fermat–Torricelli problem, as well as its generalizations, still draw the attention of many mathematicians as they are mathematically interesting and have a wide range of applications including those connected to network optimization and wireless communications.

Some of the reference books that contain surveys on the Fermat–Torricelli problem are Boltyanski et al. [3], Drezner et al. [10], Rădulescu et al. [16] and Mordukhovich and Nam [21].

In [25] Weiszfeld proposed an algorithm that approximates the minimum point for the generalized Fermat problem for $n$ given points. Martelli [19] and Sokolowski [23] studied generalized methods to find the solution for the weighted Fermat–Weber problem. Klamroth [15] was concerned with the facility location problem with barriers. Such barriers, as rivers, mountains or motorways are usually met in practice. In this sense, there are presented structural results and also algorithms for this problem of non-convex optimization, which depends on the distance function and the number and the location of the passages on the barrier. Mordukhovich and Nam [20,21] developed new applications of the variational analysis to the following optimization problem: "*given n closed subsets in a Banach space, find a point such that the sum of distances to these subsets is minimum*". Mordukhovich and Nam considered that the study of this generalized problem is of great interest and has many applications in location theory and in optimal networks. Several authors studied the facility location problem on a sphere [2,8,13,29]. Drezner and Wesolowsky [8] formulated the following problem: "*given n points on a sphere, find a point such that the weighted sum of distances on the geodesics to the n given points is minimum* ". They proposed some algorithms of Weiszfeld type whose convergence to the global minimum is not proved.

The following papers studied the Fermat–Torricelli problem in normed linear spaces: Alexandrescu [1], Durier and Michelot [9,11] and Vesely [27]. Durier and Michelot in [9,11] studied geometric properties of the Fermat–Weber point in spaces with scalar product, and Vesely studied generalized Chebyschev centers in [27]. Alexandrescu [1] studied the Fermat point problem for a system of $n$ distinct points in Hilbert spaces. The author established the existence and uniqueness of the solution and the location of the Fermat point in the convex hull of the given $n$ points. There are given closed form formulas for the Fermat point in the case of three and four distinct points. Iterative methods for finding the Fermat point were developed.

In the paper of Brimberg and Love [5], the Fermat–Torricelli problem is generalized replacing the sum of distances from the point $x$ to the points $a_1, \dots, a_n$ in $\mathbb{R}^N$ with the function

$$f(x) = \sum_{i=1}^{n} \phi_i(\|x - a_i\|), \quad x \in \mathbb{R}^N. \tag{1}$$

The generalized Fermat–Torricelli problem consists in minimization of the function (1). The results obtained in Brimberg and Love [5] are in the setting of $\mathbb{R}^N$ with the $l_p$ norm.

In the case $\phi_i(t) \equiv w_i t$, $i = 1, \dots, n$ the generalized problem is the Weber problem. In the case when in (1) all functions $\phi_i$ are equal to the function $\phi(t) = 2\arcsin(t/2)$, we obtain the Fermat–Torricelli problem on the unit sphere of a Hilbert space. In this problem one considers the distance between two points $x$ and $y$ on the unit sphere to be the geodesic length that connect the points. Thus

$$d(x, y) = \phi(\|x - y\|) = 2\arcsin(\|x - y\|/2).$$

In the present paper we complement the results from Brimberg and Love [5] by formulating a more general Fermat–Torricelli problem in normed linear spaces of arbitrary dimension and we obtain conditions for existence and uniqueness of the minimizer. More precisely if $A_1, \dots, A_n \in L(E, F)$ are invertible operators we denote

$$d_i(x, y) = \|A_i x - A_i y\|, \quad x, y \in E, i = 1, \dots, n.$$

Let $a_1, \dots, a_n$ be points in $F$. We study the more general facility location problem, that is to find the minimizer of the function:

$$f(x) = \sum_{i=1}^{n} \varphi_i(d_i(x, A^{-1} a_i)), \quad x \in E. \tag{2}$$

This is an extension of the Fermat–Torricelli problem. We introduced the operators $A_i$, $i = 1, \dots, n$ with the purpose of characterizing the anisotropy of the space. They lead to an useful extension of the Fermat–Torricelli problem that generate more generalized structures of the space $E$.

In Sects. 3 and 4 is considered the Fermat–Torricelli problem with constraints on a hyperplane and on a sphere of a Hilbert space. This issues are considered here for the first time.

## 2 Existence of the minimum point

Let $E$, $F$ be normed linear spaces, $L(E, F)$ is the set of linear continuous operators from $E$ to $F$, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$, $i = 1, 2, \dots, n$, $n \in \mathbb{N}$. We consider the distinct points $a_1, a_2, \dots, a_n \in F$, $A_i \in L(E, F)$, $i = 1, \dots, n$ and $f : E \to \mathbb{R}$, given by

$$f(x) = \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|). \tag{3}$$

Our purpose is to find conditions on the spaces $E$ and $F$, on the functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ and on the operators $A_i$, $i = 1, \ldots, n$ such that the point of global minimum of the function $f$ exists. We also give conditions under which this point is unique. We start with some preliminary results.

**Lemma 2.1** *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}$ be a convex, increasing function. Then $\varphi$ is continuous.*

*Proof* It is known that a convex function on an interval $[a, b]$, where $a < b$, is continuous on $(a, b)$. As $\varphi$ is increasing, $\varphi(+0)$ exists. We prove that $\varphi(0) = \varphi(+0)$. Indeed, for every $\lambda \in (0, 1)$ and for every $x > 0$ we have

$$\varphi(\lambda x) = \varphi((1 - \lambda)0 + \lambda x) \leq (1 - \lambda)\varphi(0) + \lambda\varphi(x).$$

Letting $x \to 0+$ we obtain

$$\varphi(+0) \leq (1 - \lambda)\varphi(0) + \lambda\varphi(+0), \quad \forall\lambda \in (0, 1).$$

Hence

$$(1 - \lambda)\varphi(+0) \leq (1 - \lambda)\varphi(0).$$

Thus, $\varphi(+0) \leq \varphi(0)$. But $\varphi$ is increasing, so $\varphi(0) \leq \varphi(+0)$. We deduce that $\varphi(0) = \varphi(+0)$. □

**Lemma 2.2** *Let $\varphi_1, \varphi_2, \ldots, \varphi_n : \mathbb{R}_+ \to \mathbb{R}$ be real functions, $A_i \in L(E, F), i = 1, 2, \ldots, n$, $f : E \to \mathbb{R}$ be given by relation (3) with the properties:*

(i) *$\varphi_i$ is increasing, $i = 1, \ldots, n$;*
(ii) *for all $j \in \{1, \ldots, n\}$ there exists $c_j \in (0, \infty)$ such that*

$$\|A_j x\| \geq c_j \|x\|, \quad \forall x \in E;$$

(iii) $\lim_{t \to \infty} \left( \sum_{i=1}^{n} \varphi_i \right)(t) = \infty.$

*Then* $\lim_{\|x\| \to \infty} f(x) = \infty.$

*Proof* Let $t_0 = \max_{1 \leq j \leq n} \|a_j\|$. For all $x \in E$ with $\|x\| \geq \frac{t_0}{\min c_j}$ we have

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|) \geq \sum_{i=1}^{n} \varphi_i(\|A_i x\| - \|a_i\|) \\
&\geq \sum_{i=1}^{n} \varphi_i \left( \min_{1\leq j\leq n} \|A_j x\| - \max_{1\leq j\leq n} \|a_j\| \right) \\
&= \sum_{i=1}^{n} \varphi_i \left( \min_{1\leq j\leq n} \|A_j x\| - t_0 \right) \geq \sum_{i=1}^{n} \varphi_i \left( \left( \min_{1\leq j\leq n} c_j \right) \cdot \|x\| - t_0 \right).
\end{aligned}
$$

This relation implies that $\lim_{\|x\|\to\infty} f(x) = \infty$. □

**Lemma 2.3** *Let $\varphi_1, \varphi_2, \ldots, \varphi_n : \mathbb{R}_+ \to \mathbb{R}$ be increasing functions, $A_i \in L(E, F)$, $i \in \{1, 2, \ldots, n\}$. Suppose there exists $k \in \{1, 2, \ldots, n\}$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty$ and $A_k$ is invertible. Then:*

$$\lim_{\|x\|\to\infty} f(x) = \infty.$$

*Proof* Note that there exists a constant $L > 0$ such that $\|x\| \geq L$ implies:

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|) \geq \sum_{i\neq k} \varphi_i(0) + \varphi_k(\|A_k x - a_k\|) \\
&\geq \sum_{i\neq k} \varphi_i(0) + \varphi_k(\|A_k x\| - \|a_k\|).
\end{aligned}
$$

As $\lim_{\|x\|\to\infty} \|A_k x\| = \infty$ results that $\lim_{\|x\|\to\infty} f(x) = \infty$.

The next result is a strong reason why convex functions and reflexive spaces are so important in optimization. □

**Theorem 2.1** *Let $E$ be a reflexive Banach space and let $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ ($i = 1, 2, \ldots, n$) be convex and increasing functions for which there exists $k \in \{1, 2, \ldots, n\}$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty$ and $A_k$ is invertible. Then the function $f$ defined by (3) admits at least one minimum point.*

*Proof* Let $x_0 \in E$ and define $M = \{x \in E \mid f(x) \leq f(x_0)\}$. Note that:

- (i) $M$ is closed, since $f$ is continuous;
- (ii) $M$ is convex, since $f$ is a sum of convex functions;
- (iii) $M$ is bounded, since $\lim_{\|x\|\to\infty} f(x) = \infty$ (see Lemma 2.3).

Now, since $M \subset E$ and $E$ is a reflexive space, we deduce that $M$ is weakly compact. The continuity and convexity of $f$ imply that $f$ is weakly lower semicontinuous on the weakly compact set $M$, hence $f$ attains its minimum, which is also its global minimum. □

## 3 Uniqueness of the minimum point

We start with some auxiliary results that provide sufficient conditions for the strict convexity of the function $f$ defined by (3).

**Definition 3.1** A norm is strictly convex if and only if $\|x+y\| = \|x\| + \|y\|$ and $x \neq 0$, $y \neq 0$ imply that there exists $\lambda \geq 0$ such that $y = \lambda x$.

**Lemma 3.1** *Let $E$ and $F$ be normed linear spaces such that the norm on $F$ is strictly convex, $A_i \in L(E,F)$, $i = 1,\dots,n$, let $\varphi_i : \mathbb{R}_+ \to \mathbb{R} (i = 1,2,\dots,n)$ be convex increasing functions, $a_1, a_2, \dots, a_n \in F$, and*

$$f(x) = \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|), \qquad x \in E.$$

*If there exists $k \in \{1,2,\dots,n\}$ such that $\varphi_k$ is strictly increasing and strictly convex and $Ker A_k = \{0\}$, then $f$ is strictly convex.*

*Proof* Note that $f$ is continuous and convex since it is the sum of convex functions. Suppose that $f$ is not strictly convex. This implies that there are $x, y \in E$, $x \neq y$, such that

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y).$$

We have

$$2\sum_{i=1}^{n} \varphi_i\left(\left\|A_i\left(\frac{x+y}{2}\right) - a_i\right\|\right) = \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|) + \sum_{i=1}^{n} \varphi_i(\|A_i y - a_i\|),$$

which is equivalent to:

$$\sum_{i=1}^{n}\left[\varphi_i(\|A_i x - a_i\|) + \varphi_i(\|A_i y - a_i\|) - 2\varphi_i\left(\left\|\frac{A_i x - a_i + A_i y - a_i}{2}\right\|\right)\right] = 0.$$

All the terms of the above sum are nonnegative, and $\varphi_i$ are convex functions, for all $i \in \{1,\dots,n\}$. Therefore

$$\varphi_i(\|A_i x - a_i\|) + \varphi_i(\|A_i y - a_i\|) = 2\varphi_i\left(\left\|\frac{A_i x - a_i + A_i y - a_i}{2}\right\|\right), \quad i = 1,\dots,n.$$

We have

$$\begin{aligned} 2\varphi_k\left(\left\|\frac{A_k x - a_k + A_k y - a_k}{2}\right\|\right) &\leq 2\varphi_k\left(\frac{\|A_k x - a_k\| + \|A_k y - a_k\|}{2}\right) \\ &\leq \varphi_k(\|A_k x - a_k\|) + \varphi_k(\|A_k y - a_k\|) = 2\varphi_k\left(\left\|\frac{A_k x - a_k + A_k y - a_k}{2}\right\|\right). \end{aligned}$$

We used here the fact that $\varphi_k$ is increasing and convex.

Since $\varphi_k$ is strictly increasing, and hence injective, it follows that

$$2\left\|\frac{A_k x - a_k + A_k y - a_k}{2}\right\| = \|A_k x - a_k\| + \|A_k y - a_k\|.$$

Combining this relation with the fact that $\varphi_k$ is strictly convex, we deduce that $\|A_k y - a_k\| = \|A_k x - a_k\|$.

Denote $A_k y - a_k = u$, $A_k x - a_k = v$. Since $\|u + v\| = \|u\| + \|v\|$, $\|u\| = \|v\|$ and $\|\cdot\|$ is a strictly convex norm, it follows that $A_k y - a_k = A_k x - a_k$ and hence $A_k x = A_k y$ which implies that $x = y$ as $Ker A_k = \{0\}$. This contradicts the fact that $f$ is not strictly convex. □

**Lemma 3.2** *Let $F$ be a strictly convex space, $\varphi_i$ be convex, strictly increasing functions, $\forall i \in \{1, \ldots, n\}$, $A_i$ be invertible operators, $\forall i \in \{1, \ldots, n\}$ and $A_i^{-1} a_i$, $(i = 1, \ldots, n)$ be noncolinear points. Then the function defined by relation (3) is strictly convex.*

*Proof* Suppose that $f$ is not strictly convex. Then there exist $x, y \in E$, $x \neq y$ such that $2f\left(\frac{x+y}{2}\right) = f(x) + f(y)$. This implies that there exists $k \in \{1, 2, \ldots, n\}$ such that $x$, $y$, $A_k^{-1} a_k$ are non-collinear.

From relation

$$\begin{aligned} 0 &= f(x) + f(y) - 2f\left(\frac{x+y}{2}\right) \\ &= \sum_{i=1}^{n} \left[\varphi_i(\|A_i x - a_i\|) + \varphi_i(\|A_i y - a_i\|) - 2\varphi_i\left(\|A_i \frac{x+y}{2} - a_i\|\right)\right] \end{aligned} \tag{4}$$

and $\varphi_i$ convex, $i = 1, 2, \ldots, n$, results that all the terms are nonnegative and hence they are null. Therefore we further obtain that

$$2\varphi_i\left(\left\|A_i \frac{x+y}{2} - a_i\right\|\right) = \varphi_i(\|A_i x - a_i\|) + \varphi_i(\|A_i y - a_i\|), \quad \forall i = 1, \ldots, n, \tag{5}$$

but

$$\begin{aligned} 2\varphi_i\left(\|A_i \frac{x+y}{2} - a_i\|\right) &= 2\varphi_i\left(\left\|\frac{A_i x - a_i + A_i y - a_i}{2}\right\|\right) \\ &\leq 2\varphi_i\left(\frac{\|A_i x - a_i\| + \|A_i y - a_i\|}{2}\right) \\ &\leq \varphi_i(\|A_i x - a_i\|) + \varphi_i(\|A_i y - a_i\|), \quad \forall i = 1, \ldots, n. \end{aligned}$$

From relation (5) results that the first term is equal to the last one and hence all are equal.

As $\varphi_i$ is strictly increasing for $i = 1, 2, \ldots, n$ we obtain:

$$\|A_i x - a_i + A_i y - a_i\| = \|A_i x - a_i\| + \|A_i y - a_i\|, \quad i = 1, 2, \ldots, n.$$

From here and from the fact that $F$ is strictly convex, results that the points $0$, $A_k x - a_k$, $A_k y - a_k$ are collinear, which is equivalent to $a_k$, $A_k x$, $A_k y$ collinear, which is further equivalent to $A_k^{-1} a_k$, $x$, $y$ are collinear. This is a contradiction.

The following theorem establishes the existence and uniqueness of the minimum of $f$. □

**Theorem 3.1** *Let $E$ be a reflexive Banach space, $F$ be a strictly convex Banach space and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, \ldots, n)$ be increasing convex functions. Assume that there exists $k \in \{1, \ldots, n\}$ such that $\varphi_k$ is strictly increasing, strictly convex and $Ker A_k = \{0\}$. Then $f$ admits a point of global minimum and this is unique.*

*Proof* The existence is proved in Sect. 2. In the following, we deal with the uniqueness of the minimum point. Suppose that there exist two distinct points of global minimum, say $x_1$ and $x_2$. Let

$$L = \{x \in E |\ f(x) = f(x_1)\}.$$

Then $x_2 \in L$ and $(x_1 + x_2)/2 \in L$. Using the strict convexity of $f$ we deduce that

$$f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2} = f(x_1).$$

This relation contradicts the fact that $x_1$ is a global minimum point . Therefore the minimum of $f$ is unique. □

The following theorem establishes similar conditions, which ensure the existence and uniqueness of the global minimum point for the function $f$.

**Theorem 3.2** *Let $E$ be a reflexive Banach space, $F$ be a strictly convex Banach space and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ be convex, strictly increasing functions. Suppose that $A_i^{-1} a_i \in E$, $i = 1, \ldots, n$, are non-collinear. Then there exists and is unique a minimum point of $f$.*

The proof of this theorem is similar to the proof of the theorem above.

*Remark* (i) The functions $\varphi_i(t) = c_i t^{p_i}$, $p_i \in [1, \infty)$, $c_i > 0$ satisfy the conditions in the above theorem.

(ii) The spaces $L^p(\mu)$ with $1 < p < \infty$ are reflexive and strictly convex.

**Theorem 3.3** *Let $E$ be a Hilbert space, $A_i \in L(E, E)$, $i = 1, \ldots, n$, be linear continuous and invertible operators with the property that for all $i = 1, \ldots, n$ there exists $\beta_i > 0$ such that $A_i^* A_i = \beta_i I$, $i \in \{1, \ldots, n\}$. Let the functions $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$, $i \in \{1, \ldots, n\}$ be differentiable on $(0, \infty)$ with $\varphi_i'(t) > 0$, $\forall t \in (0, \infty)$ and $i \in \{1, \ldots, n\}$ and $a_1, \ldots, a_n \in E$.*

*If $\overline{x}$ is a global minimum point of the function*

$$f(x) = \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|)$$

*then* $\overline{x} \in co(A_1^{-1}a_1, A_2^{-1}a_2, \ldots, A_n^{-1}a_n)$.

*Proof* If $\overline{x} \in \{A_1^{-1}a_1, \ldots, A_n^{-1}a_n\}$ then, obviously $\overline{x} \in co\{A_1^{-1}a_1, \ldots, A_n^{-1}a_n\}$.

We consider

$$\nabla f(x) = \sum_{i=1}^{n} \varphi_i'(\|A_i x - a_i\|) \frac{A_i^* A_i x - A_i^* a_i}{\|A_i x - a_i\|}$$

for all $x \in E \backslash \{A_1^{-1}a_1, \ldots, A_n^{-1}a_n\}$.

If $\overline{x} \in E \backslash \{A_1^{-1}a_1, \ldots, A_n^{-1}a_n\}$ is the minimum point of $f$, then

$$\nabla f(\overline{x}) = 0.$$

Denote by $\alpha_i = \dfrac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|}, \quad i \in \{1, \ldots, n\}$.

Notice that $\alpha_i \in (0, \infty)$, $i = 1, \ldots, n$ and

$$\nabla f(\overline{x}) = \sum_{i=1}^{n} \alpha_i (A_i^* A_i \overline{x} - A_i^* a_i) = 0$$

which is equivalent to

$$\sum_{i=1}^{n} \alpha_i A_i^* A_i \overline{x} = \sum_{i=1}^{n} \alpha_i A_i^* a_i.$$

Taking into account that $A_i^* A_i = \beta_i I$, $i = 1, \ldots, n$, then the relation below becomes:

$$\sum_{i=1}^{n} \alpha_i \beta_i \overline{x} = \sum_{i=1}^{n} \alpha_i A_i^* a_i$$

which is further equivalent to:

$$\overline{x} \sum_{i=1}^{n} \alpha_i \beta_i = \sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i$$

where from we get

$$\overline{x} = \frac{\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i}{\sum_{i=1}^{n} \alpha_i \beta_i}$$

that is $\overline{x} \in co(A_1^{-1}a_1, \ldots, A_n^{-1}a_n)$. □

## 4 Facility location problem with constraint in the hyperplane

### 4.1 Existence of the minimum

As so far, we present a qualitative property in the most general context. The following result is a refined version of Corollary 3.23 in Brezis [4].

**Theorem 4.1** *Let $E$ be a reflexive Banach space with $dim E \geq 2$, $\varphi_1, \ldots, \varphi_n : \mathbb{R}_+ \to \mathbb{R}$ be convex, increasing functions with the property that there exists $k \in \{1, \ldots, n\}$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty$, $A_k$ be invertible, $\eta \in E'$, $\eta \neq 0$, $d \in \mathbb{R}$. Consider the set $H = \{x \in E | \eta(x) = d\}$. Then the following properties hold:*

(i) $\lim_{\|x\|\to\infty} f(x) = \infty$
(ii) *there exists $\overline{x} \in H$ such that $f(x) \geq f(\overline{x})$, for all $x \in H$.*

*Proof* Let $x_0 \in H$. Since $dim E \geq 2$ there exists $y \in E \backslash H$ such that $y \neq 0$ and $\eta(y) = 0$. Then $x_0 + ty \in H$ for all $t \in \mathbb{R}$ and

$$\lim_{t\to\infty} \|x_0 + ty\| = \infty.$$

We take into account that

$$\lim_{\|x\|\to\infty, x\in H} \|A_k x\| = \infty.$$

Then, there exists $L > 0$ such that $\|x\| \geq L$ implies

$$f(x) = \sum_{i=1}^{n} \varphi_i(\|A_i x - a_i\|) \geq \sum_{i\neq k} \varphi_i(0) + \varphi_k(\|A_k x\| - \|a_k\|),$$

where from we get $\lim_{\|x\|\to\infty, x\in H} f(x) = \infty$.

(ii) We define the convex, closed and bounded set:

$$M = \{x \in H | f(x) \leq f(x_0)\}.$$

Using (i) and the hypothesis that $E$ is reflexive and $f$ is continuous and convex, we deduce that $f$ is lower semicontinuous and hence weakly lower semicontinuous on $M$, that is weakly compact. This implies that $f$ attains its minimum on $H$. □

*Remark* Let $E$ be a Hilbert space. If $c \in E$ with $\|c\| = 1$ and $d \in \mathbb{R}$, we define the following closed hyperplane

$$H = \{x \in E | \langle c, x\rangle = d\}. \tag{6}$$

If $a \in E$, let $\pi_H a$ denote the projection of $a$ on $H$. It follows that (see also Corollary 5.4 in Brezis [4])

$$\pi_H a = a + (d - \langle c, a\rangle) \cdot c.$$

**Theorem 4.2** *Let $E$ be a Hilbert space, $a_1, \ldots, a_n \in E$ be distinct points, $\varphi_1, \ldots, \varphi_n : \mathbb{R}_+ \to \mathbb{R}$ be convex, increasing and differentiable functions on $(0, \infty)$ and $\varphi_i'(t) > 0$, for all $t \in (0, \infty)$, $i \in \{1, \ldots, n\}$. Suppose that there exists $k \in \{1, \ldots, n\}$ such that $\lim_{t\to\infty} \varphi_k(t) = \infty$. We consider $A_i \in L(E, E)$ invertible and suppose that there exists $\beta_i > 0$, $\forall i \in \{1, \ldots, n\}$ such that $A_i^* A_i = \beta_i I$, $i \in \{1, \ldots, n\}$. If $H$ is given by relation (6), then there exists $\overline{x} \in H$ such that $f(x) \geq f(\overline{x})$, for all $x \in H$ and $\overline{x} \in co(\pi_H A_1^{-1} a_1, \ldots, \pi_H A_n^{-1} a_n)$.*

*Proof* The existence of the minimum point results from Theorem 4.1. We define the function:

$$F(x, \lambda) = f(x) - \lambda(\langle c, x\rangle - d), \quad x \in E, \lambda \in \mathbb{R}.$$

For $x \in E \backslash \{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\}$ and $\lambda \in \mathbb{R}$, we have

$$\nabla_x F(x, \lambda) = \sum_{i=1}^{n} \varphi_i'(\|A_i x - a_i\|) \cdot \frac{A_i^* A_i x - A_i^* a_i}{\|A_i x - a_i\|} - \lambda c.$$

Then $\nabla_x F(\overline{x}, \lambda) = 0$. Hence we have

$$\sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} \cdot A_i^* A_i \overline{x} = \sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} A_i^* a_i + \lambda c.$$

Multiplying the relation above by $c$ we get:

$$\sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} \cdot \langle A_i^* A_i \overline{x}, c\rangle = \sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} \cdot \langle c, A_i^* a_i\rangle + \lambda.$$

We further obtain:

$$\lambda = \sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} \langle A_i^* A_i \overline{x} - A_i^* a_i, c\rangle,$$

end hence

$$\sum_{i=1}^{n} \frac{\varphi_i'(\|A_i\overline{x} - a_i\|)}{\|A_i\overline{x} - a_i\|} \cdot A_i^* A_i \overline{x} = \sum_{i=1}^{n} \frac{\varphi_i'(\|A_i\overline{x} - a_i\|)}{\|A_i\overline{x} - a_i\|} A_i^* a_i - \sum_{i=1}^{n} \frac{\varphi_i'(\|A_i\overline{x} - a_i\|)}{\|A_i\overline{x} - a_i\|} \langle A_i^* a_i - A_i^* A_i \overline{x}, c\rangle \cdot c.$$

Denote by: $\alpha_i = \dfrac{\varphi_i'(\|A_i\overline{x} - a_i\|)}{\|A_i\overline{x} - a_i\|}, \quad \forall i = 1, \ldots, n.$

We get: $\sum_{i=1}^{n} \alpha_i A_i^* A_i \overline{x} = \sum_{i=1}^{n} \alpha_i A_i^* a_i - \sum_{i=1}^{n} \alpha_i \langle A_i^* a_i - A_i^* A_i \overline{x}, c\rangle \cdot c$ which is equivalent to

$$\sum_{i=1}^{n} \alpha_i A_i^* A_i \overline{x} = \sum_{i=1}^{n} \alpha_i A_i^* a_i - \left\langle \sum_{i=1}^{n} \alpha_i (A_i^* a_i - A_i^* A_i \overline{x}), c \right\rangle \cdot c.$$

Taking into account that $A_i^* A_i = \beta_i I$, that is $A_i^* = \beta_i A_i^{-1}$, $i = 1, \ldots, n$, we obtain:

$$\begin{aligned}
\overline{x} &= \frac{\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i - \left\langle \sum_{i=1}^{n} \alpha_i \beta_i (A_i^{-1} a_i - \overline{x}), c \right\rangle \cdot c}{\sum_{i=1}^{n} \alpha_i \beta_i} \\
&= \frac{\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i - \sum_{i=1}^{n} \alpha_i \beta_i \langle A_i^{-1} a_i - \overline{x}, c\rangle \cdot c}{\sum_{i=1}^{n} \alpha_i \beta_i} \\
&= \frac{\sum_{i=1}^{n} \alpha_i \beta_i \left( A_i^{-1} a_i - \langle A_i^{-1} a_i - \overline{x}, c\rangle \right) \cdot c}{\sum_{i=1}^{n} \alpha_i \beta_i} \\
&= \frac{\sum_{i=1}^{n} \alpha_i \beta_i \left( A_i^{-1} a_i + \langle c, \overline{x} - A_i^{-1} a_i\rangle \right) \cdot c}{\sum_{i=1}^{n} \alpha_i \beta_i}
\end{aligned}$$

$$= \frac{\sum_{i=1}^{n} \alpha_i \beta_i \left[ A_i^{-1} a_i + (d - \langle A_i^{-1} a_i, c \rangle) \cdot c \right]}{\sum_{i=1}^{n} \alpha_i \beta_i} = \frac{\sum_{i=1}^{n} \alpha_i \beta_i \pi_H A_i^{-1} a_i}{\sum_{i=1}^{n} \alpha_i \beta_i}.$$

This representation of $\overline{x}$ shows that if $\overline{x} \in E \setminus \{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\}$, then $\overline{x} \in co(\pi_H A_1^{-1} a_1, \ldots, \pi_H A_n^{-1} a_n)$, this means that the existence of a minimum point $\overline{x}$ implies that $\overline{x}$ belongs to the convex hull of the projections on the hyperplane $H$. If $\overline{x} \in \{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\}$, we suppose that $\overline{x} = A_k^{-1} a_k$, $a_k \in H$. As $A_k^{-1} a_k = \pi_H A_k^{-1} a_k$, implies that $\overline{x} \in co(\pi_H A_1^{-1} a_1, \ldots, \pi_H A_n^{-1} a_n)$. □

## 5 Conditions on the existence of the minimum point on the sphere

In this section we establish sufficient conditions for the existence of the minimum point on the unit sphere.

**Theorem 5.1** *Let $E$ be a reflexive Banach space, $a_1, \ldots, a_n \in E$ be distinct points and $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ be convex and increasing functions ($i = 1, 2, \ldots, n$). If $\widetilde{x}$ is a global minimum point for the function $f$ defined by (3) and $\|\widetilde{x}\| \geq 1$, then $f$ attains its minimum on $S = \{x \in E \,|\, \|x\| = 1\}$.*

*Proof* The existence of $\widetilde{x}$ results from Theorem 2.1. Indeed, if $\|\widetilde{x}\| = 1$ then $\widetilde{x} \in S$. Suppose that $\|\widetilde{x}\| > 1$. The set $B = \{x \in E \,|\, \|x\| \leq 1\}$ is convex, closed and bounded in the space $E$ and hence it is weakly compact. Next, since $f$ is continuous and convex, it follows that $f$ is weakly lower semicontinuous on weakly compact sets and therefore it attains its minimum in $B$. Let $\overline{x} \in B$ be the minimum point of $f$ on $B$. We consider the function

$$u(t) = f((1-t)\widetilde{x} + t\overline{x}), \quad t \in [0, 1],$$

where $\widetilde{x}$ is the point of global minimum of $f$ on $E$ and $\overline{x}$ is the minimum point of $f$ on $B$.

Let $x^*$ be the intersection of the segment $[\overline{x}, \widetilde{x}]$ with the sphere $S$. Therefore there exists $t_0 \in [0, 1]$ such that $x^* = (1-t)\widetilde{x} + t_0 \overline{x}$. We notice that $u(t_0) = f(x^*)$.

Note that the convexity of $f$ implies that $u$ is convex and $u(t) \geq u(0) = f(\widetilde{x})$ for all $t \in [0, 1]$, which implies that $u$ is increasing. Thus, $u(0) \leq u(t_0) \leq u(1)$, which implies

$$f(\widetilde{x}) \leq f(x^*) \leq f(\overline{x}) \leq f(x^*).$$

We further conclude that $f(x^*) = f(\overline{x})$, that is, $f$ attains its minimum on $S$ at $x^* \in S$. □

### 5.1 Facility location problem subject to constraint on the sphere

If $E$ is a normed space, we denote by $S = \{x \in E | \|x\| = 1\}$. For $x \in E$ and $x \neq 0$, we define the projection of $x$ on the sphere $S$ by $\pi_S x = \dfrac{x}{\|x\|}$.

**Theorem 5.2** *Let $E$ be a Hilbert space, $a_i \in E$, $i = 1, \ldots, n$ be distinct points, $\varphi_i : \mathbb{R}_+ \to \mathbb{R}$ be convex, continuous and differentiable functions on $(0, \infty)$ with $\varphi_i'(t) > 0$ on $(0, \infty)$, for all $i = 1, \ldots, n$, $A_i \in L(E, E)$ be linear, continuous and invertible operators, $i = 1, \ldots, n$, with the following properties:*

(1) $\langle A_i^{-1} a_i, A_j^{-1} a_j \rangle > 0, \quad \forall i, j \in \{1, \ldots n\}$;

(2) *for all $i = 1, \ldots, n$, there exists $\beta_i > 0$ such that $A_i^* A_i = \beta_i I$, where $I$ is the unit operator. If $\overline{x} \in S$ has the property that*

$$f(\overline{x}) \leq f(x), \quad \forall x \in S,$$

*then*

$$\overline{x} \in \pi_S co \left( A_1^{-1} a_1, \ldots, A_n^{-1} a_n \right).$$

*Proof* Consider the auxiliary function

$$F(x, \lambda) = f(x) - \lambda(\|x\|^2 - 1), \quad x \in E, \ \lambda \in \mathbb{R}. \tag{7}$$

Then

$$\nabla_x F(x, \lambda) = \sum_{i=1}^{n} \frac{\varphi_i'(\|A_i x - a_i\|)}{\|A_i x - a_i\|} (A_i^* A_i x - A_i^* a_i) - 2\lambda x,$$
$$x \in E \backslash \{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\}, \quad \lambda \in \mathbb{R}.$$

If $\overline{x} \in S \backslash \{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\}$, then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla_x F(\overline{x}, \lambda) = 0,$$

which is further equivalent to

$$\sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} (A_i^* A_i \overline{x} - A_i^* a_i) - 2\lambda \overline{x} = 0. \tag{8}$$

Multiplying relation (8) by $\overline{x}$ we obtain

$$\sum_{i=1}^{n} \frac{\varphi_i'(\|A_i \overline{x} - a_i\|)}{\|A_i \overline{x} - a_i\|} \left( \beta_i - \langle A_i^{-1} a_i \beta_i, \overline{x} \rangle \right) = 2\lambda. \tag{9}$$

We denote by $\alpha_i = \dfrac{\varphi_i'(\|A_i\overline{x} - a_i\|)}{\|A_i\overline{x} - a_i\|}$, $\quad i = 1, \ldots, n$. Then we have:

$$\sum_{i=1}^{n} \alpha_i \beta_i \left(1 - \langle A_i^{-1} a_i, \overline{x}\rangle\right) = 2\lambda$$

and hence we have

$$\overline{x} = \frac{\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i}{\sum_{i=1}^{n} \alpha_i \beta_i - 2\lambda} = \frac{\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i}{\sum_{i=1}^{n} \alpha_i \beta_i \langle A_i^{-1} a_i, \overline{x}\rangle}. \tag{10}$$

Equation (10) is equivalent to:

$$\overline{x} \cdot \sum_{i=1}^{n} \alpha_i \beta_i \langle A_i^{-1} a_i, \overline{x}\rangle = \sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i. \tag{11}$$

Multiplying Eq. (11) by $\alpha_j \beta_j A_j^{-1} a_j$ and summing after $j$ we obtain:

$$\sum_{i,j=1}^{n} \alpha_i \beta_i \langle A_i^{-1} a_i, \overline{x}\rangle \alpha_j \beta_j \langle A_j^{-1} a_j, \overline{x}\rangle = \sum_{i,j=1}^{n} \alpha_i \beta_i \alpha_j \beta_j \langle A_i^{-1} a_i, A_j^{-1} a_j\rangle. \tag{12}$$

We have:

$$\left(\sum_{i=1}^{n} \alpha_i \beta_i \langle A_i^{-1} a_i, \overline{x}\rangle\right)^2 = \left\|\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i\right\|^2,$$

wherefrom we deduce that :

$$\left\langle \overline{x}, \sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i \right\rangle = \epsilon \left\|\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i\right\|, \quad \epsilon \in \{-1, 1\}. \tag{13}$$

From Eq. (13) and taking into account that $\left\|\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i\right\|^2 = \sum_{i,j=1}^{n} \alpha_i \beta_i \alpha_j \beta_j \langle A_i^{-1} a_i, A_j^{-1} a_j\rangle > 0$ results that:

$$\overline{x} = \frac{\epsilon \sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i}{\left\|\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i\right\|}.$$

Let $u = \dfrac{\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i}{\left\| \sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i \right\|}$. This means that $\overline{x} = \epsilon u$.

We prove that

$$\|u - A_i^{-1} a_i\| \leq \| - u - A_i^{-1} a_i\|, \quad \forall i \in \{1, \ldots, n\}. \tag{14}$$

We square relation (14) and we obtain:

$$\|u\|^2 - 2\left\langle u, A_i^{-1} a_i \right\rangle + \|A_i^{-1} a_i\|^2 \leq \|u\|^2 + 2\left\langle u, A_i^{-1} a_i \right\rangle + \|A_i^{-1} a_i\|^2$$

which is further equivalent to:

$$\left\langle u, A_i^{-1} a_i \right\rangle \geq 0, \quad i \in \{1, \ldots, n\}.$$

The functions $\varphi_i$ are increasing and hence

$$\varphi_i \left( \|u - A_i^{-1} a_i\| \right) \leq \varphi_i \left( \| - u - A_i^{-1} a_i\| \right), \quad i = 1, \ldots, n.$$

From here results that $f(u) \leq f(-u)$, but $\overline{x} \in \{-u, u\}$, therefore we have:

$$\overline{x} = \frac{\displaystyle\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i}{\left\| \displaystyle\sum_{i=1}^{n} \alpha_i \beta_i A_i^{-1} a_i \right\|}.$$

This shows that $\overline{x}$ is the projection on the sphere of the point belonging to $co\left\{A_1^{-1} a_1, \ldots A_n^{-1} a_n\right\}$. If $\overline{x} \in \left\{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\right\}$, then $\overline{x} = A_k^{-1} a_k$, which implies that $\overline{x} = \pi_S A_k^{-1} a_k$ and hence we have $\overline{x} \in \pi_S co\left\{A_1^{-1} a_1, \ldots, A_n^{-1} a_n\right\}$. The proof is now complete. □

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