MULTIPLICITY OF CONCENTRATING SOLUTIONS FOR (p, q)-SCHRÖDINGER EQUATIONS WITH LACK OF COMPACTNESS

VINCENZO AMBROSIO AND VICENŢIU D. RĂDULESCU *

Abstract. We study the multiplicity of concentrating solutions for the following class of (p, q)-Laplacian problems:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = f(u) + \gamma u^{q^*-1} \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0 \text{ in } \mathbb{R}^N, \end{cases}$$

where $\varepsilon > 0$ is a small parameter, $\gamma \in \{0, 1\}$, $1 < p < q < N$, $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent, $\Delta_s u = \mathrm{div}(|\nabla u|^{s-2}\nabla u)$, with $s \in \{p, q\}$, is the s-Laplacian operator, $V : \mathbb{R}^N \to \mathbb{R}$ is a positive continuous potential such that $\inf_{\partial\Lambda} V > \inf_\Lambda V$ for some bounded open set $\Lambda \subset \mathbb{R}^N$, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity with subcritical growth. The main results are obtained by combining minimax theorems, penalization technique and Ljusternik-Schnirelmann category theory. We also provide a multiplicity result for a supercritical version of the above problem by combining a truncation argument with a Moser-type iteration. As far as we know, all these results are new.

1. Introduction

In this paper we investigate the multiplicity and concentration phenomenon of positive solutions for the following (p, q)-Laplacian problem:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = f(u) + \gamma u^{q^*-1} \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $\gamma \in \{0, 1\}$, $1 < p < q < N$, $q^* = \frac{Nq}{N-q}$ is the critical Sobolev exponent, $\Delta_s u = \mathrm{div}(|\nabla u|^{s-2}\nabla u)$, with $s \in \{p, q\}$, is the s-Laplacian operator, the potential $V : \mathbb{R}^N \to \mathbb{R}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ are continuous functions.

The problem (1.1) is related to the study of stationary solutions of reaction diffusion systems of the form

$$u_t = \mathrm{div}[D(u)\nabla u] + c(x, u), \quad D(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}. \tag{1.2}$$

This equation has a wide range of applications in physical and related sciences, e.g. in biophysics, plasma physics, and chemical reaction design; see [14]. In such applications, the function u in (1.2) describes a concentration, $\mathrm{div}[D(u)\nabla u]$ corresponds to the diffusion with a diffusion coefficient $D(u)$, and the reaction term $c(x, u)$ relates to source and loss processes. Tipically, in chemical and biological applications, the reaction term $c(x, u)$ has a polynomial form with respect to the concentration u. We refer to [6, 19, 24, 25, 27, 34–37] for some existence and multiplicity results for (p, q)-Laplacian problems in bounded or unbounded domains. For completeness, we also observe

2010 *Mathematics Subject Classification.* 35A15, 35J62, 35Q55, 55M30, 35B33.

Key words and phrases. (p, q)-Laplacian problem; penalization technique; critical exponent; Ljusternik-Schnirelmann theory.

* Corresponding author: Vicenţiu D. Rădulescu (radulescu@inf.ucv.ro).

that the functional associated to the (p,q)-Laplacian operator falls in the realm of the following double-phase functional

$$\mathcal{P}_{p,q}(u;\Omega)=\int_{\Omega}|\nabla u|^{p}+a(x)|\nabla u|^{q}\,dx$$

where $\Omega\subset\mathbb{R}^N$ is an open set and $a(x)\geq 0$, introduced by Zhikov [45] to provide models for strongly anisotropic materials in the context of homogenization phenomena. We refer to [10, 16, 30–32] for some remarkable regularity results for functionals with non-standard growth of (p,q)-type.

Note that, if $p=q=2$, after rescaling, equation (1.1) reduces to the classical nonlinear Schrödinger equation

$$-\varepsilon^2\Delta u+V(x)u=f(u)+\gamma|u|^{2^*-2} \text{ in } \mathbb{R}^N, \tag{1.3}$$

for which several existence, multiplicity and concentration results of positive solutions have been established by different authors, under suitable conditions on the potential V and the nonlinearity f. In [39] Rabinowitz proved via a mountain pass argument, the existence of positive solutions of (1.3) for small $\varepsilon>0$ whenever

$$\liminf_{|x|\to\infty}V(x)>\inf_{x\in\mathbb{R}^N}V(x). \tag{1.4}$$

These solutions concentrate around the global minimum points of V when $\varepsilon\to 0$, as it was shown by Wang [43]. Later, del Pino and Felmer [17], by introducing a penalization approach, proved a localized version of the result of Rabinowitz and Wang. They assumed that V is a positive locally Hölder-continuous function and that there exists a bounded open set $\Omega\subset\mathbb{R}^N$ such that

$$\inf_{\Omega}V<\inf_{\partial\Omega}V. \tag{1.5}$$

In [3], the authors studied the existence and the concentration behavior of positive bound-state solutions to (1.3) with $\gamma=1$ and assuming that V satisfies (1.5). Cingolani and Lazzo [15], under the assumption (1.4), used Ljusternik-Schnirelmann theory to relate the multiplicity of solutions for (1.3) with $\gamma=0$, $f(u)=|u|^{p-2}u$ and $p\in(2,\frac{2N}{N-2})$, to the richness of the set of minimum points of V. On the other hand, when $p=q>1$ in (1.1), then we obtain the following class of p-Laplacian equations:

$$-\varepsilon^p\Delta_p u+V(x)|u|^{p-2}u=f(u)+\gamma|u|^{p^*-2}u \text{ in } \mathbb{R}^N, \tag{1.6}$$

which has been extensively considered in literature; see for instance [4, 5, 18, 21]. In particular, inspired by [15], Alves and Figueiredo [5] proved a multiplicity result for (1.6) whenever $\gamma=0$, assuming that V satisfies (1.4) and that f is a C^1-subcritical nonlinearity. Later, in [4] the authors extended this result by assuming del Pino-Felmer type assumptions on V. These results have been generalized in the critical case in [18,21]. Concerning the (p,q)-case, when $\gamma=0$ and $f\in C^1$ in (1.1), the authors in [6] (see also [2]) generalized the multiplicity result in [5] under the assumption (1.4) on V, while in [1] the authors dealt with a class of quasilinear problems including the (p,q)-case under the condition (1.5) on V. More recently, the multiplicity result in [6] has been improved in [9] by considering continuous nonlinearities. We also mention [8] in which a multiplicity result for a class of subcritical fractional (p,q)-Laplacian problems is proved. We emphasize that in [1,2,6,8,9] the authors focused only on the subcritical case (that is $\gamma=0$).

Particularly motivated by [1, 2, 4–6, 8, 9, 18, 21], in the first part of this paper we are interested in the multiplicity and concentration behavior as $\varepsilon\to 0$ of positive solutions to (1.1), when we assume a local condition on the potential V, f is merely continuous and $\gamma\in\{0,1\}$. More precisely, we suppose that $V\in C(\mathbb{R}^N,\mathbb{R})$ satisfies the following conditions:

(V_1) there exists $V_0 > 0$ such that $V_0 = \inf_{x\in\mathbb{R}^N} V(x)$;
(V_2) there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad 0 \in M = \{x \in \Lambda : V(x) = V_0\}.$$

Concerning the nonlinearity f, we require that $f \in C(\mathbb{R},\mathbb{R})$ fulfills the following hypotheses:

(f_1) $\lim_{|t|\to 0} \frac{|f(t)|}{|t|^{p-1}} = 0$;
(f_2) when $\gamma = 0$, there exists $\nu \in (q, q^*)$ such that $\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{\nu-1}} = 0$;
(f_2') when $\gamma = 1$, there exist $\sigma_1, \sigma_2 \in (q, q^*)$ and $\lambda > 1$ such that

$$f(t) \geq \lambda t^{\sigma_1 - 1} \quad \forall t > 0, \quad \lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{\sigma_2-1}} = 0;$$

(f_3) there exists $\vartheta \in (q, q^*)$ such that $0 < \vartheta F(t) = \vartheta \int_0^t f(\tau)\, d\tau \leq t f(t)$ for all $t > 0$;
(f_4) the map $t \mapsto \frac{f(t)}{t^{q-1}}$ is increasing for $t > 0$.

Due to the fact that we look for positive solutions to (1.1), we assume that $f(t) = 0$ for $t \leq 0$.

In order to make a precise statement let us recall that, for any closed subset Y of a topological space X, the Ljusternik-Schnirelmann category of Y in X, $\mathrm{cat}_X(Y)$, stands for the least number of closed and contractible sets in X which cover Y; see [44].

The main result of this work is stated in the following multiplicity and concentration property.

Theorem 1.1. *Assume that* $1 < p < q < N$ *and that* V *satisfies* (V_1)-(V_2). *Let*

$$M_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leq \delta\}.$$

- *When* $\gamma = 0$, *we suppose that* f *satisfies* (f_1), (f_2), (f_3), (f_4). *Then, for any* $\delta > 0$ *such that* $M_\delta \subset \Lambda$, *there exists* $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *problem* (1.1) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *positive solutions.*
- *When* $\gamma = 1$, *we suppose that* f *satisfies* (f_1), (f_2'), (f_3), (f_4). *Then there exists* $\lambda^* > 1$ *such that, for any* $\lambda \geq \lambda^*$ *and for any* $\delta > 0$ *such that* $M_\delta \subset \Lambda$, *there exists* $\varepsilon_{\delta,\lambda} > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_{\delta,\lambda})$, *problem* (1.1) *has at least* $\mathrm{cat}_{M_\delta}(M)$ *positive solutions.*

Moreover, if u_ε *denotes one of these solutions and* $x_\varepsilon \in \mathbb{R}^N$ *is a global maximum point of* u_ε, *then*

$$\lim_{\varepsilon\to 0} V(\varepsilon\, x_\varepsilon) = V_0,$$

and there exist $C_1, C_2 > 0$ *such that*

$$u_\varepsilon(x) \leq C_1 e^{-C_2|x - x_\varepsilon|} \quad \forall x \in \mathbb{R}^N.$$

The proof of Theorem 1.1 will be obtained by combining suitable variational and topological arguments inspired by [1,6,8,9,17,22]. Concerning our variational approach, as in [17], we modify in a convenient way the nonlinearity outside of the set Λ and we consider an auxiliary problem. The main feature of the corresponding modified energy functional $\mathcal{J}_\varepsilon$ is that it satisfies all the assumptions of the mountain-pass theorem [7]. Note that in the critical case, we assume λ sufficiently large in order to obtain an upper bound for the mountain-pass level c_ε. Indeed, differently from the critical problems considered in [12,18,21], we can not use cut-off functions of the extremal functions for the best constant in the Sobolev inequality, because the lack of homogenity of the (p,q)-Laplacian operator does not permit to use this trick. To circumvent this obstacle, we use a different strategy inspired by [19]; see Lemma 2.3. The estimate for c_ε combined with the concentration-compactness principle of Lions [28,29] will play a fundamental role in proving a local Palais-Smale condition for $\mathcal{J}_\varepsilon$; see Lemma 2.5. To obtain multiple solutions for the modified problem, we use a technique due to Benci and Cerami [11] based on precise comparisons between the category of some sublevel sets of $\mathcal{J}_\varepsilon$ and the category of the set M. We emphasize that f is merely continuous, so standard C^1-Nehari manifold arguments as in [1,2,4–6] do not work in our setting because the Nehari manifold associated with $\mathcal{J}_\varepsilon$ is non-differentiable. For this reason, motivated by [8,9,22], we use some variants of critical point theorems established in [41]. Clearly, the lack of homogeneity caused by the (p,q)-Laplacian operators combined with the presence of the critical exponent make our analysis more delicate and intriguing with respect to the above mentioned works, and some refined estimates will be carried out to implement our variational machinery. Finally, we need to show that, for $\varepsilon > 0$ small enough, the solutions of the modified problem are indeed solutions of the original one. Since we deal with a general class of quasilinear operators, a standard Moser iteration procedure [33] as in [3–6,18] does not work well, and then we use an appropriate De Giorgi iteration argument inspired by [1,2,23,26]. We also verify that our solutions decay exponentially at infinity by means of a comparison argument.

In the second part of this paper, we consider a supercritical version of problem (1.1). In this case, we deal with the sum of two homogeneous nonlinearities and add a new positive parameter μ. More precisely, we consider the following problem:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = u^{s-1} + \mu u^{\tau-1} \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $\varepsilon, \mu > 0$ and $1 < p < q < s < q^* < \tau$. Our multiplicity result for the supercritical case can be stated as follows.

Theorem 1.2. *Assume that (V_1)-(V_2) hold. Then there exists $\mu_0 > 0$ such that, for any for any $\mu \in (0, \mu_0)$ and for any $\delta > 0$ satisfying $M_\delta \subset \Lambda$, there exists $\varepsilon_{\delta,\mu} > 0$ such that, for any $\varepsilon \in (0, \varepsilon_{\delta,\mu})$, problem (1.7) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions. Moreover, if u_ε denotes one of these solutions and $x_\varepsilon \in \mathbb{R}^N$ is a global maximum point of u_ε, then*

$$\lim_{\varepsilon\to 0} V(\varepsilon x_\varepsilon) = V_0.$$

The main difficulty in the study of (1.7) is due to the fact that $\tau > q^*$ is supercritical, and we cannot directly use variational techniques because the corresponding functional is not well-defined on the space $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$. In order to overcome this obstacle, we use some arguments inspired by [13,20,38] which can be summarized as follows. We first truncate in a suitable way the nonlinearity on the right hand side of (1.7), so we deal with a new problem but with subcritical growth. In the light of Theorem 1.1, we know that a multiplicity result for this truncated problem is available. Then we deduce a priori bound (independent of μ) for these solutions and by using an appropriate Moser iteration technique [33], we show that, for $\mu > 0$ sufficiently small, the solutions of the truncated problem also solve the original one. We would like to point out that, since the hypotheses on V and f are different from [1,2,4–6], our arguments are totally distinct, and improve the previous results for the (p,q)-case because here we obtain multiplicity results for subcritical, critical and supercritical (p,q)-problems involving continuous nonlinearities and imposing a local condition on the potential V. Moreover, we believe that the ideas contained here can be applied in other situations to study problems driven by more general quasilinear operators, under local conditions on the potential V and the non-differentiability of the nonlinearity f.

An outline of the paper is as follows. In Section 2, we study the modified problem. In Section 3 we analyze the limiting problem associated with (1.1) and we introduce some tools needed to obtain a multiplicity result for the auxiliary problem. The Section 4 is devoted to the proof of Theorem 1.1. In the last section we deal with the supercritical problem (1.7).

Notations: Let $p \in [1,\infty]$ and $A \subset \mathbb{R}^N$ be a measurable set. We will use $|\cdot|_{L^p(A)}$ for the norm in $L^p(A)$, and $|\cdot|_p$ when $A = \mathbb{R}^N$. By $S_* = S_*(N,q) > 0$ we will denote the best constant in the Sobolev inequality related to the continuous embedding $D^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$.

2. The modified problem

We use a del Pino-Felmer penalization type approach [17] to deal with problem (1.1). Take $K > \frac{q}{p} > 1$ and $a > 0$ such that $f(a) + \gamma a^{q^*-1} = \frac{V_0}{K} a^{q-1}$. We define

$$\tilde{f}(t) = \begin{cases} f(t) + \gamma (t^+)^{q^*-1} & \text{if } t \leq a, \\ \frac{V_0}{K} t^{q-1} & \text{if } t > a, \end{cases}$$

and

$$g(x,t) = \chi_\Lambda(x)(f(t) + \gamma (t^+)^{q^*-1}) + (1-\chi_\Lambda(x))\tilde{f}(t) \quad \text{ for } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

where χ_A denotes the characteristic function of $A \subset \mathbb{R}^N$. By (f_1)-(f_4), we deduce that $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and it fulfills the following assumptions:

(g_1) $\lim_{t\to 0} \frac{g(x,t)}{t^{p-1}} = 0$ uniformly with respect to $x \in \mathbb{R}^N$,

(g_2) $g(x,t) \leq f(t) + \gamma t^{q^*-1}$ for all $x \in \mathbb{R}^N$ and $t > 0$,

(g_3) (i) $0 < \vartheta G(x,t) \leq g(x,t)t$ for all $x \in \Lambda$ and $t > 0$,
(ii) $0 < qG(x,t) \leq g(x,t)t \leq \frac{V_0}{K} t^q$ for all $x \in \Lambda^c$ and $t > 0$,

(g_4) for each $x \in \mathbb{R}^N$ the function $t \mapsto \frac{g(x,t)}{t^{q-1}}$ is increasing in $(0,\infty)$.

Let us introduce the following auxiliary problem:

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = g(\varepsilon x, u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0 \text{ in } \mathbb{R}^N. \end{cases} \tag{2.1}$$

We observe that if u_ε is a solution to (2.1) such that $u_\varepsilon(x) \leq a$ for all $x \in \Lambda_\varepsilon^c$, where $\Lambda_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$, then u_ε is also a solution to (1.1). Then we consider the functional $\mathcal{J}_\varepsilon : \mathbb{X}_\varepsilon \to \mathbb{R}$ associated to (2.1), that is

$$\mathcal{J}_\varepsilon(u) = \frac{1}{p}\|u\|^p_{V_\varepsilon,p} + \frac{1}{q}\|u\|^q_{V_\varepsilon,q} - \int_{\mathbb{R}^N} G(\varepsilon x, u)\, dx,$$

where the space

$$\mathbb{X}_\varepsilon = \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\varepsilon x)\, (|u|^p + |u|^q)\ dx < \infty \right\}$$

is endowed with the norm

$$\|u\|_{\mathbb{X}_\varepsilon} = \|u\|_{V_\varepsilon,p} + \|u\|_{V_\varepsilon,q},$$

and

$$\|u\|_{V_\varepsilon,t} = \left(|\nabla u|_t^t + \int_{\mathbb{R}^N} V(\varepsilon x)|u|^t\, dx \right)^{\frac{1}{t}} \quad \forall t \in \{p,q\}.$$

For $t \in \{p, q\}$, we set

$$\langle u, \varphi\rangle_{V_\varepsilon,t} = \int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla\varphi\, dx + \int_{\mathbb{R}^N} V(\varepsilon\, x)|u|^{t-2}u\, \varphi\, dx \quad \forall u, \varphi \in \mathbb{Y}_{V_0}.$$

Clearly, $\mathcal{J}_\varepsilon \in C^1(\mathbb{X}_\varepsilon, \mathbb{R})$ and it holds

$$\langle \mathcal{J}_\varepsilon'(u), \varphi\rangle = \langle u, \varphi\rangle_{V_\varepsilon,p} + \langle u, \varphi\rangle_{V_\varepsilon,q} - \int_{\mathbb{R}^N} g(\varepsilon\, x, u)\varphi\, dx$$

for any $u, \varphi \in \mathbb{X}_\varepsilon$. We denote by $\mathcal{N}_\varepsilon$ the Nehari manifold associated with $\mathcal{J}_\varepsilon$, namely

$$\mathcal{N}_\varepsilon = \{u \in \mathbb{X}_\varepsilon \setminus \{0\} : \langle \mathcal{J}_\varepsilon'(u), u\rangle = 0\},$$

and we set $c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u)$. Let $\mathbb{X}_\varepsilon^+ = \{u \in \mathbb{X}_\varepsilon : |\mathrm{supp}(u^+) \cap \Lambda_\varepsilon| > 0\}$ and $\mathbb{S}_\varepsilon^+ = \mathbb{S}_\varepsilon \cap \mathbb{X}_\varepsilon^+$, where $\mathbb{S}_\varepsilon = \{u \in \mathbb{X}_\varepsilon : \|u\|_{\mathbb{X}_\varepsilon} = 1\}$. Note that $\mathbb{S}_\varepsilon^+$ is an incomplete $C^{1,1}$-manifold of codimension one. Hence, $\mathbb{X}_\varepsilon = T_u\mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$ for all $u \in \mathbb{S}_\varepsilon^+$, where

$$T_u\mathbb{S}_\varepsilon^+ = \{v \in \mathbb{X}_\varepsilon : \langle u, v\rangle_{V_\varepsilon,p} + \langle u, v\rangle_{V_\varepsilon,q} = 0\}.$$

The next lemma shows that $\mathcal{J}_\varepsilon$ possesses a mountain pass geometry [7].

Lemma 2.1. *The functional $\mathcal{J}_\varepsilon$ has the following properties:*

(i) *There exist $\alpha, \rho > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ for $\|u\|_{\mathbb{X}_\varepsilon} = \rho$.*

(ii) *There exists $e \in \mathbb{X}_\varepsilon$ with $\|e\|_{\mathbb{X}_\varepsilon} > \rho$ and $\mathcal{J}_\varepsilon(e) < 0$.*

Proof. (i) Fix $\zeta \in (0, V_0)$. Using (g_2), (f_1), (f_2) and (f_2'), we can find $C_\zeta > 0$ such that

$$|g(x, t)| \leq \zeta|t|^{p-1} + C_\zeta|t|^{q^*-1} \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.2}$$

Consequently,

$$\mathcal{J}_\varepsilon(u) \geq \frac{1}{p}\|u\|^p_{V_\varepsilon,p} + \frac{1}{q}\|u\|^q_{V_\varepsilon,q} - \frac{\zeta}{p}|u|^p_p - \frac{C_\zeta}{q^*}|u|^{q^*}_{q^*} \geq C_1\|u\|^p_{V_\varepsilon,p} + \frac{1}{q}\|u\|^q_{V_\varepsilon,q} - \frac{C_\zeta}{q^*}|u|^{q^*}_{q^*}.$$

Choosing $\|u\|_{\mathbb{X}_\varepsilon} = \rho \in (0, 1)$ and using $1 < p < q$, we have $\|u\|_{V_\varepsilon,p} < 1$ and thus $\|u\|^p_{V_\varepsilon,p} \geq \|u\|^q_{V_\varepsilon,p}$. Recalling that $a^t + b^t \geq C_t(a + b)^t$ for all $a, b \geq 0$ and $t > 1$, and using the Sobolev embedding $W^{1,r}(\mathbb{R}^N) \hookrightarrow L^\kappa(\mathbb{R}^N)$ for $\kappa \in [r, r^*]$, we find

$$\mathcal{J}_\varepsilon(u) \geq C_2\|u\|^q_{\mathbb{X}_\varepsilon} - \frac{C_\zeta}{q^*}|u|^{q^*}_{q^*} \geq C_2\|u\|^q_{\mathbb{X}_\varepsilon} - C_3\|u\|^{q^*}_{\mathbb{X}_\varepsilon}.$$

Then there exists $\alpha > 0$ such that $\mathcal{J}_\varepsilon(u) \geq \alpha$ for $\|u\|_{\mathbb{X}_\varepsilon} = \rho$.
(ii) Fix $\varphi \in C_c^\infty(\mathbb{R}^N)$ such that $\varphi \geq 0$, $\varphi \not\equiv 0$ and $\mathrm{supp}(\varphi) \subset \Lambda_\varepsilon$. By (f_3) we deduce that $F(t) \geq At^\vartheta - B$ for all $t > 0$. Then, for all $t > 0$, we have

$$\mathcal{J}_\varepsilon(t\varphi) \leq \frac{t^p}{p}\|\varphi\|^p_{V_\varepsilon,p} + \frac{t^q}{q}\|\varphi\|^q_{V_\varepsilon,q} - At^\vartheta\int_{\Lambda_\varepsilon} (\varphi)^\vartheta\, dx + B|\mathrm{supp}(\varphi) \cap \Lambda_\varepsilon| \to -\infty \quad \text{as } t \to \infty$$

thanks to $\vartheta > q > p$. Then we take $e = t\varphi$ with $t > 0$ sufficiently large. □

The next two results are very useful since they allow us to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}_\varepsilon^+$.

Lemma 2.2. *The following properties hold:*

(i) *For each $u \in \mathbb{X}_\varepsilon^+$, let $h_u : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{J}_\varepsilon(tu)$. Then, there is a unique $t_u > 0$ such that $h_u'(t) > 0$ for all $t \in (0, t_u)$ and $h_u'(t) < 0$ for all $t \in (t_u, \infty)$.*

(ii) *There exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}_\varepsilon^+$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$, there is a positive constant $C_\mathbb{K}$ such that $t_u \leq C_\mathbb{K}$ for any $u \in \mathbb{K}$.*

(iii) *The map* $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ *given by* $\hat{m}_\varepsilon(u) = t_u u$ *is continuous and* $m_\varepsilon = \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ *is a homeomorphism between* $\mathbb{S}_\varepsilon^+$ *and* $\mathcal{N}_\varepsilon$*. Moreover,* $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}}$.

(iv) *If there is a sequence* $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ *such that* $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$ *then* $\|m_\varepsilon(u_n)\|_{\mathbb{X}_\varepsilon} \to \infty$ *and* $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof. (i) From the proof of Lemma 2.1, we see that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Then there exists a global maximum point $t_u > 0$ for h_u in $[0,\infty)$ such that $h_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. We claim that $t_u > 0$ is unique. Let $t_1, t_2 > 0$ be such that $h_u'(t_1) = h_u'(t_2) = 0$. Consequently,

$$\left(\frac{1}{t_1^{q-p}} - \frac{1}{t_2^{q-p}}\right)\|u\|_{V_\varepsilon,p}^p = \int_{\mathbb{R}^N}\left(\frac{g(\varepsilon x, t_1 u)}{(t_1 u)^{q-1}} - \frac{g(\varepsilon x, t_2 u)}{(t_2 u)^{q-1}}\right)u^q dx,$$

which combined with (g_4) and $q > p$ yields $t_1 = t_2$.

(ii) Let $u \in \mathbb{S}_\varepsilon^+$. Using (i), we can find $t_u > 0$ such that $h_u'(t_u) = 0$, that is

$$t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} g(\varepsilon x, t_u u)\, u\, dx.$$

Fix $\xi > 0$. By (g_2), (f_1), (f_2), (f_2') and the Sobolev embedding $W^{1,r}(\mathbb{R}^N) \hookrightarrow L^\kappa(\mathbb{R}^N)$ for $\kappa \in [r, r^*]$, we have

$$t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q \le \int_{\mathbb{R}^3} g(\varepsilon x, t_u u)\, u\, dx \le \xi t_u^{p-1}\|u\|_{V_\varepsilon,p}^p + C_\xi t_u^{q^*-1}\|u\|_{V_\varepsilon,q}^{q^*}.$$

Taking $\xi > 0$ sufficiently small and recalling that $1 = \|u\|_{\mathbb{X}_\varepsilon} \ge \|u\|_{V_\varepsilon,q}$, we find

$$Ct_u^{p-1}\|u\|_{V_\varepsilon,p}^p + t_u^{q-1}\|u\|_{V_\varepsilon,q}^q \le Ct_u^{q^*-1}\|u\|_{V_\varepsilon,q}^{q^*} \le Ct_u^{q^*-1}.$$

Now, if $t_u \le 1$, then $t_u^{q-1} \le t_u^{p-1}$, and using the facts that $1 = \|u\|_{\mathbb{X}_\varepsilon} \ge \|u\|_{V_\varepsilon,p}$ and $q > p$ imply that $\|u\|_{V_\varepsilon,p}^p \ge \|u\|_{V_\varepsilon,p}^q$, we can see that

$$Ct_u^{q-1} = Ct_u^{q-1}\|u\|_{\mathbb{X}_\varepsilon}^q \le t_u^{q-1}(C\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \le t_u^{q-1}(C\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \le Ct_u^{q^*-1}.$$

Thanks to $q^* > q$, we can find $\tau > 0$, independent of u, such that $t_u \ge \tau$.

When $t_u > 1$, then $t_u^{q-1} > t_u^{p-1}$, and noting that $1 = \|u\|_{\mathbb{X}_\varepsilon} \ge \|u\|_{V_\varepsilon,p}$ and $q > p$ imply $\|u\|_{V_\varepsilon,p}^p \ge \|u\|_{V_\varepsilon,p}^q$, we obtain

$$Ct_u^{p-1} = Ct_u^{p-1}\|u\|_{\mathbb{X}_\varepsilon}^q \le t_u^{p-1}(C\|u\|_{V_\varepsilon,p}^q + \|u\|_{V_\varepsilon,q}^q) \le t_u^{p-1}(C\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q) \le Ct_u^{q^*-1}.$$

Since $q^* > q > p$, there exists $\tau > 0$, independent of u, such that $t_u \ge \tau$.
Now, let $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ be a compact set, and assume by contradiction that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{K}$ such that $t_n = t_{u_n} \to \infty$. Then there exists $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. From (ii) of Lemma 2.1, we have that

$$\mathcal{J}_\varepsilon(t_n u_n) \to -\infty. \tag{2.3}$$

On the other hand, if $v \in \mathcal{N}_\varepsilon$, by $\langle \mathcal{J}_\varepsilon'(v), v\rangle = 0$ and (g_3), we have that

$$\mathcal{J}_\varepsilon(v) = \mathcal{J}_\varepsilon(v) - \frac{1}{\vartheta}\langle \mathcal{J}_\varepsilon'(v), v\rangle \ge \tilde{C}(\|v\|_{V_\varepsilon,p}^p + \|v\|_{V_\varepsilon,q}^q).$$

Taking $v_n = t_{u_n} u_n \in \mathcal{N}_\varepsilon$ in the above inequality, we find

$$\mathcal{J}_\varepsilon(t_n u_n) \ge \tilde{C}(\|v_n\|_{V_\varepsilon,p}^p + \|v_n\|_{V_\varepsilon,q}^q).$$

Since $\|v_n\|_{\mathbb{X}_\varepsilon} = t_n \to \infty$ and $\|v_n\|_{\mathbb{X}_\varepsilon} = \|v_n\|_{V_\varepsilon,p} + \|v_n\|_{V_\varepsilon,q}$, we can use (2.3) to get a contradiction.

(iii) Let us observe that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (i), for each $u \in \mathbb{X}_\varepsilon^+$ there is a unique $\hat{m}_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in \mathbb{X}_\varepsilon^+$. Otherwise, we have $|\,\mathrm{supp}(u^+) \cap \Lambda_\varepsilon| = 0$, and by (g3)-(ii) we deduce that

$$\begin{aligned}\|u\|_{V_\varepsilon,p}^p + \|u\|_{V_\varepsilon,q}^q &= \int_{\mathbb{R}^N} g(\varepsilon x, u)\, u\, dx = \int_{\Lambda_\varepsilon^c} g(\varepsilon x, u^+)\, u^+\, dx \\ &\leq \frac{1}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u|^q\, dx \leq \frac{1}{K}\|u\|_{V_\varepsilon,q}^q,\end{aligned} \tag{2.4}$$

which is impossible since $K > 1$ and $u \neq 0$. Therefore, $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} \in \mathbb{S}_\varepsilon^+$ is well defined and continuous. From $u \in \mathbb{S}_\varepsilon^+$

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_{\mathbb{X}_\varepsilon}} = \frac{u}{\|u\|_{\mathbb{X}_\varepsilon}} = u$$

we infer that m_ε is a bijection. To prove that $\hat{m}_\varepsilon : \mathbb{X}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ is continuous, let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon^+$ and $u \in \mathbb{X}_\varepsilon^+$ such that $u_n \to u$ in $\mathbb{X}_\varepsilon$. Since $\hat{m}(tu) = \hat{m}(u)$ for all $t > 0$, we may assume that $\|u_n\|_{\mathbb{X}_\varepsilon} = \|u\|_{\mathbb{X}_\varepsilon} = 1$ for all $n \in \mathbb{N}$. By (ii), there exists $t_0 > 0$ such that $t_n = t_{u_n} \to t_0$. Using $t_n u_n \in \mathcal{N}_\varepsilon$, that is

$$t_n^p \|u_n\|_{V_\varepsilon,p}^p + t_n^q \|u_n\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} g(\varepsilon x, t_n u_n)\, t_n u_n\, dx,$$

and passing to the limit as $n \to \infty$ we obtain

$$t_0^p \|u\|_{V_\varepsilon,p}^p + t_0^q \|u\|_{V_\varepsilon,q}^q = \int_{\mathbb{R}^N} g(\varepsilon x, t_0 u)\, t_0 u\, dx,$$

which means that $t_0 u \in \mathcal{N}_\varepsilon$. From (i), $t_u = t_0$ and this means that $\hat{m}_\varepsilon(u_n) \to \hat{m}_\varepsilon(u)$ in $\mathbb{X}_\varepsilon^+$. Thus, $\hat{m}_\varepsilon$ and m_ε are continuous maps.
(iv) Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}_\varepsilon^+$ be a sequence such that $\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Then for each $v \in \partial\mathbb{S}_\varepsilon^+$ and $n \in \mathbb{N}$, we have $u_n^+ \leq |u_n - v|$ a.e. in Λ_ε. Therefore, by (V_1), (V_2) and Sobolev embedding, we can see that for each $r \in [p, q^*]$ there exists $C_r > 0$ such that

$$|u_n^+|_{L^r(\Lambda_\varepsilon)} \leq \inf_{v\in\partial\mathbb{S}_\varepsilon^+} |u_n - v|_{L^r(\Lambda_\varepsilon)} \leq C_r \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_{\mathbb{X}_\varepsilon} \quad \forall n \in \mathbb{N}.$$

By virtue of (g_2), (f_1), (f_2), (f_2'), (g_3)-(ii), and $q > p$, we get, for all $t > 0$,

$$\begin{aligned}\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\, dx &= \int_{\Lambda_\varepsilon^c} G(\varepsilon x, tu_n)\, dx + \int_{\Lambda_\varepsilon} G(\varepsilon x, tu_n)\, dx \\ &\leq \frac{V_0}{Kq}\int_{\Lambda_\varepsilon^c} t^q |u_n|^q\, dx + \int_{\Lambda_\varepsilon} \left(F(tu_n) + \frac{\gamma}{q^*}(tu_n^+)^{q^*}\right) dx \\ &\leq \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx + C_1 t^p \int_{\Lambda_\varepsilon} (u_n^+)^p dx + C_2 t^{q^*}\int_{\Lambda_\varepsilon} (u_n^+)^{q^*} dx \\ &\leq \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx + C_p' t^p \mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^p + C_\nu' t^{q^*}\mathrm{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^{q^*}.\end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^N} G(\varepsilon x, tu_n)\, dx \leq \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\, dx + o_n(1). \tag{2.5}$$

Now, we note that $K > \frac{q}{p} > 1$, and that $1 = \|u_n\|_{\mathbb{X}_\varepsilon} \geq \|u_n\|_{V_\varepsilon,p}$ implies that $\|u_n\|^p_{V_\varepsilon,p} \geq \|u_n\|^q_{V_\varepsilon,p}$. Then, for all $t > 1$, we obtain that

$$\begin{aligned}
&\frac{t^p}{p}\|u_n\|^p_{V_\varepsilon,p} + \frac{t^q}{q}\|u_n\|^q_{V_\varepsilon,q} - \frac{t^q}{Kp}\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,dx \\
&= \frac{t^p}{p}\|u_n\|^p_{V_\varepsilon,p} + \frac{t^q}{q}|\nabla u_n|^q_q + t^q\left(\frac{1}{q} - \frac{1}{Kp}\right)\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\,dx \\
&\geq C_1 t^p\|u_n\|^p_{V_\varepsilon,p} + C_2 t^q\|u_n\|^q_{V_\varepsilon,q} \\
&\geq C_1 t^p\|u_n\|^q_{V_\varepsilon,p} + C_2 t^q\|u_n\|^q_{V_\varepsilon,q} \\
&\geq C_1 t^p\|u_n\|^q_{V_\varepsilon,p} + C_2 t^p\|u_n\|^q_{V_\varepsilon,q} \\
&\geq C_3 t^p(\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^q = C_3 t^p.
\end{aligned} \tag{2.6}$$

Bearing in mind the definition of $m_\varepsilon(u_n)$ and using (2.5), (2.6), we find

$$\liminf_{n\to\infty} \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \geq \liminf_{n\to\infty} \mathcal{J}_\varepsilon(tu_n) \geq C_3 t^p \quad \forall t > 1,$$

which yields

$$\liminf_{n\to\infty}\left\{\frac{1}{p}\|m_\varepsilon(u_n)\|^p_{V_\varepsilon,p} + \frac{1}{q}\|m_\varepsilon(u_n)\|^q_{V_\varepsilon,q}\right\} \geq \liminf_{n\to\infty} \mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \geq C_3 t^p \quad \forall t > 1.$$

By sending $t \to \infty$, we get $\|m_\varepsilon(u_n)\|_{\mathbb{X}_\varepsilon} \to \infty$ and $\mathcal{J}_\varepsilon(m_\varepsilon(u_n)) \to \infty$ as $n \to \infty$. This completes the proof of the lemma. □

Remark 2.1. *There exists $\kappa > 0$, independent of ε, such that $\|u\|_{\mathbb{X}_\varepsilon} \geq \kappa$ for all $u \in \mathcal{N}_\varepsilon$. Indeed, if $u \in \mathcal{N}_\varepsilon$, we can use (g_2), (f_1), (f_2), (f_2') and the Sobolev embeddings to see that*

$$\|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} = \int_{\mathbb{R}^N} g(\varepsilon x, u)\, u\, dx \leq \zeta|u|^p_p + C_\zeta|u|^{q^*}_{q^*} \leq \frac{\zeta}{V_0}\|u\|^p_{V_\varepsilon,p} + C'_\zeta\|u\|^{q^*}_{V_\varepsilon,q}.$$

Choosing $\zeta \in (0, V_0)$ we find $\|u\|_{V_\varepsilon,q} \geq \kappa = (C'_\zeta)^{-\frac{1}{q^*-q}}$ which implies that $\|u\|_{\mathbb{X}_\varepsilon} \geq \|u\|_{V_\varepsilon,q} \geq \kappa$.

Now we define the maps $\hat{\psi}_\varepsilon : \mathbb{X}^+_\varepsilon \to \mathbb{R}$ and $\psi_\varepsilon : \mathbb{S}^+_\varepsilon \to \mathbb{R}$ given by by $\hat{\psi}_\varepsilon(u) = \mathcal{J}_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon = \hat{\psi}_\varepsilon|_{\mathbb{S}^+_\varepsilon}$, respectively. From Lemma 2.2 and arguing as in the proofs of Proposition 9 and Corollary 10 in [41], we may obtain the following result.

Proposition 2.1. *The following properties hold:*

(a) $\hat{\psi}_\varepsilon \in C^1(\mathbb{X}^+_\varepsilon, \mathbb{R})$ *and*

$$\langle \hat{\psi}'_\varepsilon(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_{\mathbb{X}_\varepsilon}}{\|u\|_{\mathbb{X}_\varepsilon}}\langle \mathcal{J}'_\varepsilon(\hat{m}_\varepsilon(u)), v\rangle \quad \forall u \in \mathbb{X}^+_\varepsilon,\ \forall v \in \mathbb{X}_\varepsilon.$$

(b) $\psi_\varepsilon \in C^1(\mathbb{S}^+_\varepsilon, \mathbb{R})$ *and*

$$\langle \psi'_\varepsilon(u), v\rangle = \|m_\varepsilon(u)\|_{\mathbb{X}_\varepsilon}\langle \mathcal{J}'_\varepsilon(m_\varepsilon(u)), v\rangle \quad \forall v \in T_u\mathbb{S}^+_\varepsilon.$$

(c) *If $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence for ψ_ε, then $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$, then $\{m^{-1}_\varepsilon(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_c$ sequence for ψ_ε.*

(d) *u is a critical point of ψ_ε if, and only if, $m_\varepsilon(u)$ is a critical point for $\mathcal{J}_\varepsilon$. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}^+_\varepsilon} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u).$$

Remark 2.2. *As in [41], we have the following variational characterization of the infimum of $\mathcal{J}_\varepsilon$ over $\mathcal{N}_\varepsilon$:*

$$c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} \mathcal{J}_\varepsilon(u) = \inf_{u\in\mathbb{X}_\varepsilon^+} \max_{t>0} \mathcal{J}_\varepsilon(tu) = \inf_{u\in\mathbb{S}_\varepsilon^+} \max_{t>0} \mathcal{J}_\varepsilon(tu) > 0.$$

Moreover, if $c'_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} \mathcal{J}_\varepsilon(\gamma(t))$, where $\Gamma_\varepsilon = \{\gamma \in C([0,1],\mathbb{X}_\varepsilon) : \gamma(0) = 0$ and $\mathcal{J}_\varepsilon(\gamma(1)) < 0\}$, then we can argue as in [17, 39, 44] to verify that $c_\varepsilon = c'_\varepsilon$.

Next we prove a very useful upper bound for the minimax level c_ε for the case $\gamma = 1$.

Lemma 2.3. *Let $\gamma = 1$. Then it holds $0 < c_\varepsilon < \frac{1}{N} S_*^{\frac{N}{q}}$.*

Proof. The proof is inspired by an argument found in the proof of Lemma 2.2 in [19]. For simplicity, we take $\varepsilon = 1$ and we use the notations $\mathcal{J}_1 = \mathcal{J}$, $\mathbb{X}_1 = \mathbb{X}$, $\Lambda_1 = \Lambda$, $c_1 = c$, $\Gamma_1 = \Gamma$. Let $e \in \mathbb{X}$ be the function given in Lemma 2.1-(ii). Note that $\mathrm{supp}(e) \subset \Lambda$, $e \geq 0$ and $e \not\equiv 0$ in $\mathbb{R}^N$. Accordingly, $\langle \mathcal{J}'(t_\lambda e), t_\lambda e\rangle = 0$, that is

$$t_\lambda^p \|e\|_{V_1,p}^p + t_\lambda^q \|e\|_{V_1,q}^q = \int_\Lambda f(t_\lambda e) t_\lambda e \, dx + t_\lambda^{q^*} |e|_{L^{q^*}(\Lambda)}^{q^*} \tag{2.7}$$

which combined with (f_2') yields $t_\lambda^p \|e\|_{V_1,p}^p + t_\lambda^q \|e\|_{V_1,q}^q \geq t_\lambda^{q^*} |e|_{L^{q^*}(\Lambda)}^{q^*}$. Since $p < q < q^*$, we can infer that t_λ is bounded and that there exists a sequence $\lambda_n \to \infty$ such that $t_{\lambda_n} \to t_0 \geq 0$. Let us observe that if $t_0 > 0$ then we have

$$t_{\lambda_n}^p \|e\|_{V_1,p}^p + t_{\lambda_n}^q \|e\|_{V_1,q}^q \to L \in (0,\infty),$$

and

$$\int_\Lambda f(t_{\lambda_n} e) t_{\lambda_n} e \, dx + t_{\lambda_n}^{q^*} |e|_{L^{q^*}(\Lambda)}^{q^*} \geq \lambda_n \int_\Lambda (t_{\lambda_n} e)^{\sigma_1} \, dx + t_{\lambda_n}^{q^*} |e|_{L^{q^*}(\Lambda)}^{q^*} \to \infty,$$

which gives a contradiction in view of (2.7). Therefore, $t_0 = 0$. Let us now define $\gamma(t) = te$ with $t \in [0,1]$. Then, $\gamma \in \Gamma$ and we get

$$0 < c \leq \max_{t\in[0,1]} \mathcal{J}(te) = \mathcal{J}(t_\lambda e) \leq t_\lambda^p \|e\|_{V_1,p}^p + t_\lambda^q \|e\|_{V_1,q}^q. \tag{2.8}$$

Taking λ sufficiently large, we obtain that

$$t_\lambda^p \|e\|_{V_1,p}^p + t_\lambda^q \|e\|_{V_1,q}^q < \frac{1}{N} S_*^{\frac{N}{q}},$$

hence $0 < c < \frac{1}{N} S_*^{\frac{N}{q}}$. Moreover, since $t_\lambda \to 0$ as $\lambda \to \infty$, it follows from (2.8) that $c \to 0$ as $\lambda \to \infty$. □

The main feature of the modified functional is that it satisfies a compactness condition. We start by proving the boundedness of Palais-Smale sequences.

Lemma 2.4. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$ at the level c. Then $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$.*

Proof. From (g_3) and $\vartheta > q > p$, we have that

$$\begin{aligned}
C(1+\|u_n\|_\varepsilon) &\geq \mathcal{J}_\varepsilon(u_n) - \frac{1}{\vartheta}\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle \\
&= \left(\frac{1}{p}-\frac{1}{\vartheta}\right)\|u_n\|^p_{V_\varepsilon,p} + \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\|u_n\|^q_{V_\varepsilon,q} + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon^c} (g(\varepsilon x, u_n)u_n - \vartheta G(\varepsilon x, u_n))\,dx \\
&\quad + \frac{1}{\vartheta}\int_{\Lambda_\varepsilon} (g(\varepsilon x, u_n)u_n - \vartheta G(\varepsilon x, u_n))\,dx \\
&\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\left(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}\right) - \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\frac{1}{K}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)(|u_n|^p + |u_n|^q)\,dx \\
&\geq \left(\frac{1}{q}-\frac{1}{\vartheta}\right)\left(1-\frac{1}{K}\right)(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}) = \tilde{C}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q}),
\end{aligned}$$

where $\tilde{C} > 0$ since $K > 1$.

Now, we assume by contradiction that $\|u_n\|_{\mathbb{X}_\varepsilon} \to \infty$ and consider the following cases.
Case 1 $\|u_n\|_{V_\varepsilon,p} \to \infty$ and $\|u_n\|_{V_\varepsilon,q} \to \infty$. Then, for n large, we have $\|u_n\|^{q-p}_{V_\varepsilon,q} \geq 1$, that is $\|u_n\|^q_{V_\varepsilon,q} \geq \|u_n\|^p_{V_\varepsilon,q}$. Therefore,

$$C_0(1+\|u_n\|_{\mathbb{X}_\varepsilon}) \geq \tilde{C}(\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^p_{V_\varepsilon,q}) \geq C_1(\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q})^p = C_1\|u_n\|^p_{\mathbb{X}_\varepsilon}$$

that is an absurd.
Case 2 $\|u_n\|_{V_\varepsilon,p} \to \infty$ and $\|u_n\|_{V_\varepsilon,q}$ is bounded. We have

$$C_0(1+\|u_n\|_{V_\varepsilon,p} + \|u_n\|_{V_\varepsilon,q}) = C_0(1+\|u_n\|_{\mathbb{X}_\varepsilon}) \geq \tilde{C}\|u_n\|^p_{V_\varepsilon,p}$$

and consequently

$$C_0\left(\frac{1}{\|u_n\|^p_{V_\varepsilon,p}} + \frac{1}{\|u_n\|^{p-1}_{V_\varepsilon,p}} + \frac{\|u_n\|_{V_\varepsilon,q}}{\|u_n\|^p_{V_\varepsilon,p}}\right) \geq \tilde{C}.$$

Since $p > 1$ and passing to the limit as $n \to \infty$, we obtain $0 < \tilde{C} \leq 0$ which is impossible.
Case 3 $\|u_n\|_{V_\varepsilon,q} \to \infty$ and $\|u_n\|_{V_\varepsilon,p}$ is bounded. This is similar to the case 2, so we omit the details. Consequently, $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$. □

Remark 2.3. *We may always assume that any* (PS) *sequence* $\{u_n\}_{n\in\mathbb{N}}$ *of* $\mathcal{J}_\varepsilon$ *is nonnegative. In fact, by using* $\langle \mathcal{J}_\varepsilon'(u_n), u_n\rangle = o_n(1)$, *where* $u_n^- = \min\{u_n, 0\}$, *and* $g(x,t) = 0$ *for* $t \leq 0$, *we have that* $\langle u_n, u_n^-\rangle_{\varepsilon,p} + \langle u_n, u_n^-\rangle_{\varepsilon,q} = o_n(1)$ *from which* $\|u_n^-\|^p_{V_\varepsilon,p} + \|u_n^-\|^q_{V_\varepsilon,q} = o_n(1)$, *that is* $u_n^- \to 0$ *in* $\mathbb{X}_\varepsilon$. *Moreover,* $\{u_n^+\}$ *is bounded in* $\mathbb{X}_\varepsilon$. *Clearly,* $\|u_n\|_{V_\varepsilon,t} = \|u_n^+\|_{V_\varepsilon,t} + o_n(1)$ *for* $t \in \{p,q\}$. *Thus, we can easily check that* $\mathcal{J}_\varepsilon(u_n) = \mathcal{J}_\varepsilon(u_n^+) + o_n(1)$ *and* $\mathcal{J}_\varepsilon'(u_n) = \mathcal{J}_\varepsilon'(u_n^+) + o_n(1)$, *so we get* $\mathcal{J}_\varepsilon(u_n^+) \to c$ *and* $\mathcal{J}_\varepsilon'(u_n^+) = o_n(1)$.

Lemma 2.5. $\mathcal{J}_\varepsilon$ *satisfies the Palais-Smale condition at any level* $c \in \mathbb{R}$ *if* $\gamma = 0$, *and at any level* $c < \frac{1}{N}S_*^{\frac{N}{q}}$ *if* $\gamma = 1$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ be a $(PS)_c$ sequence for $\mathcal{J}_\varepsilon$. In view of Lemma 2.4, we may assume that $u_n \rightharpoonup u$ in $\mathbb{X}_\varepsilon$ and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q^*)$. It is standard to verify that the weak limit u is a critical point of $\mathcal{J}_\varepsilon$. Indeed, taking into account that for all $\phi \in C_c^\infty(\mathbb{R}^N)$

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla \phi \, dx \to \int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla \phi \, dx, \quad \forall t \in \{p,q\},\\
&\int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^{t-2}u_n\phi \, dx \to \int_{\mathbb{R}^N} V(\varepsilon x)|u|^{t-2}u\phi \, dx, \quad \forall t \in \{p,q\},\\
&\int_{\mathbb{R}^N} g(\varepsilon x, u_n)\phi \, dx \to \int_{\mathbb{R}^N} g(\varepsilon x, u)\phi \, dx,
\end{aligned}$$

and that $\langle \mathcal{J}_\varepsilon'(u_n), \phi\rangle = o_n(1)$, we can deduce that $\langle \mathcal{J}_\varepsilon'(u), \phi\rangle = 0$ for any $\phi \in C_c^\infty(\mathbb{R}^N)$. By the density of $C_c^\infty(\mathbb{R}^N)$ in $\mathbb{X}_\varepsilon$, we obtain that u is a critical point of $\mathcal{J}_\varepsilon$. In particular, $\langle \mathcal{J}_\varepsilon'(u), u\rangle = 0$.

Now, we show that for any $\eta > 0$ there exists $R = R(\eta) > 0$ such that

$$\limsup_{n\to\infty} \int_{B_R^c(0)} \left(|\nabla u_n|^p + |\nabla u_n|^q + V(\varepsilon x)(|u_n|^p + |u_n|^q)\right) \, dx < \eta. \tag{2.9}$$

For $R > 0$, let $\psi_R \in C^\infty(\mathbb{R}^N)$ be such that $0 \leq \psi_R \leq 1$, $\psi_R = 0$ in $B_{\frac{R}{2}}(0)$, $\psi_R = 1$ in $B_R^c(0)$, and $|\nabla \psi_R| \leq \frac{C}{R}$, for some constant $C > 0$ independent of R. Since $\{\psi_R u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{X}_\varepsilon$, it follows that $\langle \mathcal{J}_\varepsilon'(u_n), \psi_R u_n\rangle = o_n(1)$, namely

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^p\psi_R \, dx + \int_{\mathbb{R}^N} |\nabla u_n|^q\psi_R \, dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\psi_R \, dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\psi_R \, dx\\
&= o_n(1) + \int_{\mathbb{R}^N} g(\varepsilon x, u_n)\psi_R u_n \, dx - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_R u_n \, dx - \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla \psi_R u_n \, dx.
\end{aligned}$$

Take $R > 0$ such that $\Lambda_\varepsilon \subset B_{\frac{R}{2}}(0)$. From the definition of ψ_R and (g_3)-(ii), we see that

$$\begin{aligned}
&\int_{B_R^c(0)} |\nabla u_n|^p \, dx + \int_{B_R^c(0)} |\nabla u_n|^q \, dx + \left(1 - \frac{1}{K}\right) \int_{B_R^c(0)} V(\varepsilon x)(|u_n|^p + |u_n|^q) \, dx\\
&\leq \int_{\mathbb{R}^N} |\nabla u_n|^p\psi_R \, dx + \int_{\mathbb{R}^N} |\nabla u_n|^q\psi_R \, dx + \left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^p + |u_n|^q)\psi_R \, dx\\
&\leq o_n(1) - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi_R u_n \, dx - \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n \cdot \nabla \psi_R u_n \, dx.
\end{aligned} \tag{2.10}$$

Now, using the Hölder inequality and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}_\varepsilon$, we have, for $t \in \{p,q\}$,

$$\left|\int_{\mathbb{R}^N} |\nabla u_n|^{t-2}\nabla u_n \cdot \nabla \psi_R u_n \, dx\right| \leq \frac{C}{R}|\nabla u_n|_t^{t-1}|u_n|_t \leq \frac{C}{R}.$$

which combined with (2.10) implies that

$$\begin{aligned}
&\int_{B_R^c(0)} |\nabla u_n|^p \, dx + \int_{B_R^c(0)} |\nabla u_n|^q \, dx + \left(1 - \frac{1}{K}\right) \int_{B_R^c(0)} V(\varepsilon x)(|u_n|^p + |u_n|^q) \, dx\\
&\leq \int_{\mathbb{R}^N} |\nabla u_n|^p\psi_R \, dx + \int_{\mathbb{R}^N} |\nabla u_n|^q\psi_R \, dx + \left(1 - \frac{1}{K}\right) \int_{\mathbb{R}^N} V(\varepsilon x)(|u_n|^p + |u_n|^q)\psi_R \, dx\\
&\leq o_n(1) + \frac{C}{R}.
\end{aligned} \tag{2.11}$$

Consequently,

$$\limsup_{n\to\infty} \left(\int_{B_R^c(0)} |\nabla u_n|^p \, dx + \int_{B_R^c(0)} |\nabla u_n|^q \, dx + \left(1 - \frac{1}{K}\right) \int_{B_R^c(0)} V(\varepsilon x)(|u_n|^p + |u_n|^q) \, dx\right) \leq \frac{C}{R} < \eta$$

provided that $R = R(\eta) > \frac{C}{\eta}$. This proves the assertion (2.9). Next we show that (2.9) is useful to infer that $u_n \to u$ in $L^r(\mathbb{R}^N)$ for any $r \in [p, q^*)$. Fixed $\eta > 0$ we can find $R = R(\eta) > 0$ such that (2.9) holds true. Using the compact embedding $W^{1,p}(\mathbb{R}^N) \Subset L^p_{loc}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned}
\limsup_{n\to\infty} |u_n - u|_p^p &= \limsup_{n\to\infty} [|u_n - u|^p_{L^p(B_R(0))} + |u_n - u|^p_{L^p(B_R^c(0))}] \\
&= \lim_{n\to\infty} |u_n - u|^p_{L^p(B_R(0))} + \limsup_{n\to\infty} |u_n - u|^p_{L^p(B_R^c(0))} \\
&\leq 2^{p-1} \limsup_{n\to\infty} (|u_n|^p_{L^p(B_R^c(0))} + |u|^p_{L^p(B_R^c(0))}) \\
&\leq \frac{2^{p-1}}{V_0} \limsup_{n\to\infty} \left(\int_{B_R^c(0)} (|\nabla u_n|^p + V(\varepsilon x)|u_n|^p) \, dx + \int_{B_R^c(0)} (|\nabla u|^p + V(\varepsilon x)|u|^p) \, dx \right) \\
&< \frac{2^p}{V_0} \eta = \kappa \eta.
\end{aligned}$$

The arbitrariness of η implies the strong convergence in L^p-norm. By interpolation, we can see that $u_n \to u$ in $L^r(\mathbb{R}^N)$ for any $r \in [p, q^*)$.

Now, in order to prove the strong convergence in $\mathbb{X}_\varepsilon$, we distinguish two cases. First, we assume that $\gamma = 0$. Then, from (f_1), (f_2) and (g_2), we have that

$$\int_{\mathbb{R}^N} g(\varepsilon x, u_n) u_n \, dx \to \int_{\mathbb{R}^N} g(\varepsilon x, u) u \, dx. \tag{2.12}$$

On the other hand, using $\langle \mathcal{J}_\varepsilon'(u_n), u_n \rangle = o_n(1)$ and $\langle \mathcal{J}_\varepsilon'(u), u \rangle = 0$, we have

$$\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q} = \int_{\mathbb{R}^N} g(\varepsilon x, u_n) u_n \, dx + o_n(1) \quad \text{and} \quad \|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} = \int_{\mathbb{R}^N} g(\varepsilon x, u) u \, dx.$$

Putting together the above relations with (2.12), we find

$$\|u_n\|^p_{V_\varepsilon,p} + \|u_n\|^q_{V_\varepsilon,q} = \|u\|^p_{V_\varepsilon,p} + \|u\|^q_{V_\varepsilon,q} + o_n(1).$$

Since the Brezis-Lieb lemma gives

$$\|u_n - u\|^p_{V_\varepsilon,p} = \|u_n\|^p_{V_\varepsilon,p} - \|u\|^p_{V_\varepsilon,p} + o_n(1) \quad \text{and} \quad \|u_n - u\|^q_{V_\varepsilon,q} = \|u_n\|^q_{V_\varepsilon,q} - \|u\|^q_{V_\varepsilon,q} + o_n(1),$$

we can infer that $\|u_n - u\|^p_{V_\varepsilon,p} + \|u_n - u\|^q_{V_\varepsilon,q} = o_n(1)$. This fact implies that $u_n \to u$ in $\mathbb{X}_\varepsilon$ as $n \to \infty$.

Second, we consider the case $\gamma = 1$. The main difference with respect to the previous case, is that we cannot directly prove that (2.12) holds due to the presence of the critical exponent. For this reason, a more accurate analysis is needed.

Note that the Sobolev inequality, $0 \leq \psi_R \leq 1$, $|\nabla \psi_R| \leq \frac{C}{R}$, (2.11) and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $L^q(\mathbb{R}^N)$ yield

$$\begin{aligned}
|u_n|^q_{L^{q^*}(B_R^c(0))} &\leq |u_n \psi_R|^q_{q^*} \leq C |\nabla(u_n \psi_R)|^q_q \leq C \left(\int_{\mathbb{R}^N} |\nabla u_n|^q \psi_R \, dx + \int_{\mathbb{R}^N} |u_n|^q |\nabla \psi_R|^q \, dx \right) \\
&\leq o_n(1) + \frac{C}{R} + \frac{C}{R^q}.
\end{aligned}$$

Consequently,

$$\lim_{R\to\infty} \limsup_{n\to\infty} |u_n|^q_{L^{q^*}(B_R^c(0))} = 0. \tag{2.13}$$

Clearly, the strong convergence in $L^r(\mathbb{R}^N)$ for all $r \in [p, q^*)$ gives

$$\lim_{R\to\infty} \limsup_{n\to\infty} |u_n|^r_{L^r(B^c_R(0))} = 0. \tag{2.14}$$

Then, using the growth assumption on g, (2.13) and (2.14), for all $\eta > 0$ there exists $R = R(\eta) > 0$ such that

$$\limsup_{n\to\infty} \int_{B^c_R(0)} g(\varepsilon x, u_n)u_n \, dx \le C \limsup_{n\to\infty} \int_{B^c_R(0)} (|u_n|^p + |u_n|^{\sigma_2} + |u_n|^{q^*}) \, dx \le C\eta. \tag{2.15}$$

On the other hand, choosing $R > 0$ large enough, we may assume that

$$\int_{B^c_R(0)} g(\varepsilon x, u)u \, dx < \eta. \tag{2.16}$$

Then, (2.15) and (2.16) yield

$$\limsup_{n\to\infty} \left| \int_{B^c_R(0)} g(\varepsilon x, u_n)u_n \, dx - \int_{B^c_R(0)} g(\varepsilon x, u)u \, dx \right| < C\eta \quad \forall \eta > 0,$$

which implies that

$$\lim_{n\to\infty} \int_{B^c_R(0)} g(\varepsilon x, u_n)u_n \, dx = \int_{B^c_R(0)} g(\varepsilon x, u)u \, dx. \tag{2.17}$$

Using the definition of g it follows that

$$g(\varepsilon x, u_n)u_n \le f(u_n)u_n + a^{q^*} + \frac{V_0}{K}|u_n|^q \quad \forall x \in \mathbb{R}^N \setminus \Lambda_\varepsilon.$$

Since $B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)$ is bounded, we can use the above estimate, (f_1), (f_2'), the compact embedding $W^{1,r}(\mathbb{R}^N) \Subset L^\kappa_{loc}(\mathbb{R}^N)$ for $\kappa \in [1, r^*)$, and the dominated convergence theorem to infer that

$$\lim_{n\to\infty} \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} g(\varepsilon x, u_n)u_n \, dx = \int_{B_R(0)\cap(\mathbb{R}^N\setminus\Lambda_\varepsilon)} g(\varepsilon x, u)u \, dx. \tag{2.18}$$

At this point, we aim to show that

$$\lim_{n\to\infty} \int_{\Lambda_\varepsilon} (u_n^+)^{q^*} \, dx = \int_{\Lambda_\varepsilon} (u^+)^{q^*} \, dx. \tag{2.19}$$

Indeed, if we assume that (2.19) is true, from (g_2), (f_1), (f_2'), the compact Sobolev embedding $W^{1,r}(\mathbb{R}^N) \Subset L^\kappa_{loc}(\mathbb{R}^N)$ for $\kappa \in [1, r^*)$, and the dominated convergence theorem, we deduce that

$$\lim_{n\to\infty} \int_{\Lambda_\varepsilon\cap B_R(0)} g(\varepsilon x, u_n)u_n \, dx = \int_{\Lambda_\varepsilon\cap B_R(0)} g(\varepsilon x, u)u \, dx. \tag{2.20}$$

Putting together (2.17), (2.18) and (2.20) we conclude that (2.12) holds. It remains to prove that (2.19) holds true. Firstly, we may suppose that

$$|\nabla u_n|^q \rightharpoonup \mu, \; |u_n|^{q^*} \rightharpoonup \nu \tag{2.21}$$

weakly in the sense of measures. Using the concentration-compactness principle of Lions [28, 29], we have an at most countable index set I, sequences $\{x_i\}_{i\in I} \subset \mathbb{R}^N$, $\{\mu_i\}_{i\in I}$, $\{\nu_i\}_{i\in I}$ in $(0, \infty)$ such that

$$\mu \ge |\nabla u|^q + \sum_{i\in I} \mu_i \delta_{x_i}, \quad \nu = |u|^{q^*} + \sum_{i\in I} \nu_i \delta_{x_i}, \quad S_* \nu_i^{\frac{q}{q^*}} \le \mu_i \quad \forall i \in I. \tag{2.22}$$

It is enough to prove that $\{x_i\}_{i\in I} \cap \Lambda_\varepsilon = \emptyset$. Suppose, by contradiction, that $x_i \in \Lambda_\varepsilon$ for some $i \in I$. For $\rho > 0$, define $\zeta_\rho(x) = \zeta(\frac{x-x_i}{\rho})$ where $\zeta \in C^\infty_c(\mathbb{R}^N)$ is such that $0 \le \zeta \le 1$, $\zeta = 1$ in $B_1(0)$, $\zeta = 0$ in

$B_2^c(0)$ and $|\nabla\zeta|_\infty \le 2$. We suppose that ρ is chosen in such way that the support of ζ_ρ is contained in Λ_ε. Since $\{\zeta_\rho u_n\}_{n\in\mathbb{N}}$ is bounded, $\langle \mathcal{J}_\varepsilon'(u_n), u_n\zeta_\rho\rangle = o_n(1)$ and we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla u_n|^q\zeta_\rho\,dx &\le \int_{\mathbb{R}^N} |\nabla u_n|^p\zeta_\rho\,dx + \int_{\mathbb{R}^N} |\nabla u_n|^q\zeta_\rho\,dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^p\zeta_\rho\,dx + \int_{\mathbb{R}^N} V(\varepsilon x)|u_n|^q\zeta_\rho\,dx\\ &= o_n(1) - \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla\zeta_\rho u_n\,dx - \int_{\mathbb{R}^N} |\nabla u_n|^{q-2}\nabla u_n\cdot\nabla\zeta_\rho u_n\,dx\\ &\quad + \int_{\mathbb{R}^N} f(u_n)\zeta_\rho u_n\,dx + \int_{\mathbb{R}^N} |u_n|^{q^*}\zeta_\rho\,dx. \end{aligned}\tag{2.23}$$

Due to the fact that f has subcritical growth and ζ_ρ has compact support, we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}\zeta_\rho f(u_n)u_n\,dx = \lim_{\rho\to 0}\int_{\mathbb{R}^N}\zeta_\rho f(u)u\,dx = 0.\tag{2.24}$$

Now, we verify that, for $t\in\{p,q\}$, we have

$$\lim_{\rho\to 0}\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{t-2}\nabla u_n\cdot\nabla\zeta_\rho u_n\,dx = 0.\tag{2.25}$$

In fact, applying the Hölder inequality, we get

$$\left|\int_{\mathbb{R}^N}|\nabla u_n|^{t-2}\nabla u_n\cdot\nabla\zeta_\rho u_n\,dx\right| \le |\nabla u_n|_t\left(\int_{B_\rho(x_i)}|u_n|^t|\nabla\zeta_\rho|^t\,dx\right)^{\frac1t} \le C\left(\int_{B_\rho(x_i)}|u_n|^t|\nabla\zeta_\rho|^t\,dx\right)^{\frac1t}.$$

Using again the Hölder inequality, we see that

$$\limsup_{n\to\infty}\left|\int_{\mathbb{R}^N}|\nabla u_n|^{t-2}\nabla u_n\cdot\nabla\zeta_\rho u_n\,dx\right| \le C\left(\int_{B_\rho(x_i)}|u|^t|\nabla\zeta_\rho|^t\,dx\right)^{\frac1t} \le \frac{C}{\rho}|u|_{t^*}|B_\rho(x_i)|^{\frac{t^*-t}{tt^*}} \le C\rho^{\frac1N}\to 0$$

as $\rho\to 0$. Therefore, (2.25) holds. Putting together (2.23), (2.24), (2.25), and using (2.21), we find $\mu_i\le\nu_i$. This fact combined with the last statement in (2.22) yield $\nu_i \ge S_*^{\frac{N}{q}}$. Then, by (g_3), and recalling that $q>p$, we obtain

$$\begin{aligned} c &= \mathcal{J}_\varepsilon(u_n) - \frac1q\langle\mathcal{J}_\varepsilon'(u_n),u_n\rangle + o_n(1)\\ &= \left(\frac1p-\frac1q\right)\|u_n\|^p_{V_\varepsilon,p} + \int_{\mathbb{R}^N\setminus\Lambda_\varepsilon}\left(\frac1q g(\varepsilon x,u_n)u_n - G(\varepsilon x,u_n)\right)dx\\ &\quad + \int_{\Lambda_\varepsilon}\left(\frac1q f(u_n)u_n - F(u_n)\right)dx + \left(\frac1q-\frac1{q^*}\right)\int_{\Lambda_\varepsilon}|u_n|^{q^*}\,dx + o_n(1)\\ &\ge \frac1N\int_{\Lambda_\varepsilon}|u_n|^{q^*}\,dx + o_n(1) \ge \frac1N\int_{\Lambda_\varepsilon}|u_n|^{q^*}\zeta_\rho\,dx + o_n(1). \end{aligned}$$

Taking the limit and using (2.22), we deduce that

$$c \ge \frac1N\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\zeta_\rho(x_i)\nu_i = \frac1N\sum_{\{i\in I: x_i\in\Lambda_\varepsilon\}}\nu_i \ge \frac1N S_*^{\frac Nq}$$

which is an absurd because $c<\frac1N S_*^{\frac Nq}$. ☐

Corollary 2.1. *The functional ψ_ε satisfies the Palais-Smale condition on $\mathbb{S}_\varepsilon^+$ at any level $c\in\mathbb{R}$ if $\gamma=0$ and at any level $c<\frac1N S_*^{\frac Nq}$ if $\gamma=1$.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^+_\varepsilon$ be a Palais-Smale sequence for ψ_ε at the level c. Then $\psi_\varepsilon(u_n) \to c$ and $\psi'_\varepsilon(u_n) \to 0$ in $(T_{u_n}\mathbb{S}^+_\varepsilon)'$. By Proposition 2.1-(c), we see that $\{m_\varepsilon(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{X}_\varepsilon$ is a Palais-Smale sequence for $\mathcal{J}_\varepsilon$ at the level c. Then, by Lemma 2.5, we deduce that $\mathcal{J}_\varepsilon$ satisfies the $(PS)_c$ condition in $\mathbb{X}_\varepsilon$ and thus there exists $u \in \mathbb{S}^+_\varepsilon$ such that, up to a subsequence, $m_\varepsilon(u_n) \to m_\varepsilon(u)$ in $\mathbb{X}_\varepsilon$. By Lemma 2.2-(iii), we obtain that $u_n \to u$ in $\mathbb{S}^+_\varepsilon$. □

We end this section by proving an existence result for (2.1).

Theorem 2.1. *Suppose that* (V_1)-(V_2) *and* (f_1)-(f_4) *hold. Then, for any* $\varepsilon > 0$, (2.1) *has a positive ground state solution.*

Proof. In view of Lemma 2.1, Remark 2.2 and Lemma 2.5, we can apply the mountain pass theorem [7] to deduce that, for all $\varepsilon > 0$, there exists a nontrivial critical point $u_\varepsilon \in \mathbb{X}_\varepsilon$ for $\mathcal{J}_\varepsilon$. Since $\langle \mathcal{J}'_\varepsilon(u_\varepsilon), u^-_\varepsilon\rangle = 0$, where $u^-_\varepsilon = \min\{u_\varepsilon, 0\}$, $g(\varepsilon\,\cdot, t) = 0$ for $t \leq 0$, we can see that $\|u^-_\varepsilon\|^p_{V_\varepsilon,p} + \|u^-_\varepsilon\|^q_{V_\varepsilon,q} = 0$ which implies that $u^-_\varepsilon = 0$, that is $u_\varepsilon \geq 0$ in $\mathbb{R}^N$. By the regularity results in [24], we have that $u_\varepsilon \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{loc}(\mathbb{R}^N)$ and $u_\varepsilon(x) \to 0$ as $|x| \to \infty$. Using the Harnack inequality [42], we deduce that $u_\varepsilon > 0$ in $\mathbb{R}^N$. □

3. The autonomous problem

Since we are interested in giving a multiplicity result for the auxiliary problem (2.1), it is important to analyze the limiting problem associated with (1.1), namely

$$\begin{cases} -\Delta_p u - \Delta_q u + V_0(u^{p-1} + u^{q-1}) = f(u) + \gamma u^{q^*-1} \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\, u > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{3.1}$$

whose energy functional $\mathcal{L}_{V_0} : \mathbb{Y}_{V_0} \to \mathbb{R}$ is given by

$$\mathcal{L}_{V_0}(u) = \frac{1}{p}\|u\|^p_{V_0,p} + \frac{1}{q}\|u\|^q_{V_0,q} - \int_{\mathbb{R}^N} \left(F(u) + \frac{\gamma}{q^*}(u^+)^q\right) dx,$$

and $\mathbb{Y}_{V_0} = W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ is equipped with the norm $\|u\|_{\mathbb{Y}_{V_0}} = \|u\|_{V_0,p} + \|u\|_{V_0,q}$, where

$$\|u\|_{V_0,t} = \left(|\nabla u|^t_t + V_0|u|^t_t\right)^{\frac{1}{t}} \quad \forall t \in \{p, q\}.$$

For $t \in \{p, q\}$, we set

$$\langle u, \varphi\rangle_{V_0,t} = \int_{\mathbb{R}^N} |\nabla u|^{t-2}\nabla u \cdot \nabla\varphi\, dx + \int_{\mathbb{R}^N} V_0|u|^{t-2}u\,\varphi\, dx \quad \forall u, \varphi \in \mathbb{Y}_{V_0}.$$

Standard arguments show that $\mathcal{L}_{V_0} \in C^1(\mathbb{Y}_{V_0}, \mathbb{R})$ and that

$$\langle \mathcal{L}'_{V_0}(u), \varphi\rangle = \langle u, \varphi\rangle_{V_0,p} + \langle u, \varphi\rangle_{V_0,q} - \int_{\mathbb{R}^N} (f(u) + \gamma(u^+)^{q^*-1})\varphi\, dx$$

for any $u, \varphi \in \mathbb{Y}_{V_0}$. We also consider the Nehari manifold $\mathcal{M}_{V_0}$ associated with $\mathcal{L}_{V_0}$, that is $\mathcal{M}_{V_0} = \{u \in \mathbb{Y}_{V_0} \setminus \{0\} : \langle \mathcal{L}'_{V_0}(u), u\rangle = 0\}$, and we set $d_{V_0} = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u)$. Now we define $\mathbb{Y}^+_{V_0} = \{u \in \mathbb{Y}_{V_0} : |\operatorname{supp}(u^+)| > 0\}$, and $\mathbb{S}^+_{V_0} = \mathbb{S}_{V_0} \cap \mathbb{Y}^+_{V_0}$, where $\mathbb{S}_{V_0}$ is the unit sphere of $\mathbb{Y}_{V_0}$. As in section 2, $\mathbb{S}^+_{V_0}$ is an incomplete $C^{1,1}$-manifold of codimension one and contained in $\mathbb{Y}^+_{V_0}$. Thus, $\mathbb{Y}_{V_0} = T_u\mathbb{S}^+_{V_0} \oplus \mathbb{R}u$ for each $u \in \mathbb{S}^+_{V_0}$, where

$$T_u\mathbb{S}^+_{V_0} = \{v \in \mathbb{Y}_{V_0} : \langle u, v\rangle_{V_0,p} + \langle u, v\rangle_{V_0,q} = 0\}.$$

In the sequel we state without proofs the following results which can be obtained arguing as in section 2.

Lemma 3.1. *The following properties hold:*

(i) *For each $u \in \mathbb{Y}^+_{V_0}$, let $h : \mathbb{R}^+ \to \mathbb{R}$ be defined by $h_u(t) = \mathcal{L}_{V_0}(tu)$. Then, there is a unique $t_u > 0$ such that $h'_u(t) > 0$ for all $t \in (0, t_u)$ and $h'_u(t) < 0$ for all $t \in (t_u, \infty)$.*

(ii) *There exists $\tau > 0$ independent of u such that $t_u \geq \tau$ for any $u \in \mathbb{S}^+_{V_0}$. Moreover, for each compact set $\mathbb{K} \subset \mathbb{S}^+_{V_0}$ there is a positive constant $C_{\mathbb{K}}$ such that $t_u \leq C_{\mathbb{K}}$ for any $u \in \mathbb{K}$.*

(iii) *The map $\hat{m}_{V_0} : \mathbb{Y}^+_{V_0} \to \mathcal{M}_{V_0}$ given by $\hat{m}_{V_0}(u) = t_u u$ is continuous and $m_{V_0} = \hat{m}_{V_0}|_{\mathbb{S}^+_{V_0}}$ is a homeomorphism between $\mathbb{S}^+_{V_0}$ and $\mathcal{M}_{V_0}$. Moreover $m^{-1}_{V_0}(u) = \frac{u}{\|u\|_{\mathbb{Y}_{V_0}}}$.*

(iv) *If there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^+_{V_0}$ such that $\operatorname{dist}(u_n, \partial\mathbb{S}^+_{V_0}) \to 0$ then $\|m_{V_0}(u_n)\|_{\mathbb{Y}_{V_0}} \to \infty$ and $\mathcal{L}_{V_0}(m_{V_0}(u_n)) \to \infty$.*

Let us consider the maps $\hat{\psi}_{V_0} : \mathbb{Y}^+_{V_0} \to \mathbb{R}$ and $\psi_{V_0} : \mathbb{S}^+_{V_0} \to \mathbb{R}$ given by by $\hat{\psi}_{V_0}(u) = \mathcal{L}_{V_0}(\hat{m}_{V_0}(u))$ and $\psi_{V_0} = \hat{\psi}_{V_0}|_{\mathbb{S}^+_{V_0}}$, respectively.

Proposition 3.1. *The following properties hold:*

(a) *$\hat{\psi}_{V_0} \in C^1(\mathbb{Y}^+_{V_0}, \mathbb{R})$ and*

$$\langle \hat{\psi}'_{V_0}(u), v\rangle = \frac{\|\hat{m}_{V_0}(u)\|_{\mathbb{Y}_{V_0}}}{\|u\|_{\mathbb{Y}_{V_0}}} \langle \mathcal{L}'_{V_0}(\hat{m}_{V_0}(u)), v\rangle \quad \forall u \in \mathbb{Y}^+_{V_0}, \forall v \in \mathbb{Y}_{V_0}.$$

(b) *$\psi_{V_0} \in C^1(\mathbb{S}^+_{V_0}, \mathbb{R})$ and*

$$\langle \psi'_{V_0}(u), v\rangle = \|m_{V_0}(u)\|_{\mathbb{Y}_{V_0}} \langle \mathcal{L}'_{V_0}(m_{V_0}(u)), v\rangle \quad \forall v \in T_u\mathbb{S}^+_{V_0}.$$

(c) *If $\{u_n\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for ψ_{V_0}, then $\{m_{V_0}(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for $\mathcal{L}_{V_0}$. If $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ is a bounded $(PS)_d$ sequence for $\mathcal{L}_{V_0}$, then $\{m^{-1}_{V_0}(u_n)\}_{n\in\mathbb{N}}$ is a $(PS)_d$ sequence for ψ_{V_0}.*

(d) *u is a critical point of ψ_{V_0} if, and only if, $m_{V_0}(u)$ is a nontrivial critical point for $\mathcal{L}_{V_0}$. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}^+_{V_0}} \psi_{V_0}(u) = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u).$$

Remark 3.1. *As in section 2, we have the following characterization of the infimum of $\mathcal{L}_{V_0}$ over $\mathcal{M}_{V_0}$:*

$$0 < d_{V_0} = \inf_{u\in\mathcal{M}_{V_0}} \mathcal{L}_{V_0}(u) = \inf_{u\in\mathbb{Y}^+_{V_0}} \max_{t>0} \mathcal{L}_{V_0}(tu) = \inf_{u\in\mathbb{S}^+_{V_0}} \max_{t>0} \mathcal{L}_{V_0}(tu).$$

Moreover, when $\gamma = 1$, we can argue as in the proof of Lemma 2.3 to see that $0 < d_{V_0} < \frac{1}{N} S_*^{\frac{N}{q}}$.

The next lemma allows us to assume that the weak limit of a $(PS)_{d_{V_0}}$ sequence of $\mathcal{L}_{V_0}$ is nontrivial.

Lemma 3.2. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ be a $(PS)_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$ such that $u_n \rightharpoonup 0$ in $\mathbb{Y}_{V_0}$. Then,*

(a) *either $u_n \to 0$ in $\mathbb{Y}_{V_0}$, or*

(b) *there is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^q \, dx \geq \beta.$$

Proof. Assume that (b) does not true. Since $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, we can apply Lions Lemma [28] to see that $u_n \to 0$ in $L^r(\mathbb{R}^N)$ for all $r \in (p, q^*)$. In particular, by (f_1)-(f_2), we have that $\int_{\mathbb{R}^N} F(u_n)\,dx = \int_{\mathbb{R}^N} f(u_n)u_n\,dx = o_n(1)$ as $n \to \infty$. Recalling that $\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle = o_n(1)$, we have

$$\|u_n\|^p_{V_0,p} + \|u_n\|^q_{V_0,q} = \int_{\mathbb{R}^N} f(u_n)u_n\,dx + \gamma|u_n^+|^{q^*}_{q^*} = o_n(1) + \gamma|u_n^+|^{q^*}_{q^*}.$$

When $\gamma = 0$, we have $\|u_n\|_{\mathbb{Y}_{V_0}} \to 0$ as $n \to \infty$ and the item (a) holds true. Now we consider the case $\gamma = 1$. Then, up to a subsequence, there exists $\ell \geq 0$ such that $\|u_n\|^p_{V_0,p} + \|u_n\|^q_{V_0,q} \to \ell$ and $|u_n^+|^{q^*}_{q^*} \to \ell$. Assume by contradiction that $\ell > 0$. Since $\mathcal{L}_{V_0}(u_n) \to d_{V_0}$, $\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle = 0$ and $q > p$, we deduce that

$$\begin{aligned} d_{V_0} + o_n(1) &= \mathcal{L}_{V_0}(u_n) - \frac{1}{q}\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p_{V_0,p} + o_n(1) + \left(\frac{1}{q} - \frac{1}{q^*}\right)|u_n^+|^{q^*}_{q^*} \geq \frac{1}{N}\ell + o_n(1), \end{aligned}$$

which implies that $d_{V_0} \geq \frac{1}{N}\ell$. Using the Sobolev inequality, we see that

$$\|u_n\|^p_{V_0,p} + \|u_n\|^q_{V_0,q} \geq S_*|u_n|^q_{q^*} \geq S_*(|u_n^+|^{q^*}_{q^*})^{\frac{q}{q^*}}$$

and taking the limit as $n \to \infty$ we get $\ell \geq S_*\ell^{\frac{q}{q^*}}$ that is $\ell \geq S_*^{\frac{N}{q}}$. Consequently, $d_{V_0} \geq \frac{1}{N}\ell \geq \frac{1}{N}S_*^{\frac{N}{q}}$ and this contradicts Remark 3.1. □

Remark 3.2. *As it has been mentioned earlier, if u is the weak limit of a $(PS)_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$, then we can assume $u \neq 0$. Otherwise, $u_n \rightharpoonup 0$ and, if $u_n \not\to 0$ in $\mathbb{Y}_{V_0}$, we conclude from the Lemma 3.2 that there are $\{y_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and $R, \beta > 0$ such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^q\,dx \geq \beta.$$

Set $v_n(x) = u_n(x + y_n)$. Then, using the invariance of $\mathbb{R}^N$ by translation, we see that $\{v_n\}_{n\in\mathbb{N}}$ is a bounded $(PS)_{d_{V_0}}$ sequence for $\mathcal{L}_{V_0}$ such that $v_n \rightharpoonup v$ in $\mathbb{Y}_{V_0}$ with $v \neq 0$.

In the next result we obtain a positive ground state solution for the autonomous problem (3.1).

Theorem 3.1. *Problem (3.1) admits a positive ground state solution.*

Proof. Using a variant of the mountain-pass theorem without (PS)-condition (see [44]), there exists a Palais-Smale sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{Y}_{V_0}$ for $\mathcal{L}_{V_0}$ at the level d_{V_0}. Proceeding as in the proof of Lemma 2.5, we can prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$ so, going if necessary to a subsequence, we may assume that $u_n \rightharpoonup u$ in $\mathbb{Y}_{V_0}$ and $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$ for all $r \in [1, q^*)$. Standard arguments (see proof of Lemma 2.5) show that $\mathcal{L}'_{V_0}(u) = 0$. From Remark 3.2, we may assume that $u \neq 0$. On the other hand, by Fatou's lemma and (f_3), we can see that

$$\mathcal{L}_{V_0}(u) - \frac{1}{q}\langle \mathcal{L}'_{V_0}(u), u\rangle \leq \liminf_{n\to\infty}\left(\mathcal{L}_{V_0}(u_n) - \frac{1}{q}\langle \mathcal{L}'_{V_0}(u_n), u_n\rangle\right) = d_{V_0}$$

which yields $d_{V_0} = \mathcal{L}_{V_0}(u)$. Finally, arguing as at the end of the proof of Theorem 2.1, we can prove that $u_\varepsilon > 0$ in $\mathbb{R}^N$. □

The next lemma is a compactness result on autonomous problem which we will use later.

Lemma 3.3. *Let $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ be a sequence such that $\mathcal{L}_{V_0}(u_n) \to d_{V_0}$. Then, $\{u_n\}_{n\in\mathbb{N}}$ has a convergent subsequence in $\mathbb{Y}_{V_0}$.*

Proof. Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}$ and $\mathcal{L}_{V_0}(u_n) \to d_{V_0}$, it follows from Lemma 3.1-(iii), Proposition 3.1-(d) and the definition of d_{V_0} that $v_n = m_{V_0}^{-1}(u_n) = \frac{u_n}{\|u_n\|_{\mathbb{Y}_{V_0}}} \in \mathbb{S}^+_{V_0}$ for all $n \in \mathbb{N}$, and $\psi_{V_0}(v_n) = \mathcal{L}_{V_0}(u_n) \to d_{V_0} = \inf_{v\in\mathbb{S}^+_{V_0}} \psi_{V_0}(v)$. Let us define $\mathcal{G} : \overline{\mathbb{S}}^+_{V_0} \to \mathbb{R}\cup\{\infty\}$ as $\mathcal{G}(u) = \psi_{V_0}(u)$ if $u \in \mathbb{S}^+_{V_0}$, and $\mathcal{G}(u) = \infty$ if $u \in \partial\mathbb{S}^+_{V_0}$. Note that $(\overline{\mathbb{S}}^+_{V_0}, \delta_{V_0})$, where $\delta_{V_0}(u,v) = \|u-v\|_{\mathbb{Y}_{V_0}}$, is a complete metric space, $\mathcal{G} \in C(\overline{\mathbb{S}}^+_{V_0}, \mathbb{R}\cup\{\infty\})$ (by Lemma 3.1-(iv)), $\mathcal{G}$ is bounded below (by Proposition 3.1-(d)). Hence, applying the Ekeland variational principle to $\mathcal{G}$, there exists $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset \mathbb{S}^+_{V_0}$ such that $\{\hat{v}_n\}_{n\in\mathbb{N}}$ is a $(PS)_{d_{V_0}}$ sequence for ψ_{V_0} at the level d_{V_0} and $\|\hat{v}_n - v_n\|_{\mathbb{Y}_{V_0}} = o_n(1)$. Now the remainder of the proof follows from Proposition 3.1, Theorem 3.1, and arguing as in the proof of Corollary 2.1. □

We conclude this section by showing the following useful relation between the minimax levels c_ε and d_{V_0}.

Lemma 3.4. *It holds* $\lim_{\varepsilon\to 0} c_\varepsilon = d_{V_0}$.

Proof. Let $\omega_\varepsilon(x) = \psi_\varepsilon(x)\omega(x)$, where ω is a positive ground state of (3.1) which is given by Theorem 3.1, and $\psi_\varepsilon(x) = \psi(\varepsilon x)$ with $\psi \in C^\infty_c(\mathbb{R}^N)$ such that $0 \le \psi \le 1$, $\psi(x) = 1$ if $|x| \le r$ and $\psi(x) = 0$ if $|x| \ge 2r$. For simplicity, we assume that $\mathrm{supp}(\psi) \subset B_{2r} \subset \Lambda$ for some $r > 0$. Using the dominated convergence theorem we see that

$$\omega_\varepsilon \to \omega \quad \text{in } \mathcal{W} \quad \text{and} \quad \mathcal{L}_{V_0}(\omega_\varepsilon) \to \mathcal{L}_{V_0}(\omega) = d_{V_0} \tag{3.2}$$

as $\varepsilon \to 0$. Now, for each $\varepsilon > 0$ there exists $t_\varepsilon > 0$ such that $\mathcal{J}_\varepsilon(t_\varepsilon\omega_\varepsilon) = \max_{t\ge0} \mathcal{J}_\varepsilon(t\omega_\varepsilon)$. Therefore, $\langle \mathcal{J}'_\varepsilon(t_\varepsilon\omega_\varepsilon), \omega_\varepsilon\rangle = 0$ and this implies that

$$t_\varepsilon^p\|\omega_\varepsilon\|^p_{V_\varepsilon,p} + t_\varepsilon^q\|\omega_\varepsilon\|^q_{V_\varepsilon,q} = \int_{\mathbb{R}^N} \left(f(t_\varepsilon\omega_\varepsilon)t_\varepsilon\omega_\varepsilon + \gamma(t_\varepsilon\omega_\varepsilon)^{q^*} \right) dx.$$

If $t_\varepsilon \to \infty$ then

$$t_\varepsilon^{p-q}\|\omega_\varepsilon\|^p_{V_\varepsilon,p} + \|\omega_\varepsilon\|^q_{V_\varepsilon,q} = \int_{\mathbb{R}^N} \frac{f(t_\varepsilon\omega_\varepsilon) + \gamma(t_\varepsilon\omega_\varepsilon)^{q^*-1}}{(t_\varepsilon\omega_\varepsilon)^{q-1}} \omega_\varepsilon^q \, dx, \tag{3.3}$$

and using (3.2), $p < q$ and (f_3), we obtain that $\|\omega\|^q_{V_0,q} = \infty$ which gives a contradiction. Then $t_\varepsilon \to t_0 \in [0,\infty)$. If $t_0 = 0$, using (f_1), (f_2) and (f_2'), we see that for $\zeta \in (0, V_0)$, it holds

$$\left(1 - \frac{\zeta}{V_0}\right)\|\omega_\varepsilon\|^p_{V_\varepsilon,p} + t_\varepsilon^{q-p}\|\omega_\varepsilon\|^q_{V_\varepsilon,q} \le C_\zeta t_\varepsilon^{q-p}\|\omega_\varepsilon\|^{q^*}_{V_\varepsilon,q}$$

which yields $\|\omega\|^p_{V_0,p} = 0$ and this is an absurd. Hence, $t_\varepsilon \to t_0 > 0$. Taking the limit as $\varepsilon \to 0$ in (3.3) we get

$$t_0^{p-q}\|\omega\|^p_{V_0,p} + \|\omega\|^q_{V_0,q} = \int_{\mathbb{R}^N} \frac{f(t_0\omega) + \gamma(t_0\omega)^{q^*-1}}{(t_0\omega)^{q-1}} \omega^q \, dx$$

which together with (f_4) and $\omega \in \mathcal{M}_{V_0}$ implies that $t_0 = 1$. On the other hand, we can note that

$$c_\varepsilon \le \max_{t\ge0} \mathcal{J}_\varepsilon(t\omega_\varepsilon) = \mathcal{J}_\varepsilon(t_\varepsilon\omega_\varepsilon) = \mathcal{L}_{V_0}(t_\varepsilon\omega_\varepsilon) + \frac{t_\varepsilon^p}{p}\int_{\mathbb{R}^N}(V_\varepsilon(x) - V_0)\omega_\varepsilon^p \, dx + \frac{t_\varepsilon^q}{q}\int_{\mathbb{R}^N}(V_\varepsilon(x) - V_0)\omega_\varepsilon^q \, dx.$$

Since $V(\varepsilon\,\cdot)$ is bounded on the support of ω_ε, we use the dominated convergence theorem, (3.2) and the above inequality to obtain that $\limsup_{\varepsilon\to0} c_\varepsilon \le d_{V_0}$. By (V_1) we deduce that $\liminf_{\varepsilon\to0} c_\varepsilon \ge d_{V_0}$ and thus $\lim_{\varepsilon\to0} c_\varepsilon = d_{V_0}$. This ends the proof of the lemma. □

4. Multiplicity of solutions to (2.1)

In this section we collect some technical results which will be used to implement the barycenter machinery below. Let $\delta > 0$ be such that

$$M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\} \subset \Lambda, \tag{4.1}$$

and $w \in \mathbb{Y}_{V_0}$ be a positive ground state solution to the autonomous problem (3.1) which exists by virtue of Theorem 3.1. Let $\eta \in C^\infty([0,\infty),[0,1])$ be a non increasing function satisfying $\eta(t) = 1$ if $0 \leq t \leq \frac{\delta}{2}$, $\eta(t) = 0$ if $t \geq \delta$ and $|\eta'(t)| \leq c$ for some $c > 0$. For any $y \in M$, we define

$$\Psi_{\varepsilon,y}(x) = \eta(|\,\varepsilon\, x - y|) w\left(\frac{\varepsilon\, x - y}{\varepsilon}\right),$$

and $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ given by $\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}$, where $t_\varepsilon > 0$ satisfies $\max_{t\geq 0} \mathcal{J}_\varepsilon(t\Psi_{\varepsilon,y}) = \mathcal{J}_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y})$. By construction, $\Phi_\varepsilon(y)$ has compact support for any $y \in M$.

Lemma 4.1. *The functional Φ_ε verifies the following limit:*

$$\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y \in M.$$

Proof. Suppose that the thesis of the lemma is false. Then we can find $\delta_0 > 0$, $\{y_n\}_{n\in\mathbb{N}} \subset M$ and $\varepsilon_n \to 0$ such that

$$|\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - d_{V_0}| \geq \delta_0. \tag{4.2}$$

Now, for each $n \in \mathbb{N}$ and for all $z \in B_{\frac{\delta}{\varepsilon_n}}(0)$, we have $\varepsilon_n z \in B_\delta(0)$, and thus $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Taking the change of variable $z = \frac{\varepsilon_n x - y_n}{\varepsilon_n}$ and using the fact that $G(x,t) = F(x,t) + \frac{\gamma}{q^*} t^{q^*}$ for $(x,t) \in \Lambda \times [0,\infty)$, we can write

$$\begin{aligned}
\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) &= \frac{t_{\varepsilon_n}^p}{p} \|\Psi_{\varepsilon_n,y_n}\|^p_{V_{\varepsilon_n},p} + \frac{t_{\varepsilon_n}^q}{q} \|\Psi_{\varepsilon_n,y_n}\|^q_{V_{\varepsilon_n},q} - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_{\varepsilon_n} \Psi_{\varepsilon_n,y_n})\, dx \\
&= \frac{t_{\varepsilon_n}^p}{p} \left(|\nabla(\eta(|\,\varepsilon_n \cdot|) w)|_p^p + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\,\varepsilon_n z|) w(z))^p \, dz \right) \\
&\quad + \frac{t_{\varepsilon_n}^q}{q} \left(|\nabla(\eta(|\,\varepsilon_n \cdot|) w)|_q^q + \int_{\mathbb{R}^N} V(\varepsilon_n z + y_n)(\eta(|\,\varepsilon_n z|) w(z))^q \, dz \right) \\
&\quad - \int_{\mathbb{R}^N} \left(F(t_{\varepsilon_n} \eta(|\,\varepsilon_n z|) w(z)) + \frac{\gamma}{q^*} (t_{\varepsilon_n} \eta(|\,\varepsilon_n z|) w(z))^{q^*} \right) dz.
\end{aligned} \tag{4.3}$$

We claim that $t_{\varepsilon_n} \to 1$ as $\varepsilon_n \to 0$. We start by proving that $t_{\varepsilon_n} \to t_0 \in [0,\infty)$. Since $\langle \mathcal{J}'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$ and $g = f$ on $\Lambda \times \mathbb{R}$, we have

$$\begin{aligned}
&\frac{1}{t_{\varepsilon_n}^{q-p}} \|\Psi_{\varepsilon_n,y_n}\|^p_{V_{\varepsilon_n},p} + \|\Psi_{\varepsilon_n,y_n}\|^q_{V_{\varepsilon_n},q} \\
&= \int_{\mathbb{R}^N} \left(\frac{f(t_{\varepsilon_n} \eta(|\,\varepsilon_n z|) w(z)) + \gamma (t_{\varepsilon_n} \eta(|\,\varepsilon_n z|) w(z))^{q^*-1}}{(t_{\varepsilon_n} \eta(|\,\varepsilon_n z|) w(z))^{q-1}} \right) (\eta(|\,\varepsilon_n z|) w(z))^q \, dz.
\end{aligned} \tag{4.4}$$

Observing that $\eta(|x|) = 1$ for $x \in B_{\frac{\delta}{2}}(0)$ and $B_{\frac{\delta}{2}}(0) \subset B_{\frac{\delta}{\varepsilon_n}}(0)$ for all n large enough, the identity (4.4) yields

$$t_{\varepsilon_n}^{p-q} \|\Psi_{\varepsilon_n,y_n}\|^p_{V_{\varepsilon_n},p} + \|\Psi_{\varepsilon_n,y_n}\|^q_{V_{\varepsilon_n},q} \geq \int_{B_{\frac{\delta}{2}}(0)} \left(\frac{f(t_{\varepsilon_n} w(z)) + \gamma (t_{\varepsilon_n} w(z))^{q^*-1}}{(t_{\varepsilon_n} w(z))^{q-1}} \right) (w(z))^q \, dz,$$

which combined with (f_4) gives

$$t_{\varepsilon_n}^{p-q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^{p}+\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^{q}\geq\left(\left(\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{q-1}}\right)(w(\hat{z}))^{q}+\gamma t_{\varepsilon_n}^{q^*-q}(w(\hat{z}))^{q^*}\right)|B_{\frac{\delta}{2}}(0)|, \tag{4.5}$$

where $w(\hat{z})=\min_{z\in\overline{B_{\frac{\delta}{2}}(0)}}w(z)>0$ (we remark that w is continuous and positive in $\mathbb{R}^N$). If $t_{\varepsilon_n}\to\infty$, using the fact that $q>p$ and that the dominated convergence theorem yields

$$\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},r}\to\|w\|_{V_0,r}\in(0,\infty)\quad\forall r\in\{p,q\}, \tag{4.6}$$

we find

$$t_{\varepsilon_n}^{p-q}\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},p}^{p}+\|\Psi_{\varepsilon_n,y_n}\|_{V_{\varepsilon_n},q}^{q}\to\|w\|_{V_0,q}^{q}. \tag{4.7}$$

On the other hand, by (f_3), we get

$$\lim_{n\to\infty}\frac{f(t_{\varepsilon_n}w(\hat{z}))}{(t_{\varepsilon_n}w(\hat{z}))^{q-1}}=\infty. \tag{4.8}$$

Gathering (4.5), (4.7), (4.8) and using $q^*>q$, we achieve a contradiction. Consequently, $\{t_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is bounded and we may suppose that $t_{\varepsilon_n}\to t_0$ for some $t_0\geq 0$. From (4.4), (4.6), (f_1)-(f_2), we can see that $t_0>0$. Now we claim that $t_0=1$. Passing to the limit as $n\to\infty$ in (4.4), and using (4.6) and the dominated convergence theorem, we have that

$$t_0^{p-q}\|w\|_{V_0,p}^{p}+\|w\|_{V_0,q}^{q}=\int_{\mathbb{R}^N}\frac{f(t_0w)+\gamma(t_0w)^{q^*-1}}{(t_0w)^{q-1}}\,w^q\,dx.$$

Recalling that $w\in\mathcal{M}_{V_0}$, we obtain

$$(t_0^{p-q}-1)\|w\|_{V_0,p}^{p}=\int_{\mathbb{R}^N}\left(\frac{f(t_0w)}{(t_0w)^{q-1}}-\frac{f(w)}{w^{q-1}}\right)w^q\,dx+\gamma(t_0^{q^*-q}-1)|w|_{q^*}^{q^*}.$$

Using assumption (f_4), we conclude that $t_0=1$. Therefore, letting $n\to\infty$ in (4.3), we deduce that $\lim_{n\to\infty}\mathcal{J}_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n})=\mathcal{L}_{V_0}(w)=d_{V_0}$, which contradicts (4.2). □

For any $\delta>0$ given by (4.1), let $\rho=\rho(\delta)>0$ be such that $M_\delta\subset B_\rho(0)$. Define $\Upsilon:\mathbb{R}^N\to\mathbb{R}^N$ by setting $\Upsilon(x)=x$ if $|x|<\rho$ and $\Upsilon(x)=\frac{\rho x}{|x|}$ if $|x|\geq\rho$. Let us consider the barycenter map $\beta_\varepsilon:\mathcal{N}_\varepsilon\to\mathbb{R}^N$ given by

$$\beta_\varepsilon(u)=\frac{\int_{\mathbb{R}^N}\Upsilon(\varepsilon x)(|u(x)|^p+|u(x)|^q)\,dx}{\int_{\mathbb{R}^N}(|u(x)|^p+|u(x)|^q)\,dx}.$$

Since $M\subset B_\rho(0)$, by the definition of Υ and applying the dominated convergence theorem, we conclude that

$$\lim_{\varepsilon\to 0}\beta_\varepsilon(\Phi_\varepsilon(y))=y \text{ uniformly in } y\in M. \tag{4.9}$$

The next compactness result plays an important role to verify that the solutions of the modified problem are also solutions of the original one.

Lemma 4.2. *Let $\varepsilon_n\to 0$ and $\{u_n\}_{n\in\mathbb{N}}\subset\mathcal{N}_{\varepsilon_n}$ be such that $\mathcal{J}_{\varepsilon_n}(u_n)\to d_{V_0}$. Then there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^N$ such that $v_n(x)=u_n(x+\tilde{y}_n)$ has a convergent subsequence in $\mathbb{Y}_{V_0}$. Moreover, up to a subsequence, $\{y_n\}_{n\in\mathbb{N}}=\{\varepsilon_n\tilde{y}_n\}_{n\in\mathbb{N}}$ is such that $y_n\to y_0\in M$.*

Proof. Proceeding as in the proof of Lemma 2.4, it is easy to see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$. Clearly, $\|u_n\|_{\mathbb{X}_{\varepsilon_n}} \not\to 0$ since $d_{V_0} > 0$. Consequently, we can argue as in the proof of Lemma 3.2 and Remark 3.2, to obtain a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ and constants $R, \beta > 0$ such that $\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^q dx \geq \beta$. Set $v_n(x) = u_n(x + \tilde{y}_n)$. Then, $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$, and, going if necessary to a subsequence, we may suppose that $v_n \rightharpoonup v \not\equiv 0$ in $\mathbb{Y}_{V_0}$. Take $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{M}_{V_0}$ and set $y_n = \varepsilon_n \tilde{y}_n$. Using $u_n \in \mathcal{N}_{\varepsilon_n}$ and (g_2), we have

$$\begin{aligned}
d_{V_0} \leq \mathcal{L}_{V_0}(\tilde{v}_n) &\leq \frac{1}{p}|\nabla \tilde{v}_n|_p^p + \frac{1}{q}|\nabla \tilde{v}_n|_q^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) \left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right) dx \\
&\quad - \int_{\mathbb{R}^N} \left(F(\tilde{v}_n) + \frac{\gamma}{q^*}(\tilde{v}_n^+)^{q^*}\right) dx \\
&\leq \frac{t_n^p}{p}\|u_n\|^p_{V_{\varepsilon_n},p} + \frac{t_n^q}{q}\|u_n\|^q_{V_{\varepsilon_n},q} - \int_{\mathbb{R}^N} G(\varepsilon_n x, t_n u_n)\, dx \\
&= \mathcal{J}_{\varepsilon_n}(t_n u_n) \leq \mathcal{J}_{\varepsilon_n}(u_n) = d_{V_0} + o_n(1),
\end{aligned}$$

which implies that

$$\mathcal{L}_{V_0}(\tilde{v}_n) \to d_{V_0} \quad \text{and} \quad \{\tilde{v}_n\}_{n\in\mathbb{N}} \subset \mathcal{M}_{V_0}. \tag{4.10}$$

Moreover, $\{\tilde{v}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{Y}_{V_0}$ and thus $\tilde{v}_n \rightharpoonup \tilde{v}$ in $\mathbb{Y}_{V_0}$. We may assume that $t_n \to t_0 > 0$. From the uniqueness of the weak limit, we have $\tilde{v} = t_0 v \not\equiv 0$. By Lemma 3.3, we have $\tilde{v}_n \to \tilde{v}$ in $\mathbb{Y}_{V_0}$, and so $v_n \to v$ in $\mathbb{Y}_{V_0}$. Moreover, $\mathcal{L}_{V_0}(\tilde{v}) = d_{V_0}$ and $\langle \mathcal{L}'_{V_0}(\tilde{v}), \tilde{v}\rangle = 0$. Next we show that $\{y_n\}_{n\in\mathbb{N}}$ admits a bounded subsequence. Indeed, suppose by contradiction that there is a subsequence of $\{y_n\}_{n\in\mathbb{N}}$, still denoted by itself, such that $|y_n| \to \infty$. Choose $R > 0$ such that $\Lambda \subset B_R(0)$. Then, for n large enough, $|y_n| > 2R$, and for each $x \in B_{R/\varepsilon_n}(0)$ we have $|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R$. Hence, using $v_n \to v$ in $\mathbb{Y}_{V_0}$, the definition of g, and the dominated convergence theorem, we obtain

$$\begin{aligned}
\|v_n\|^p_{V_0,p} + \|v_n\|^q_{V_0,q} &\leq \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, v_n)v_n \, dx \\
&\leq \int_{B_{R/\varepsilon_n}(0)} \tilde{f}(v_n)v_n\, dx + \int_{B^c_{R/\varepsilon_n}(0)} \left(f(v_n)v_n + \gamma(v_n^+)^{q^*}\right) dx \\
&\leq \frac{1}{K}\int_{B_{R/\varepsilon_n}(0)} V_0(|v_n|^p + |v_n|^q)\, dx + o_n(1)
\end{aligned}$$

which implies that $\left(1 - \frac{1}{K}\right)\left(\|v_n\|^p_{V_0,p} + \|v_n\|^q_{V_0,q}\right) \leq o_n(1)$. Since $v_n \to v \not\equiv 0$ in $\mathbb{Y}_{V_0}$ and $K > 1$, we get a contradiction. Hence, $\{y_n\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{R}^N$ and, up to a subsequence, we can assume that $y_n \to y_0$. If $y_0 \notin \overline{\Lambda}$, we can proceed as above to get $v_n \to 0$ in $\mathbb{Y}_{V_0}$. Then we have $y \in \overline{\Lambda}$. Now, assume by contradiction that $V(y_0) > V_0$. Using $\tilde{v}_n \to \tilde{v}$ in $\mathbb{Y}_{V_0}$, Fatou's lemma and the invariance of $\mathbb{R}^N$ by translations, we have

$$\begin{aligned}
d_{V_0} = \mathcal{L}_{V_0}(\tilde{v}) < \liminf_{n\to\infty}\Bigg(&\frac{1}{p}|\nabla \tilde{v}_n|_p^p + \frac{1}{q}|\nabla \tilde{v}_n|_q^q + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)\left(\frac{1}{p}|\tilde{v}_n|^p + \frac{1}{q}|\tilde{v}_n|^q\right) dx \\
&- \int_{\mathbb{R}^N}\left(F(\tilde{v}_n) + \frac{\gamma}{q^*}(\tilde{v}_n^+)^{q^*}\right) dx\Bigg) \leq \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(t_n u_n) \leq \liminf_{n\to\infty} \mathcal{J}_{\varepsilon_n}(u_n) = d_{V_0}
\end{aligned}$$

which leads to a contradiction. Therefore, $V(y_0) = V_0$ and $y_0 \in \overline{M}$. The assumption (V_2) shows that $y_0 \notin \partial M$ and thus $y_0 \in M$. □

Let us define $\widetilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : \mathcal{J}_\varepsilon(u) \leq d_{V_0} + \pi(\varepsilon)\}$, where $\pi(\varepsilon) = \sup_{y\in M} |\mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) - d_{V_0}|$. By Lemma 4.1, we know that $\pi(\varepsilon) \to 0$ as $\varepsilon \to 0$. By the definition of $\pi(\varepsilon)$, we have that, for all $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and thus $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$. We present below an interesting relation between $\widetilde{\mathcal{N}}_\varepsilon$ and the barycenter map.

Lemma 4.3. *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} \operatorname{dist}(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $\varepsilon_n \to 0$ as $n \to \infty$. Then we can find $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\operatorname{dist}(\beta_{\varepsilon_n}(u_n), M_\delta) = \sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}} \operatorname{dist}(\beta_{\varepsilon_n}(u), M_\delta) + o_n(1).$$

Then, it suffices to find $\{y_n\}_{n\in\mathbb{N}} \subset M_\delta$ such that $\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0$. From $\mathcal{L}_{V_0}(tu_n) \leq \mathcal{J}_{\varepsilon_n}(tu_n)$ and $\{u_n\}_{n\in\mathbb{N}} \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we obtain $d_{V_0} \leq c_{\varepsilon_n} \leq \mathcal{J}_{\varepsilon_n}(u_n) \leq d_{V_0} + h(\varepsilon_n)$ which leads to $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$. By invoking Lemma 4.2, we can find $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n large enough. Hence,

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} (\Upsilon(\varepsilon_n z + y_n) - y_n)\,(|u_n(z+\tilde{y}_n)|^p + |u_n(z+\tilde{y}_n)|^q)\,dz}{\int_{\mathbb{R}^N} (|u_n(z+\tilde{y}_n)|^p + |u_n(z+\tilde{y}_n)|^q)\,dz}.$$

Taking into account that $u_n(\cdot + \tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $\varepsilon_n z + y_n \to y \in M_\delta$ for all $z \in \mathbb{R}^N$, we can see that $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$. The proof of the lemma is now complete. $\square$

We finalize the section by presenting a relation between the topology of M and the number of solutions of the modified problem (2.1). Since $\mathbb{S}^+_\varepsilon$ is not a complete metric space, we cannot use directly an abstract result as in [4–6]. However, we can invoke the abstract category result in [41] to achieve our purpose.

Theorem 4.1. *Assume that (V_1)-(V_2) and (f_1)-(f_4) are in force. Then, for any given $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\bar{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}_\delta)$, problem (2.1) has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions.*

Proof. For each $\varepsilon > 0$, we define the map $\alpha_\varepsilon : M \to \mathbb{S}^+_\varepsilon$ by setting $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$. From Lemma 4.1 we see that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(\Phi_\varepsilon(y)) = d_{V_0} \text{ uniformly in } y \in M. \tag{4.11}$$

Hence, there is a number $\hat{\varepsilon} > 0$ such that the set $\widetilde{\mathcal{S}}^+_\varepsilon = \{w \in \mathbb{S}^+_\varepsilon : \psi_\varepsilon(w) \leq d_{V_0} + \pi_0(\varepsilon)\}$ is nonempty for all $\varepsilon \in (0, \hat{\varepsilon})$, since $\psi_\varepsilon(M) \subset \widetilde{\mathcal{S}}^+_\varepsilon$. Here $\pi_0(\varepsilon) = \sup_{y\in M} |\psi_\varepsilon(\alpha_\varepsilon(y)) - d_{V_0}| \to 0$ as $\varepsilon \to 0$. From the above considerations, and taking into account Lemma 4.1, Lemma 2.2-(iii), Lemma 4.3 and (4.9), we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the following diagram is well defined for any $\varepsilon \in (0, \bar{\varepsilon})$:

$$M \overset{\Phi_\varepsilon}{\to} \Phi_\varepsilon(M) \overset{m_\varepsilon^{-1}}{\to} \alpha_\varepsilon(M) \overset{m_\varepsilon}{\to} \Phi_\varepsilon(M) \overset{\beta_\varepsilon}{\to} M_\delta.$$

From (4.9), we can choose a function $\theta(\varepsilon, y)$ with $|\theta(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in M$ and for all $\varepsilon \in (0, \bar{\varepsilon})$, such that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for all $y \in M$. Define $H(t, y) = y + (1-t)\theta(\varepsilon, y)$, with $(t, y) \in [0, 1] \times M$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Obviously, $H(0, y) = \beta_\varepsilon(\Phi_\varepsilon(y))$ and $H(1, y) = y$ for all $y \in M$. That is $H(t, y)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $\mathrm{id} : M \to M_\delta$. This fact implies that

$$\operatorname{cat}_{\alpha_\varepsilon(M)} \alpha_\varepsilon(M) \geq \operatorname{cat}_{M_\delta}(M). \tag{4.12}$$

It follows from Corollary 2.1, Lemma 3.4, and Theorem 27 in [41], with $c = c_\varepsilon \leq d_{V_0} + \pi_0(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, that ψ_ε has at least $\mathrm{cat}_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Therefore, by Proposition 2.1-(d) and (4.12), we conclude that $\mathcal{J}_\varepsilon$ admits at least $\mathrm{cat}_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

5. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We start with the following lemma which plays a fundamental role in the study of behavior of the maximum points of solutions to (1.1).

Lemma 5.1. *Let $\varepsilon_n \to 0$ and $u_n \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a solution to (2.1). Then $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$, and there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \in L^\infty(\mathbb{R}^N)$ and for some $C > 0$ it holds*

$$|v_n|_\infty \leq C \quad \text{for all } n \in \mathbb{N}.$$

Moreover,

$$v_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N}. \tag{5.1}$$

Proof. Observing that $\mathcal{J}_{\varepsilon_n}(u_n) \leq d_{V_0} + \pi(\varepsilon_n)$ with $\pi(\varepsilon_n) \to 0$ as $n \to \infty$, we can repeat the same arguments used in the proof of Lemma 4.3 to show that $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$. Then, applying Lemma 4.2, there exists $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n)$ strongly converges in $\mathbb{Y}_{V_0}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

In what follows, we obtain a suitable L^∞-estimate by using some arguments found in [2, 23, 26]. Let $x_0 \in \mathbb{R}^N$, $R_0 > 1$, $0 < t < s < 1 < R_0$ and $\xi \in C_c^\infty(\mathbb{R}^N)$ be such that $0 \leq \xi \leq 1$, $\operatorname{supp}\xi \subset B_s(x_0)$, $\xi \equiv 1$ on $B_t(x_0)$, $|\nabla \xi| \leq \frac{2}{s-t}$. For $\zeta \geq 1$, set $A_{n,\zeta,\rho} = \{x \in B_\rho(x_0) : v_n(x) > \zeta\}$ and

$$Q_n = \int_{A_{n,\zeta,s}} (|\nabla v_n|^p + |\nabla v_n|^q)\, \xi^q \, dx.$$

Note that v_n satisfies

$$\int_{\mathbb{R}^N} |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \eta \, dx + \int_{\mathbb{R}^N} |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla \eta \, dx$$
$$+ \int_{\mathbb{R}^N} V_n(x)(v_n^{p-1} + v_n^{q-1})\eta \, dx = \int_{\mathbb{R}^N} g_n(x, v_n)\eta \, dx,$$

for all $\eta \in \mathbb{X}_\varepsilon$. Using $\eta_n = \xi^q(v_n - \zeta)_+$ as test function, we obtain

$$q\int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)_+ |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \xi \, dx + \int_{A_{n,\zeta,s}} \xi^q |\nabla v_n|^p \, dx$$
$$+ q\int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)_+ |\nabla v_n|^{q-2}\nabla v_n \cdot \nabla \xi \, dx + \int_{A_{n,\zeta,s}} \xi^q |\nabla v_n|^q \, dx$$
$$+ \int_{A_{n,\zeta,s}} V_n(v_n^{p-1} + v_n^{q-1})\xi^q(v_n - \zeta)_+ \, dx = \int_{A_{n,\zeta,s}} g_n(x, v_n)\xi^q(v_n - \zeta)_+ \, dx,$$

which combined with (V_1) yields

$$Q_n \leq C\int_{A_{n,\zeta,s}} \xi^{q-1}(v_n - \zeta)_+ |\nabla \xi| \left(|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1}\right) dx$$
$$- \int_{A_{n,\zeta,s}} V_0 \xi^{q-1}(v_n - \zeta)_+ (v_n^{p-1} + v_n^{q-1}) \, dx + \int_{A_{n,\zeta,s}} g_n(x, v_n)\xi^q(v_n - \zeta)_+ \, dx.$$

By (2.2), choosing $\zeta > 0$ sufficiently small, we find

$$Q_n \le C \int_{A_{n,\zeta,s}} \xi^{q-1}(v_n-\zeta)_+|\nabla\xi| \left(|\nabla v_n|^{p-1} + |\nabla v_n|^{q-1}\right) dx + \int_{A_{n,\zeta,s}} v_n^{q^*-1}\xi^q(v_n-\zeta)_+ \, dx.$$

Proceeding similarly to Lemma 3.4 in [2], we obtain

$$Q_n \le C\left(\int_{A_{n,\zeta,s}} \left|\frac{v_n-\zeta}{s-t}\right|^{q^*} dx + (\zeta^{q^*}+1)|A_{n,\zeta,s}|\right).$$

Exploiting the definition of ξ we can infer

$$\int_{A_{n,\zeta,t}} |\nabla v_n|^q \, dx \le C\left(\int_{A_{n,\zeta,s}} \left|\frac{v_n-\zeta}{s-t}\right|^{q^*} dx + (\zeta^{q^*}+1)|A_{n,\zeta,s}|\right),$$

where $C>0$ does not depend on ζ and $\zeta \ge \zeta_0 \ge 1$, for some constant ζ_0.

Fix $R_1 > 0$ and define $\sigma_j = \frac{R_1}{2}\left(1+\frac{1}{2^j}\right)$, $\bar\sigma_j = \frac{1}{2}(\sigma_j+\sigma_{j+1})$, $\zeta_j = \frac{\zeta_0}{2}\left(1-\frac{1}{2^{j+1}}\right)$, and $Q_{j,n} = \int_{A_{n,\zeta_j,\sigma_j}} ((v_n-\zeta_j)_+)^{q^*} dx$. Then, arguing as in Step 1 in Lemma 3.5 in [2], we can see that for each $n \in \mathbb{N}$,

$$Q_{j+1,n} \le CA^\tau Q_{j,n}^{1+\tau} \quad \text{for all } j \in \mathbb{N}\cup\{0\},$$

where $C,\tau>0$ are independent of n and $A>1$. Since $v_n \to v$ in $\mathbb{Y}_{V_0}$, we have

$$\limsup_{\zeta_0\to\infty}\left(\limsup_{n\to\infty} Q_{0,n}\right) = \limsup_{\zeta_0\to\infty}\left(\limsup_{n\to\infty}\int_{A_{n,\zeta_0,\sigma_0}} \left(\left(v_n-\frac{\zeta_0}{4}\right)_+\right)^{q^*} dx\right) = 0.$$

Hence, there exists $n_0 \in \mathbb{N}$ and $\zeta_0^* > 0$ such that $Q_{0,n} \le C^{\frac{1}{\tau}}A^{-\frac{1}{\tau^2}}$ for $n \ge n_0$ and $\zeta_0 \ge \zeta_0^*$. Exploiting Lemma 4.7 in [26], $\lim_{j\to\infty} Q_{j,n} = 0$ for $n \ge n_0$. On the other hand,

$$\lim_{j\to\infty} Q_{j,n} = \lim_{j\to\infty}\int_{A_{n,\zeta_j,\sigma_j}} ((v_n-\zeta_j)_+)^{q^*} dx = \int_{A_{n,\frac{\zeta_0}{2},\frac{R_1}{2}}} \left(\left(v_n-\frac{\zeta_0}{2}\right)_+\right)^{q^*} dx.$$

Then,

$$\int_{A_{n,\frac{\zeta_0}{2},\frac{R_1}{2}}} \left(\left(v_n-\frac{\zeta_0}{2}\right)_+\right)^{q^*} dx = 0 \quad \text{for all } n \ge n_0,$$

and so $v_n(x) \le \frac{\zeta_0}{2}$ for a.e. $x \in B_{\frac{R_1}{2}}(x_0)$ and for all $n \ge n_0$. From the arbitrariness of $x_0 \in \mathbb{R}^N$, we deduce that $v_n(x) \le \frac{\zeta_0}{2}$ for a.e. $x \in \mathbb{R}^N$ and for all $n \ge n_0$, that is $|v_n|_\infty \le \frac{\zeta_0}{2}$ for all $n \ge n_0$. Setting $C = \max\left\{\frac{\zeta_0}{2}, |v_1|_\infty, \dots, |v_{n_0-1}|_\infty\right\}$, we get $|v_n|_\infty \le C$ for all $n \in \mathbb{N}$. Then, combining this estimate with the regularity results in [24], we obtain that $\{v_n\}_{n\in\mathbb{N}} \subset C^{1,\alpha}_{loc}(\mathbb{R}^N)$. Finally, we show that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. Arguing as before, we can see that for each $\delta > 0$ we have that

$$\limsup_{|x_0|\to\infty}\left(\limsup_{n\to\infty} Q_{0,n}\right) = \limsup_{|x_0|\to\infty}\left(\limsup_{n\to\infty}\int_{A_{n,\zeta_0,\sigma_0}} \left(\left(v_n-\frac{\delta}{4}\right)_+\right)^{q^*} dx\right) = 0.$$

Therefore, applying lemma Lemma 4.7 in [26], there exist $R_* > 0$ and $n_0 \in \mathbb{N}$ such that $\lim_{j\to\infty} Q_{j,n} = 0$ if $|x_0| > R_*$ and for $n \ge n_0$, which yields $v_n(x) \le \frac{\delta}{4}$ for $x \in B_{\frac{R_1}{2}}(x_0)$ and $|x_0| > R_*$, for all $n \ge n_0$. Now, increasing R_* if necessary, it holds $v_n(x) \le \frac{\delta}{4}$ for $|x| > R_*$ and for all $n \in \mathbb{N}$, and this completes the proof of lemma. □

5.1. **Proof of Theorem 1.1.** Take $\delta > 0$ such that $M_\delta \subset \Lambda$. We first show that there exists $\tilde{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$ and any solution $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$ of (2.1), it holds

$$|u_\varepsilon|_{L^\infty(\Lambda^c_\varepsilon)} < a. \tag{5.2}$$

In order to prove the claim we argue by contradiction. Suppose that for some sequence $\varepsilon_n \to 0$ we can obtain $u_n = u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $\mathcal{J}'_{\varepsilon_n}(u_{\varepsilon_n}) = 0$ and

$$|u_n|_{L^\infty(\Lambda^c_{\varepsilon_n})} \geq a. \tag{5.3}$$

As in Lemma 4.2, we have that $\mathcal{J}_{\varepsilon_n}(u_n) \to d_{V_0}$ and therefore we can apply Lemma 4.2 to obtain a sequence $\{\tilde{y}_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n) \to v$ in $\mathbb{Y}_{V_0}$ and $\varepsilon_n \tilde{y}_n \to y_0 \in M$.

Choosing $r > 0$ such that $B_r(y_0) \subset B_{2r}(y_0) \subset \Lambda$, we have $B_{\frac{r}{\varepsilon_n}}(\frac{y_0}{\varepsilon_n}) \subset \Lambda_{\varepsilon_n}$. Moreover, for any $y \in B_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$, it holds

$$\left|y - \frac{y_0}{\varepsilon_n}\right| \leq |y - \tilde{y}_n| + \left|\tilde{y}_n - \frac{y_0}{\varepsilon_n}\right| < \frac{1}{\varepsilon_n}(r + o_n(1)) < \frac{2r}{\varepsilon_n} \quad \text{for } n \text{ sufficiently large.}$$

For these values of n, we have $\Lambda^c_{\varepsilon_n} \subset B^c_{\frac{r}{\varepsilon_n}}(\tilde{y}_n)$. By using (5.1), there exists $R > 0$ such that $v_n(x) < a$ for any $|x| \geq R$ and $n \in \mathbb{N}$, and thus $u_n(x) < a$ for any $x \in B^c_R(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, there exists $\nu \in \mathbb{N}$ such that for any $n \geq \nu$ it holds $\Lambda^c_{\varepsilon_n} \subset B^c_{\frac{r}{\varepsilon_n}}(\tilde{y}_n) \subset B^c_R(\tilde{y}_n)$. Hence, $u_n(x) < a$ for any $x \in \Lambda^c_{\varepsilon_n}$ and $n \geq \nu$, which is in contrast with (5.3).

Let $\bar{\varepsilon}_\delta > 0$ be given by Theorem 4.1 and set $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \bar{\varepsilon}_\delta\}$. Take $\varepsilon \in (0, \varepsilon_\delta)$. By Theorem 4.1 we get at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions to (2.1). If u_ε is one of these solutions, we have that $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$, and we can use (5.2) and the definition of g to deduce that $g(\varepsilon x, u_\varepsilon) = f(u_\varepsilon) + \gamma u_\varepsilon^{q^*-1}$. This means that u_ε is also a solution of (1.1). Consequently, (1.1) admits at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions. Now we consider $\varepsilon_n \to 0$ and take a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathbb{X}_{\varepsilon_n}$ of solutions to (2.1) as above. In order to study the behavior of the maximum points of u_n, we first note that, by the definition of g and (g_1), there exists $\sigma \in (0, a)$ sufficiently small such that

$$g(\varepsilon x, t)t \leq \frac{V_0}{K}(t^p + t^q) \quad \forall (x, t) \in \mathbb{R}^N \times [0, \sigma]. \tag{5.4}$$

As before, we can take $R > 0$ such that

$$|u_n|_{L^\infty(B^c_R(\tilde{y}_n))} < \sigma. \tag{5.5}$$

Up to a subsequence, we may also assume that

$$|u_n|_{L^\infty(B_R(\tilde{y}_n))} \geq \sigma. \tag{5.6}$$

Otherwise, if this is not the case, we have $|u_n|_\infty < \sigma$. Then, using $\langle \mathcal{J}'_{\varepsilon_n}(u_n), u_n \rangle = 0$ and (5.4), we find

$$\|u_n\|^p_{V_{\varepsilon_n}, p} + \|u_n\|^q_{V_{\varepsilon_n}, q} \leq \int_{\mathbb{R}^N} g(\varepsilon_n x, u_n) u_n \, dx \leq \frac{V_0}{K} \int_{\mathbb{R}^N} (|u_n|^p + |u_n|^q) \, dx$$

which leads to a contradiction. Therefore, (5.6) is satisfied. In view of (5.5) and (5.6), we can deduce that if p_n is a global maximum point of u_n, then $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R(0)$. Since $\varepsilon_n \tilde{y}_n \to y_0 \in M$ and $\{q_n\}_{n\in\mathbb{N}} \subset B_R(0)$, we obtain that $\varepsilon_n p_n \to y_0$ which together with the continuity of V yields $\lim_{n\to\infty} V(\varepsilon_n p_n) = V(y_0) = V_0$. Finally, we prove the decay estimate for u_n. Since $v_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$, and using (g_1), we can find $R > 0$ such that

$$g_n(x, v_n(x)) \leq \frac{V_0}{2}(v_n^{p-1}(x) + v_n^{q-1}(x)) \quad \forall |x| \geq R.$$

Then, by using (V_1), we obtain

$$\begin{aligned} -\Delta_p v_n - \Delta_q v_n + \frac{V_0}{2}(v_n^{p-1} + v_n^{q-1}) &= g_n(x, v_n) - \left(V_n - \frac{V_0}{2}\right)(v_n^{p-1} + v_n^{q-1}) \\ &\leq g_n(x, v_n) - \frac{V_0}{2}(v_n^{p-1} + v_n^{q-1}) \leq 0 \quad \forall |x| \geq R. \end{aligned} \tag{5.7}$$

Let $\phi(x) = Me^{-c|x|}$ with $c, M > 0$ such that $c^p(p-1) < \frac{V_0}{2}$, $c^q(q-1) < \frac{V_0}{2}$ and $Me^{-cR} \geq v_n(x)$ for all $|x| = R$. We can see that

$$\begin{aligned} &- \Delta_p \phi - \Delta_q \phi + \frac{V_0}{2}(\phi^{p-1} + \phi^{q-1}) \\ &= \phi^{p-1}\left(\frac{V_0}{2} - c^p(p-1) + \frac{N-1}{|x|}c^{p-1}\right) + \phi^{q-1}\left(\frac{V_0}{2} - c^q(q-1) + \frac{N-1}{|x|}c^{q-1}\right) > 0 \quad \forall |x| \geq R. \end{aligned} \tag{5.8}$$

Using $\eta = (v_n - \phi)^+ \in W_0^{1,q}(\mathbb{R}^N \setminus B_R)$ as test function in (5.7) and (5.8), we find

$$\begin{aligned} 0 \geq \int_{\{|x|\geq R\}\cap\{v_n > \phi\}} &\Bigg(\Big((|\nabla v_n|^{p-2}\nabla v_n - |\nabla \phi|^{p-2}\nabla \phi) \cdot \nabla \eta + (|\nabla v_n|^{q-2}\nabla v_n - |\nabla \phi|^{q-2}\nabla \phi) \cdot \nabla \eta\Big) \\ &+ \frac{V_0}{2}\Big((v_n^{p-1} - \phi^{p-1}) + (v_n^{q-1} - \phi^{q-1})\Big)\eta\Bigg)\, dx. \end{aligned}$$

Recalling that for $t > 1$ it holds $(|x|^{t-2}x - |y|^{t-2}y) \cdot (x - y) \geq 0$ for all $x, y \in \mathbb{R}^N$ (see formula (2.10) in [40]), and that ϕ, v_n are continuous in $\mathbb{R}^N$, we deduce that $v_n(x) \leq \phi(x)$ for all $|x| \geq R$. Recalling that $\{v_n\}_{n\in\mathbb{N}}$ is uniformly bounded in $L^\infty(\mathbb{R}^N)$ and that $u_n(x) = v_n(x - \tilde{y}_n)$, we obtain that $u_n(x) \leq C_1 e^{-C_2|x-p_n|}$ for all $x \in \mathbb{R}^N$. This completes the proof of Theorem 1.1. □

6. The supercritical case

In this last section we focus our attention on (1.7). Firstly, we truncate the nonlinearity $\phi(u) = u^{s-1} + \mu u^{\tau-1}$ in a suitable way. Let $K > 0$ be a real number, whose value will be fixed later, and set

$$\phi_\mu(t) = \begin{cases} 0 & \text{if } t < 0, \\ t^{s-1} + \mu t^{\tau-1} & \text{if } 0 \leq t < K, \\ (1 + \mu K^{\tau-s})t^{s-1} & \text{if } t \geq K. \end{cases}$$

Clearly, ϕ_μ verifies the assumptions (f_1)-(f_4) ((f_3) with $\vartheta = s > q$). Moreover,

$$\phi_\mu(t) \leq (1 + \mu K^{\tau-s})t^{s-1} \quad \forall t \geq 0. \tag{6.1}$$

Now, we can introduce the following truncated problem

$$\left\{ \begin{array}{l} -\Delta_p u - \Delta_q u + V(\varepsilon x)(u^{p-1} + u^{q-1}) = \phi_\mu(u) \text{ in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N),\ u > 0 \text{ in } \mathbb{R}^N. \end{array} \right. \tag{6.2}$$

It is easy to check that weak solutions of (6.2) are critical points of the energy functional $\mathcal{J}_{\varepsilon,\mu} : \mathbb{X}_\varepsilon \to \mathbb{R}$ defined by

$$\mathcal{J}_{\varepsilon,\mu}(u) = \frac{1}{p}\|u\|^p_{V_\varepsilon,p} + \frac{1}{q}\|u\|^q_{V_\varepsilon,q} - \int_{\mathbb{R}^N} \Phi_\mu(u)\, dx,$$

where $\Phi_\mu(t) = \int_0^t \phi_\mu(s)ds$. We also consider the autonomous functional

$$\mathcal{J}_{0,\mu}(u) = \frac{1}{p}\|u\|^p_{V_0,p} + \frac{1}{q}\|u\|^q_{V_0,q} - \int_{\mathbb{R}^N} \Phi_\mu(u)\, dx.$$

Using Theorem 1.1, we know that for any $\mu \geq 0$ and $\delta > 0$, there exists $\bar{\varepsilon}(\delta,\mu) > 0$ such that, for any $\varepsilon \in (0, \bar{\varepsilon}(\delta,\mu))$, problem (6.2) admits at least $cat_{M_\delta}(M)$ positive solutions $u_{\varepsilon,\mu}$. Now, we prove that it is possible to estimate the $W^{1,q}$-norm of these solutions uniformly with respect to μ. More precisely:

Lemma 6.1. *There exists $\bar{C} > 0$ such that $\|u_{\varepsilon,\mu}\|_{V_\varepsilon,q} \leq \bar{C}$ for any $\varepsilon > 0$ sufficiently small and uniformly in μ.*

Proof. A simple inspection of the proof of Theorem 1.1 shows that any solution $u_{\varepsilon,\mu}$ of (6.2) satisfies the following inequality $\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leq c_{0,\mu} + h_\mu(\varepsilon)$, where $c_{0,\mu}$ is the mountain pass level related to the functional $\mathcal{J}_{0,\mu}$, and $h_\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. Then, decreasing $\bar{\varepsilon}(\delta,\mu)$ if necessary, we may suppose that $\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leq c_{0,\mu} + 1$ for any $\varepsilon \in (0, \bar{\varepsilon}(\delta,\mu))$. Using the fact that $c_{0,\mu} \leq c_{0,0}$ for any $\mu \geq 0$, we can deduce that

$$\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) \leq c_{0,0} + 1 \tag{6.3}$$

for any $\varepsilon \in (0, \bar{\varepsilon}(\delta,\mu))$. On the other hand,

$$\begin{aligned}
\mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) &= \mathcal{J}_{\varepsilon,\mu}(u_{\varepsilon,\mu}) - \frac{1}{s}\langle \mathcal{J}'_{\varepsilon,\mu}(u_{\varepsilon,\mu}), u_{\varepsilon,\mu}\rangle \\
&= \left(\frac{1}{p} - \frac{1}{s}\right)\|u_{\varepsilon,\mu}\|^p_{V_\varepsilon,p} + \left(\frac{1}{q} - \frac{1}{s}\right)\|u_{\varepsilon,\mu}\|^q_{V_\varepsilon,q} + \int_{\mathbb{R}^N}\left(\frac{1}{s}\phi_\mu(u_{\varepsilon,\mu})u_{\varepsilon,\mu} - \Phi_\mu(u_{\varepsilon,\mu})\right) dx \\
&\geq \left(\frac{1}{p} - \frac{1}{s}\right)\|u_{\varepsilon,\mu}\|^p_{V_\varepsilon,p} + \left(\frac{1}{q} - \frac{1}{s}\right)\|u_{\varepsilon,\mu}\|^q_{V_\varepsilon,q},
\end{aligned} \tag{6.4}$$

where in the last inequality we have used assumption (f_3). Putting together (6.3) and (6.4), we can infer that $\|u_{\varepsilon,\mu}\|_{V_\varepsilon,q} \leq \left(\left(\frac{sq}{s-q}\right)(c_{0,0}+1)\right)^{\frac{1}{q}} = \bar{C}$ for any $\varepsilon \in (0, \bar{\varepsilon}(\delta,\mu))$. □

Now, our plan is to prove that $u_{\varepsilon,\mu}$ is a solution of the original problem (1.7) for μ sufficiently small. To this end, we will show that we can find $K_0 > 0$ such that for any $K \geq K_0$, there exists $\mu_0 = \mu_0(K) > 0$ such that

$$|u_{\varepsilon,\mu}|_\infty \leq K \quad \forall \mu \in [0, \mu_0]. \tag{6.5}$$

In order to achieve our goal, we develop a suitable Moser iteration technique [33]. For simplicity, we set $u = u_{\varepsilon,\mu}$. For any $L > 0$, we define $u_L = \min\{u, L\} \geq 0$, where $\beta > 1$ will be chosen later, and let $w_L = u u_L^{\beta-1}$. Taking $z_L = u_L^{q(\beta-1)}u$ in (6.2), we see that

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla u|^p + |\nabla u|^q)u_L^{q(\beta-1)}\, dx + \int_{\{u\leq L\}} q(\beta-1)u^{q(\beta-1)}(|\nabla u|^p + |\nabla u|^q)\, dx \\
&+ \int_{\mathbb{R}^N} V(\varepsilon\, x)(u^p + u^q)u_L^{q(\beta-1)}\, dx = \int_{\mathbb{R}^N}\phi_\mu(u)u u_L^{q(\beta-1)}\, dx.
\end{aligned} \tag{6.6}$$

Putting together (6.6), (6.1) and (V_1), we get

$$\int_{\mathbb{R}^N}|\nabla u|^q u_L^{q(\beta-1)}\, dx \leq C_{\mu,K}\int_{\mathbb{R}^N} u^s u_L^{q(\beta-1)}\, dx, \tag{6.7}$$

where $C_{\mu,K} = 1 + \mu K^{\tau - s}$. On the other hand, by the Sobolev inequality, $(a+b)^q \le 2^{q-1}(a^q + b^q)$ for all $a, b > 0$, and $\beta > 1$, we have

$$\begin{aligned} |w_L|_{q^*}^q &\le S_*^{-1} \int_{\mathbb{R}^N} |\nabla w_L|^q \, dx = S_*^{-1} \int_{\mathbb{R}^N} |\nabla (u_L^{\beta-1} u)|^q \, dx \\ &\le S_*^{-1} 2^{q-1} \left(\int_{\mathbb{R}^N} (\beta-1)^q u_L^{q(\beta-1)} |\nabla u|^q \, dx + \int_{\mathbb{R}^N} u_L^{q(\beta-1)} |\nabla u_L|^q \, dx \right) \\ &\le S_*^{-1} 2^{q-1} ((\beta-1)^q + 1) \int_{\mathbb{R}^N} u_L^{q(\beta-1)} |\nabla u|^q \, dx \\ &\le S_*^{-1} 2^{q-1} \beta^q \left(\left(\frac{\beta-1}{\beta} \right)^q + \frac{1}{\beta^q} \right) \int_{\mathbb{R}^N} u_L^{q(\beta-1)} |\nabla u|^q \, dx \\ &\le C_1 \beta^q \int_{\mathbb{R}^N} u_L^{q(\beta-1)} |\nabla u|^q \, dx, \end{aligned} \tag{6.8}$$

with $C_1 = 2^q S_*^{-1} > 0$. Taking into account (6.7) and (6.8), and using the Hölder inequality, we deduce that

$$|w_L|_{q^*}^q \le C_1 C_{\mu,K} \beta^q \int_{\mathbb{R}^N} u^s u_L^{q(\beta-1)} \, dx = C_1 C_{\mu,K} \beta^q \int_{\mathbb{R}^N} u^{s-q} w_L^q \, dx \le C_1 \beta^q C_{\mu,K} |u|_{q^*}^{s-q} |w_L|_{\alpha^*}^q, \tag{6.9}$$

where $\alpha^* = \frac{qq^*}{q^* - (s-q)}$. In view of Lemma 6.1, the embedding $W^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$ and (6.9), we see that

$$|w_L|_{q^*}^q \le C_2 \beta^q C_{\mu,K} |w_L|_{\alpha^*}^q, \tag{6.10}$$

where $C_2 = C_1 S_*^{-\frac{s-q}{q}} \bar{C}^{s-q}$ is independent of ε and μ. Now, we observe that if $u^\beta \in L^{\alpha^*}(\mathbb{R}^N)$, it follows from the definition of w_L, that $u_L \le u$, and (6.10), that it holds

$$|w_L|_{q^*}^q \le C_2 \beta^q C_{\mu,K} |u|_{\beta\alpha^*}^{q\beta} < \infty. \tag{6.11}$$

Letting $L \to \infty$ in (6.11), the Fatou Lemma yields

$$|u|_{q^*\beta} \le (C_2 C_{\mu,K})^{\frac{1}{q\beta}} \beta^{\frac{1}{\beta}} |u|_{\beta\alpha^*} \tag{6.12}$$

whenever $u^{\beta\alpha^*} \in L^1(\mathbb{R}^N)$. Now, we set $\beta = \frac{q^*}{\alpha^*} > 1$, and observe that, since $u \in L^{q^*}(\mathbb{R}^N)$, the above inequality holds for this choice of β. Then, using the fact that $\beta^2 \alpha^* = q^* \beta$, it follows that (6.12) holds with β replaced by β^2. Consequently,

$$|u|_{q^*\beta^2} \le (C_2 C_{\mu,K})^{\frac{1}{q\beta^2}} \beta^{\frac{2}{\beta^2}} |u|_{\beta^2\alpha^*} \le (C_2 C_{\mu,K})^{\frac{1}{q}\left(\frac{1}{\beta} + \frac{1}{\beta^2}\right)} \beta^{\frac{1}{\beta} + \frac{2}{\beta^2}} |u|_{\beta\alpha^*}.$$

Iterating this process and recalling that $\beta\alpha^* = q^*$, we infer that, for every $m \in \mathbb{N}$,

$$|u|_{q^*\beta^m} \le (C_2 C_{\mu,K})^{\sum_{j=1}^m \frac{1}{q\beta^j}} \beta^{\sum_{j=1}^m j\beta^{-j}} |u|_{q^*}. \tag{6.13}$$

Sending $m \to \infty$ in (6.13) and using Lemma 6.1, we obtain

$$|u|_\infty \le (C_2 C_{\mu,K})^{\gamma_1} \beta^{\gamma_2} C_3, \tag{6.14}$$

where $C_3 = S_*^{-\frac{1}{q}} \bar{C}$, $\gamma_1 = \frac{1}{q} \sum_{j=1}^\infty \frac{1}{\beta^j} < \infty$, and $\gamma_2 = \sum_{j=1}^\infty \frac{j}{\beta^j} < \infty$. Next, we will find some suitable values of K and μ such that the following inequality holds $(C_2 C_{\mu,K})^{\gamma_1} \beta^{\gamma_2} C_3 \le K$, or equivalently,

$$1 + \mu K^{\tau - s} \le C_2^{-1} \beta^{-\frac{\gamma_2}{\gamma_1}} (K C_3^{-1})^{\frac{1}{\gamma_1}}.$$

Take $K > 0$ such that $(KC_3^{-1})^{\frac{1}{\gamma_1}} > C_2\beta^{\frac{\gamma_2}{\gamma_1}}$, and fix $\mu_0 > 0$ satisfying

$$\mu \leq \mu_0 \leq \left(\frac{(KC_3^{-1})^{\frac{1}{\gamma_1}}}{C_2\beta^{\frac{\gamma_2}{\gamma_1}}} - 1 \right) \frac{1}{K^{\tau-s}}.$$

Then, in view of (6.14), we see that (6.5) is satisfied, that is $u = u_{\varepsilon,\mu}$ is a solution of (1.7). This completes the proof of Theorem 1.2.

Acknowledgements. The first author was partly supported by the GNAMPA Project 2020 entitled: *Studi di Problemi Frazionari Nonlocali tramite Tecniche Variazionali*. The research of V.D. Rădulescu was supported by the grant "Nonlinear Differential Systems in Applied Sciences" of the Romanian Ministry of Research, Innovation and Digitization, within PNRR-III-C9-2022-I8/22.

References

[1] C.O. Alves and A.R. da Silva, *Multiplicity and concentration behavior of solutions for a quasilinear problem involving* N*-functions via penalization method*, Electron. J. Differential Equations 2016, Paper No. **158**, 24 pp.

[2] C.O. Alves and A.R. da Silva, *Multiplicity and concentration of positive solutions for a class of quasilinear problems through Orlicz–Sobolev space*, J. Math. Phys. **57** (2016), no. 11, 111502, 22 pp.

[3] C.O. Alves and J.M. do Ó, and M.A.S. Souto, *Local mountain-pass for a class of elliptic problems in* $\mathbb{R}^N$ *involving critical growth*, Nonlinear Analysis **46** (2001) 495–510.

[4] C.O. Alves and G.M. Figueiredo, *Multiplicity of positive solutions for a quasilinear problem in* $\mathbb{R}^N$ *via penalization method*, Adv. Nonlinear Stud. **5** (2005), no. 4, 551–572.

[5] C.O. Alves and G.M. Figueiredo, *Existence and multiplicity of positive solutions to a* p*-Laplacian equation in* $\mathbb{R}^N$, Differential Integral Equations **19** (2006), no. 2, 143–162.

[6] C.O. Alves and G.M. Figueiredo, *Multiplicity and concentration of positive solutions for a class of quasilinear problems*, Adv. Nonlinear Stud. **11** (2011), no. 2, 265–294.

[7] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Functional Analysis **14** (1973), 349–381.

[8] V. Ambrosio and V.D. Rădulescu, *Fractional double-phase patterns: concentration and multiplicity of solutions*, J. Math. Pures Appl. (9) **142** (2020), 101–145.

[9] V. Ambrosio and D. Repovš, *Multiplicity and concentration results for a (p,q)-Laplacian problem in* $\mathbb{R}^N$, Z. Angew. Math. Phys. **72** (2021), no. 1, 33.

[10] L. Beck, G. Mingione, *Lipschitz bounds and non-uniform ellipticity*, Comm. Pure Appl. Math. **73** (2020), 944-1034.

[11] V. Benci and G. Cerami, *Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology*, Calc. Var. Partial Differential Equations **2** (1994), 29–48.

[12] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[13] J. Chabrowski and J. Yang, *Existence theorems for elliptic equations involving supercritical Sobolev exponent*, Adv. Differential Equations **2** (1997), no. 2, 231–256.

[14] L. Cherfils and V. Il'yasov, *On the stationary solutions of generalized reaction difusion equations with* p&q*-Laplacian*, Commun. Pure Appl. Anal. **1** (4), 1–14 (2004).

[15] S. Cingolani and M. Lazzo, *Multiple semiclassical standing waves for a class of nonlinear Schrödinger equations*, Topol. Methods Nonlinear Anal. **10** (1997), no. 1, 1–13.

[16] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal. **215** (2015), no. 2, 443–496.

[17] M. Del Pino and P.L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differential Equations **4** (1996), 121–137.

[18] J.M. do Ó, *On existence and concentration of positive bound states of* p*-Laplacian equations in* $\mathbb{R}^N$ *involving critical growth*, Nonlinear Anal. **62** (2005), 777–801.

[19] G.M. Figueiredo, *Existence of positive solutions for a class of* p&q *elliptic problems with critical growth on* $\mathbb{R}^N$, J. Math. Anal. Appl. **378** (2011), 507–518.

[20] G.M. Figueiredo and M. Furtado, *Positive solutions for some quasilinear equations with critical and supercritical growth*, Nonlinear Anal. **66** (2007), no. 7, 1600–1616.

[21] G.M. Figueiredo and M. Furtado, *Positive solutions for a quasilinear Schrödinger equation with critical growth*, J. Dynam. Differential Equations **24** (2012), no. 1, 13–28.

[22] G.M. Figueiredo and J.R. Santos, *Multiplicity and concentration behavior of positive solutions for a Schrödinger-Kirchhoff type problem via penalization method*, ESAIM Control Optim. Calc. Var. **20** (2014), no. 2, 389–415.

[23] N. Fusco and C. Sbordone, *Some remarks on the regularity of minima of anisotropic integrals*, Comm. Partial Differential Equations **18** (1993), no. 1-2, 153–167.

[24] C. He and G. Li, *The regularity of weak solutions to nonlinear scalar field elliptic equations containing* p&q-*Laplacians*, Ann. Acad. Sci. Fenn. Math. **33** (2008), no. 2, 337–371.

[25] C. He and G. Li, *The existence of a nontrivial solution to the* p&q-*Laplacian problem with nonlinearity asymptotic to* u^{p-1} *at infinity in* $\mathbb{R}^N$, Nonlinear Anal. **68** (2008), no. 5, 1100–1119.

[26] O.A. Ladyzhenskaya and N.N. Ural'tseva, *Linear and Quasilinear Elliptic Equations*, (Academic Press, 1968).

[27] G. Li and G. Zhang, *Multiple solutions for the* p&q-*Laplacian problem with critical exponent*, Acta Math. Sci. Ser. B Engl. Ed. **29** (4) (2009), 903–918.

[28] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **1** (1984), no. 4, 223–283.

[29] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana **1** (1985) 145–201.

[30] P. Marcellini, *Regularity under general and* p, q-*growth conditions*, Discrete Contin. Dyn. Syst. Ser. S **13** (2020), no. 7, 2009–2031.

[31] P. Marcellini, *Growth conditions and regularity for weak solutions to nonlinear elliptic pdes*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 124408, 32 pp.

[32] G. Mingione and V.D. Rădulescu, *Recent developments in problems with nonstandard growth and nonuniform ellipticity*, J. Math. Anal. Appl. **501** (2021), no. 1, Paper No. 125197, 41 pp.

[33] J. Moser, *A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math. **13** (1960), 457–468.

[34] D. Mugnai and N.S. Papageorgiou, *Wang's multiplicity result for superlinear* (p, q)-*equations without the Ambrosetti-Rabinowitz condition*, Trans. Amer. Math. Soc. **366** (9) (2014), 4919–4937.

[35] N. Papageorgiou and V.D. Rădulescu, *Resonant* $(p, 2)$-*equations with asymmetric reaction*, Anal. Appl. (Singap.) **13** (2015), no. 5, 481–506.

[36] N.S. Papageorgiou, V.D. Rădulescu, and D.D. Repovš, *On a class of parametric* $(p, 2)$-*equations*, Appl. Math. Optim. **75** (2017), no. 2, 193–228.

[37] N.S. Papageorgiou, V.D. Rădulescu, and D.D. Repovš, *Double-phase problems with reaction of arbitrary growth* Z. Angew. Math. Phys. **69** (2018), no. 4, Art. 108, 21 pp.

[38] P.H. Rabinowitz, *Variational methods for nonlinear elliptic eigenvalue problems*, Indiana Univ. Math. J. **23** (1973/74), 729–754.

[39] P.H. Rabinowitz, *On a class of nonlinear Schrödinger equations* Z. Angew. Math. Phys. **43** (1992), no. 2, 270–291.

[40] J. Simon, *Régularité de la solution d'un problème aux limites non linéaires*, Ann. Fac. Sci. Toulouse Math. **3** (1981), 247–274.

[41] A. Szulkin and T. Weth, *The method of Nehari manifold*, in Handbook of Nonconvex Analysis and Applications, edited by D. Y. Gao and D. Motreanu (International Press, Boston, 2010), pp. 597–632.

[42] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[43] X. Wang, *On concentration of positive bound states of nonlinear Schrödinger equations*, Comm. Math. Phys. **53** (1993), 229–244.

[44] M. Willem, *Minimax Theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24, Birkhäuser Boston, Inc., Boston, MA, 1996.

[45] V.V. Zhikov, *Averaging of functionals of the calculus of variations and elasticity theory*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no. 4, 675–710; English translation in Math. USSR-Izv. **29** (1987), no. 1, 33–66.

Vincenzo Ambrosio
Dipartimento di Ingegneria Industriale e Scienze Matematiche
Università Politecnica delle Marche
Via Brecce Bianche, 12
60131 Ancona (Italy)
Email address: v.ambrosio@univpm.it

Vicenţiu D. Radulescu
Faculty of Applied Mathematics
AGH University of Kraków
al. Mickiewicza 30
30-059 Kraków (Poland)
&
Department of Mathematics
University of Craiova
200585 Craiova (Romania)
Email address: radulescu@inf.ucv.ro