Weak Solutions and Energy Estimates for a Class of Nonlinear Elliptic Neumann Problems

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Abstract

We establish existence results and energy estimates of solutions for a homogeneous Neumann problem involving the p–Laplace operator. The case of large dimensions, corresponding to the lack of compactness of $W^{1,p}(\Omega)$ in $C^0(\bar{\Omega})$ is also considered. In particular, for a precise localization of the parameter, the existence of a non-zero solution is established, without requiring any asymptotic condition at zero or at infinity of the nonlinear term. In the case of $(p-1)$–sublinear terms at the origin, we deduce the existence of solutions for small positive values of the parameter and we obtain that the corresponding solutions have smaller and smaller energies as the parameter goes to zero. Finally, a multiplicity result is obtained and concrete examples of applications are provided. A basic ingredient in our arguments is a recent local minimum theorem for differentiable functionals.

1 Introduction

The aim of this paper is to investigate the existence and the qualitative properties of nontrivial weak solutions of the following non-autonomous elliptic Neumann problem

$$\begin{cases} -\Delta_p u + \alpha(x)|u|^{p-2}u = \lambda f(x,u) & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and $\Omega \subset \mathbb{R}^N$ is a bounded open subset with boundary of class C^1.

As usual, Δ_p denotes the p-Laplace operator, namely $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$. We assume that $\alpha \in L^\infty(\Omega)$, with $\text{ess inf}_{x\in\Omega}\, \alpha(x) > 0$, λ is a positive real parameter and ν denotes the outward unit normal to $\partial\Omega$.

Throughout this paper we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

$$|f(x,t)| \le a_1 + a_2|t|^{q-1}, \qquad \forall\, (x,t) \in \Omega \times \mathbb{R}, \tag{1.2}$$

for some non-negative constants a_1, a_2, where $q \in\,]1, p^*[$ and p^* stands for the critical Sobolev exponent, that is

$$p^* := \begin{cases} \dfrac{pN}{N-p} & \text{if} \quad p < N \\ +\infty & \text{if} \quad p \ge N. \end{cases}$$

Finally, we denote by $\mathfrak{F}_q$ the class of all Carathéodory functions for which condition (1.2) holds true.

The question of existence of multiple non-zero solutions has been studied for Dirichlet problems also involving the p-Laplace operator and there are several such papers in the literature, using different hypotheses and techniques. On the other hand, recently there have been established some multiplicity results within the Neumann setting, see the papers [3, 5, 6, 7] and [9, 25, 34]. In these works, the authors establish the existence of weak solutions for certain nonlinear elliptic problems, by imposing different kinds of oscillatory behavior on the nonlinear term.

In all the above cited contributions the basic assumption $p > N$ is imposed, which corresponds to low-dimensional problems. This dimensionality condition implies that the Sobolev space $W^{1,p}(\Omega)$ is compactly embedded in $C^0(\bar{\Omega})$ and this fact is used extensively in the aforementioned works; see

Remark 3.9. In particular, in [14] and [16], the authors consider nonlinear Neumann eigenvalue problems and prove a "three solutions theorem" using an abstract multiplicity result of Ricceri [35]; see also the related papers [19, 23]. Condition $p > N$ is also assumed in a recent work of Wu and Tan [36], in which there are used minimax techniques combined with critical point theory tools.

The difficulties caused by the lack of a compact embedding of $W^{1,p}(\Omega)$ in $C^0(\bar{\Omega})$ if $p \leq N$ are overcome by a different procedure. Indeed, to apply variational techniques to Neumann problems in the case $p \leq N$, a suitable condition on the nonlinear term f, which implies that the problem admits the zero solution, is usually required. Thus, three non-zero solutions can be surely obtained only when the problem is perturbed; see [16, 32] and [33].

Very recently, by using variational methods, a precise interval of values of the parameter λ, for which problem (1.1) admits at least three non-zero weak solutions has been achieved in [15, Theorem 3.1] without explicit perturbations of the nonlinear term and in higher dimension. We also recall that in the last few years, several authors have treated Neumann problems with p-Laplace operators (with $p \leq N$), by using completely different techniques. We refer to [17, 24] and the references therein; see also the works [8, 20, 21].

We point out that problem (1.1) arises in the study of mathematical models in biological formation theory governed by diffusion and cross-diffusion systems [28]. We refer to the recent monograph by Kristály, Rădulescu and Varga [22] for several related results and examples.

The main result of this paper (Theorem 3.1) ensures the existence of precise values of the parameter λ for which (1.1) admits at least one non-zero weak solution, without assuming any asymptotic condition at zero or at infinity. A special case is also pointed out in Corollary 3.1. A related consequence, where the unique condition requested on the datum is the $(p-1)$-sublinearity at the origin, is also presented in this paper; see Theorem 3.2.

We also observe that, when the nonlinear term is $(p-1)$-sublinear at infinity, then the corresponding energy functional is coercive, hence the existence of one solution (possibly zero) comes from the direct methods theorem; see Remark 3.5. It is worth noticing that, in our cases, the potential may be also not coercive; see, for instance, Example 3.1. Moreover, also in presence of coercivity, our results ensure the existence of at least one non-zero weak solution.

A basic tool used in the proofs is a recent critical point theorem obtained by Bonanno in [1, Theorem 5.1] for functionals of the form $J_\lambda := \Phi - \lambda\Psi$, where λ is a positive parameter; see Theorem 2.1 below.

Consider the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \alpha(x)|u(x)|^p dx \right)^{1/p},$$

which is equivalent to the usual one.

We state here, as an example, the following special case of our results; see also Remark 3.8.

Theorem 1.1 *Let $p > 1$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = +\infty, \quad \text{and} \quad \lim_{|t| \to \infty} \frac{f(t)}{|t|^s} = 0, \tag{1.3}$$

for some $0 \leq s < p^* - 1$. Then, there exists $\lambda^\star > 0$ such that, for every $\lambda \in]0, \lambda^\star[$, the following autonomous Neumann problem

$$\begin{cases} -\Delta_p u + \alpha(x)|u|^{p-2}u = \lambda f(u) & \text{in} \quad \Omega \\ \partial u/\partial \nu = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

admits at least one non-zero weak solution $u_{0,\lambda} \in W^{1,p}(\Omega)$. Moreover, we have $\|u_{0,\lambda}\| \to 0$ as $\lambda \to 0^+$ and the mapping

$$\lambda \mapsto \frac{1}{p}\Big(\int_\Omega |\nabla u_{0,\lambda}(x)|^p dx + \int_\Omega \alpha(x)|u_{0,\lambda}(x)|^p dx\Big) - \lambda \int_\Omega \Big(\int_0^{u_{0,\lambda}(x)} f(x,t)dt\Big)dx,$$

is negative and strictly decreasing in $]0, \lambda^\star[$.

We observe that Theorem 1.1 ensures the existence of one non-zero weak solution even if $f(0) = 0$ provided that (1.3) holds, and, if α is not constant, the solution is not constant. Moreover, a concrete expression for $\lambda^\star$, that appears in the above result, is given in Remark 3.4. Furthermore, we just point out that in Corollary 3.2, adding to hypotheses of Theorem 3.2 the classical Ambrosetti and Rabinowitz condition, namely (AR), a second weak solution is achieved; see Example 3.2. It is worth noticing that in Corollary 3.2 the assumptions are different from those usually required when applying the mountain pass theorem; see Remark 3.10.

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our main results and examples.

2   Preliminaries

Let $W^{1,p}(\Omega)$ endowed with the norm $\|\cdot\|$. From standard variational arguments, the weak solutions of (1.1) are the critical points of the C^1-functional given by

$$J_\lambda(u) := \frac{1}{p}\Big(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \alpha(x)|u(x)|^p dx\Big) - \lambda \int_\Omega \Big(\int_0^{u(x)} f(x,t)dt\Big)dx,$$

for every $u \in W^{1,p}(\Omega)$.

A function $u : \Omega \to \mathbb{R}$ is said to be a weak solution of problem (1.1) if $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega \alpha(x)|u(x)|^{p-2}u(x)v(x)dx = \lambda \int_\Omega f(x,u(x))v(x)dx,$$

for every $v \in W^{1,p}(\Omega)$.

Let X be a real Banach space. We say that a continuously Gâteaux differentiable functional $J : X \to \mathbb{R}$ verifies the *Palais-Smale condition* (in short (PS)-condition) if any sequence $\{u_n\}$ such that

(j_1) $\{J(u_n)\}$ is bounded,

(j_2) $\lim\limits_{n\to\infty} \|J'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

For an exhaustive treatment of these topics we refer to [27, 30] and the references therein.

Let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Set

$$J = \Phi - \Psi,$$

and fix $r_1, r_2 \in [-\infty, +\infty]$, with $r_1 < r_2$; we say that function J verifies the *Palais-Smale condition cut off lower at r_1 and upper at r_2* (in short $^{[r_1]}$(PS)$^{[r_2]}$-condition) if any sequence $\{u_n\}$ such that (j_1), (j_2) hold and

(j_3) $r_1 < \Phi(u_n) < r_2$, $\forall n \in \mathbb{N}$,

has a convergent subsequence.

Clearly, if $r_1 = -\infty$ and $r_2 = +\infty$ it coincides with the classical (PS)-condition. Moreover, if $r_1 = -\infty$ and $r_2 \in \mathbb{R}$ it is denoted by $(PS)^{[r_2]}$, while if $r_1 \in \mathbb{R}$ and $r_2 = +\infty$ it is denoted by $^{[r_1]}(PS)$. Furthermore, if J satisfies $^{[r_1]}(PS)^{[r_2]}$-condition, then it satisfies $^{[\rho_1]}(PS)^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, +\infty]$ such that $r_1 \le \rho_1 < \rho_2 \le r_2$.

In particular, we deduce that if J satisfies the classical (PS)-condition, then it satisfies $^{[\rho_1]}(PS)^{[\rho_2]}$-condition for all $\rho_1, \rho_2 \in [-\infty, +\infty]$ with $\rho_1 < \rho_2$. Set

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)}, \tag{2.5}$$

and

$$\rho_2(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1}, \tag{2.6}$$

for all $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$.

A crucial role in the existence proof of one non-zero weak solution for problem (1.1) is played by the following version of an abstract local minimum theorem obtained in [1, Theorem 5.1], which we recall here for convenience.

Theorem 2.1 *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions. Assume that there are $r_1, r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1, r_2) < \rho_2(r_1, r_2),$$

where β and ρ_2 are given by (2.5) and (2.6), and for each

$$\lambda \in \left] \frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right[,$$

the function $J_\lambda := \Phi - \lambda\Psi$ satisfies $^{[r_1]}(PS)^{[r_2]}$-condition. Then, for all $\lambda \in \left] \frac{1}{\rho_2(r_1, r_2)}, \frac{1}{\beta(r_1, r_2)} \right[$ there is $u_{0,\lambda} \in \Phi^{-1}(]r_1, r_2[)$ such that $J_\lambda(u_{0,\lambda}) \le J_\lambda(u)$ for all $u \in \Phi^{-1}(]r_1, r_2[)$ and $J_\lambda'(u_{0,\lambda}) = 0$.

Remark 2.1 Theorem 2.1 has been inspired from the Ricceri variational principle; see [31, Theorem 2.5]. With respect to the mentioned principle, the above result furnishes a more precise localization of the minimum $u_{0,\lambda}$, and, in particular, since $\Phi(u_{0,\lambda}) > r_1$, in many applications we have $u_{0,\lambda} \ne 0$ as, for instance, in Theorem 3.1. Moreover, we also emphasize that in Theorem 2.1 no weak lower semicontinuity assumption is requested on the contrary of [31, Theorem 2.5].

3 Main results

Recall that p^* stands for the critical exponent of the Sobolev embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$. Thus, if $p < N$ then $p^* = Np/(N - p)$ and for every $q \in [1, p^*]$ there exists a positive constant κ_q such that

$$\|u\|_{L^q(\Omega)} \le \kappa_q \|u\|, \tag{3.7}$$

for every $u \in W^{1,p}(\Omega)$. When $p \ge N$ this inequality holds for any $q \in [1, +\infty[$.

Moreover, for every two nonnegative constants γ, δ, with $\gamma \neq \delta$, we set

$$a_\gamma(\delta) := \frac{A(\gamma) - q \displaystyle\int_\Omega F(x, \delta)dx}{\|\alpha\|_{L^1(\Omega)}(\gamma^p - \delta^p)q}, \tag{3.8}$$

where

$$A(\gamma) := q\|\alpha\|_{L^1(\Omega)}^{1/p}\gamma\kappa_1 a_1 + \|\alpha\|_{L^1(\Omega)}^{q/p}\gamma^q\kappa_q^q a_2, \tag{3.9}$$

and

$$F(x, \xi) := \int_0^\xi f(x, t)dt,$$

for every $(x, \xi) \in \Omega \times \mathbb{R}$.

The following result establishes a qualitative property of solutions to problem (1.1).

Theorem 3.1 *Let $f \in \mathfrak{F}_q$ and assume that there exist three real constants γ_1, γ_2 and δ, with $0 \leq \gamma_1 < \delta < \gamma_2$, such that*

$$a_{\gamma_2}(\delta) < a_{\gamma_1}(\delta). \tag{3.10}$$

Then, for each parameter λ belonging to

$$\left] \frac{1}{pa_{\gamma_1}(\delta)}, \frac{1}{pa_{\gamma_2}(\delta)} \right[,$$

problem (1.1) admits at least one non-zero weak solution $u_{0,\lambda} \in W^{1,p}(\Omega)$, such that

$$\|\alpha\|_{L^1(\Omega)}^{1/p}\gamma_1 < \|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p}\gamma_2.$$

Proof. Our aim is to apply Theorem 2.1. Hence, let $X := W^{1,p}(\Omega)$ and consider the functionals $\Phi, \Psi : X \to \mathbb{R}$ defined by

$$\Phi(u) := \frac{\|u\|^p}{p}, \quad \Psi(u) := \int_\Omega F(x, u(x))dx, \qquad \forall\, u \in X.$$

Clearly, $\Phi : X \to \mathbb{R}$ is a coercive and continuously Gâteaux differentiable functional. On the other hand, Ψ is well-defined and continuously Gâteaux differentiable. More precisely, we have

$$\Phi'(u)(v) = \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla v(x)dx + \int_\Omega \alpha(x)|u(x)|^{p-2}u(x)v(x)dx,$$

$$\Psi'(u)(v) = \int_\Omega f(x, u(x))v(x)dx,$$

for every $u, v \in X$.

Fix $\lambda > 0$. A critical point of the functional $J_\lambda := \Phi - \lambda\Psi$ is a function $u \in X$ such that

$$\Phi'(u)(v) - \lambda\Psi'(u)(v) = 0,$$

for every $v \in X$. Hence, the critical points of the functional J_λ are weak solutions of problem (1.1). At this point, let us observe that $\Phi(0_X) = \Psi(0_X) = 0$. Moreover, since $f \in \mathfrak{F}$, we have

$$F(x, \xi) \leq a_1|\xi| + a_2\frac{|\xi|^q}{q}, \tag{3.11}$$

for every $(x, \xi) \in \Omega \times \mathbb{R}$. Now, taking into account relation (3.11), it follows that

$$\Psi(u) = \int_{\Omega} F(x, u(x)) dx \leq a_1 \|u\|_{L^1(\Omega)} + \frac{a_2}{q} \|u\|_{L^q(\Omega)}^q.$$

Then, for every $u \in X$ such that $\Phi(u) \leq r$, owing to (3.7), we obtain

$$\Psi(u) \leq (pr)^{1/p} \kappa_1 a_1 + \frac{p^{q/p} \kappa_q^q a_2}{q} r^{q/p}.$$

Therefore

$$\sup_{u \in \Phi^{-1}(]-\infty, r])} \Psi(u) \leq (pr)^{1/p} \kappa_1 a_1 + \frac{p^{q/p} \kappa_q^q a_2}{q} r^{q/p}. \tag{3.12}$$

Next, we set

$$r_1 := \frac{\|\alpha\|_{L^1(\Omega)}}{p} \gamma_1^p, \quad r_2 := \frac{\|\alpha\|_{L^1(\Omega)}}{p} \gamma_2^p, \quad \text{and} \quad u_\delta(x) := \delta, \quad \text{for every } x \in \Omega.$$

Then $u_\delta \in X$ and

$$\Phi(u_\delta) = \frac{1}{p} \Big(\int_{\Omega} |\nabla u_\delta(x)|^p dx + \int_{\Omega} \alpha(x) |u_\delta(x)|^p dx \Big) = \frac{\delta^p}{p} \|\alpha\|_{L^1(\Omega)}. \tag{3.13}$$

Taking into account that $\gamma_1 < \delta < \gamma_2$, by a direct computation, one has $r_1 < \Phi(u_\delta) < r_2$. Moreover,

$$\Psi(u_\delta) = \int_{\Omega} F(x, u_\delta(x)) \, dx = \int_{\Omega} F(x, \delta) dx. \tag{3.14}$$

From (3.12) it follows that

$$\sup_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) \leq (pr_2)^{1/p} \kappa_1 a_1 + \frac{p^{q/p} \kappa_q^q a_2}{q} r_2^{q/p}. \tag{3.15}$$

and

$$\sup_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u) \leq (pr_1)^{1/p} \kappa_1 a_1 + \frac{p^{q/p} \kappa_q^q a_2}{q} r_1^{q/p}. \tag{3.16}$$

We have $r_1 < \Phi(u_\delta) < r_2$. On the other hand,

$$\beta(r_1, r_2) := \inf_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\sup\limits_{u \in \Phi^{-1}(]r_1, r_2[)} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)} \leq \frac{\sup\limits_{u \in \Phi^{-1}(]-\infty, r_2[)} \Psi(u) - \Psi(u_\delta)}{r_2 - \Phi(u_\delta)}$$

and

$$\rho_2(r_1, r_2) := \sup_{v \in \Phi^{-1}(]r_1, r_2[)} \frac{\Psi(v) - \sup\limits_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(v) - r_1} \geq \frac{\Psi(u_\delta) - \sup\limits_{u \in \Phi^{-1}(]-\infty, r_1])} \Psi(u)}{\Phi(u_\delta) - r_1}.$$

Hence, by using the notation (3.8), from (3.14) and (3.15) together with (3.16), it follows that

$$\beta(r_1, r_2) \leq p a_{\gamma_2}(\delta), \quad \text{and} \quad \rho_2(r_1, r_2) \geq p a_{\gamma_1}(\delta).$$

Finally, hypothesis (3.10) yields

$$\beta(r_1, r_2) < \rho_2(r_1, r_2).$$

Now, from [1, Proposition 2.1], the functional J_λ satisfies $^{[r_1]}(PS)^{[r_2]}$-condition for all r_1 and r_2 with $r_1 < r_2 < +\infty$. Therefore, owing to Theorem 2.1, for each $\lambda \in \left] \dfrac{1}{pa_{\gamma_1}(\delta)}, \dfrac{1}{pa_{\gamma_2}(\delta)} \right[$, the functional J_λ admits at least one critical point $u_{0,\lambda}$ such that

$$r_1 < \Phi(u_{0,\lambda}) < r_2,$$

that is

$$\|\alpha\|_{L^1(\Omega)}^{1/p} \gamma_1 < \|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p} \gamma_2.$$

This completes the proof.

Remark 3.1 If $p < N$, a concrete upper bound for the constants κ_q in Theorem 3.1 (hence for the values of constant A defined as in (3.9)) can be obtained considering an open and convex set $\Omega \subset \mathbb{R}^N$ of diameter d and Lebesgue measure " meas(Ω)". In this case, if $q \in [1, p^*[$, we have

$$\kappa_q \leq S_q \frac{(1 + N) \operatorname{meas}(\Omega)^{\frac{p-q}{pq}}}{\min\left\{1, \operatorname*{ess\,inf}_{x \in \Omega} \alpha(x)\right\}^{1/p}}, \tag{3.17}$$

where

$$S_q := \begin{cases} \max\left\{1, \dfrac{d}{N}\right\} & \text{if } q \in [1, p[, \\[4mm] \max\left\{1, \dfrac{d}{N}\left(\dfrac{pq - N(q - p)}{pq + (p - q)}\right)^{\frac{q - p(1+q)}{pq}}\right\} & \text{if } q \in [p, p^*[, \end{cases} \tag{3.18}$$

see, for instance, Remark 3.2 in [15] and, more generally, the exhaustive book [13]. Moreover, we point out that the following alternative holds: either $q \in [p, p^*[$ and the diameter of Ω is sufficiently small, that is,

$$d < N\left(\frac{pq - N(q - p)}{pq + (p - q)}\right)^{\frac{p(1+q)-q}{pq}},$$

or $q \in [1, p[$ and $d < N$. Inequality (3.17) can be rewritten as

$$\kappa_q \leq \frac{(1 + N) \operatorname{meas}(\Omega)^{\frac{p-q}{pq}}}{\min\{1, \operatorname{ess\,inf}_{x \in \Omega} \alpha(x)\}^{1/p}}, \tag{3.19}$$

as observed in [15, Remark 3.3].

On the other hand, in the low-dimensional case that corresponds to $p > N$, assuming Ω to be convex and by using [5, Remark 2.1], an easy computation ensures that for every $q \geq 1$, we have

$$\kappa_q \leq 2^{\frac{p-1}{p}} \max\left\{ \frac{1}{\|\alpha\|_{L^1(\Omega)}^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}}\left(\frac{p-1}{p-N}\operatorname{meas}(\Omega)\right)^{\frac{p-1}{p}} \frac{\|\alpha\|_\infty}{\|\alpha\|_{L^1(\Omega)}} \right\} \operatorname{meas}(\Omega)^{1/q}.$$

Remark 3.2 If $p > N$ and Ω is convex, by using again [5, Remark 2.1], a concrete estimate of the solution $u_{0,\lambda}$ in $C^0(\bar{\Omega})$ can be achieved. Precisely, we have

$$\|u_{0,\lambda}\|_\infty \leq 2^{\frac{p-1}{p}} \max \left\{ \frac{1}{\|\alpha\|_{L^1(\Omega)}^{\frac{1}{p}}}, \frac{d}{N^{\frac{1}{p}}} \left(\frac{p-1}{p-N} \operatorname{meas}(\Omega) \right)^{\frac{p-1}{p}} \frac{\|\alpha\|_\infty}{\|\alpha\|_{L^1(\Omega)}} \right\} \|\alpha\|_{L^1(\Omega)}^{1/p} \gamma_2.$$

Now, we point out a particular case of Theorem 3.1.

Corollary 3.1 *Let $f \in \mathfrak{F}_q$ and assume that there exist two positive constants γ and δ, with $\gamma > \delta$, such that*

$$\frac{\int_\Omega F(x,\delta)dx}{\delta^p} > \frac{A(\gamma)}{q\gamma^p}. \tag{3.20}$$

Then, for each parameter λ belonging to

$$\left] \frac{\|\alpha\|_{L^1(\Omega)}\delta^p}{p \int_\Omega F(x,\delta)dx}, \frac{\|\alpha\|_{L^1(\Omega)}\gamma^p q}{pA(\gamma)} \right[,$$

problem (1.1) admits at least one non-zero weak solution $u_{0,\lambda} \in W^{1,p}(\Omega)$, such that

$$\|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p}\gamma.$$

Proof. Our aim is to apply Theorem 3.1. To this end we pick $\gamma_1 = 0$ and $\gamma_2 := \gamma$. Bearing in mind (3.8), we obtain

$$a_\gamma(\delta) = \frac{A(\gamma) - q \int_\Omega F(x,\delta)dx}{\|\alpha\|_{L^1(\Omega)}(\gamma^p - \delta^p)q},$$

as well as

$$a_0(\delta) = \frac{\int_\Omega F(x,\delta)dx}{\delta^p\|\alpha\|_{L^1(\Omega)}}.$$

Now, inequality (3.20) immediately yields

$$a_\gamma(\delta) < a_0(\delta).$$

Hence, Theorem 3.1 ensures the conclusion. A direct consequence of Corollary 3.1 is the following property.

Theorem 3.2 *Let $f \in \mathfrak{F}_q$ and assume that*

$$\lim_{\xi \to 0^+} \frac{\int_\Omega F(x,\xi)dx}{\xi^p} = +\infty. \tag{3.21}$$

Furthermore, let $\gamma > 0$ and set

$$\lambda_\gamma^\star := \frac{q\|\alpha\|_{L^1(\Omega)}}{p} \frac{\gamma^p}{A(\gamma)}.$$

Then, for every $\lambda \in]0, \lambda_\gamma^\star[$, the problem (1.1) admits at least one non-zero weak solution $u_{0,\lambda} \in W^{1,p}(\Omega)$ such that $\|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p} \gamma$ and

$$\lim_{\lambda \to 0^+} \|u_{0,\lambda}\| = 0.$$

Proof. Fix $\lambda \in]0, \lambda_\gamma^\star[$. From (3.21) there exists a positive constant δ with $\delta < \gamma$ such that

$$\frac{\|\alpha\|_{L^1(\Omega)} \delta^p}{p \int_\Omega F(x, \delta) dx} < \lambda < \frac{\|\alpha\|_{L^1(\Omega)} \gamma^p q}{A(\gamma) p}.$$

Hence, owing to Corollary 3.1, problem (1.1) admits at least one non-zero weak solution $u_{0,\lambda}$, such that $\|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p} \gamma$. Then, for every $\lambda \in]0, \lambda_\gamma^\star[$, there exists at least one non-zero weak solution $u_{0,\lambda} \in \Phi^{-1}(]0, r_2[)$ of problem (1.1) and one has

$$\|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p} \gamma. \tag{3.22}$$

for every $\lambda \in]0, \lambda_\gamma^\star[$. Therefore, from (1.2), taking into account (3.7) and (3.22), it follows that

$$\left| \int_\Omega f(x, u_{0,\lambda}(x)) u_{0,\lambda}(x) dx \right| \leq \|\alpha\|_{L^1(\Omega)}^{1/p} \gamma \kappa_1 a_1 + \|\alpha\|_{L^1(\Omega)}^{q/p} \gamma^q \kappa_q^q a_2, \tag{3.23}$$

for every $\lambda \in]0, \lambda_\gamma^\star[$. Now, $J_\lambda'(u_{0,\lambda}) = 0$, for every $\lambda \in]0, \lambda_\gamma^\star[$ and in particular $J_\lambda'(u_{0,\lambda})(u_{0,\lambda}) = 0$, that is,

$$\|u_{0,\lambda}\|^p = \lambda \int_\Omega f(x, u_{0,\lambda}(x)) u_{0,\lambda}(x) dx,$$

for every $\lambda \in]0, \lambda_\gamma^\star[$. Then, from (3.23), it follows that

$$\lim_{\lambda \to 0^+} \|u_{0,\lambda}\|^p = \lim_{\lambda \to 0^+} \lambda \Psi'(u_{0,\lambda})(u_{0,\lambda}) = 0,$$

that implies $\lim_{\lambda \to 0^+} \|u_{0,\lambda}\| = 0$. The proof is complete.

Remark 3.3 We claim that under the above assumptions, the mapping $\lambda \mapsto J_\lambda(u_{0,\lambda})$ is negative and strictly decreasing in $]0, \lambda_\gamma^\star[$. Indeed, the restriction of the functional J_λ to $\Phi^{-1}(]0, r_2[)$, where $r_2 := (\|\alpha\|_{L^1(\Omega)}/p)\gamma_2^p$, admits a global minimum, which is a critical point (local minimum) of J_λ in X. Moreover, since $u_\delta := \delta \in \Phi^{-1}(]0, r_2[)$ and

$$\frac{\Phi(u_\delta)}{\Psi(u_\delta)} = \frac{\|\alpha\|_{L^1(\Omega)} \delta^p}{p \int_\Omega F(x, \delta) dx} < \lambda,$$

we have

$$J_\lambda(u_{0,\lambda}) \leq J_\lambda(u_\delta) = \Phi(u_\delta) - \lambda \Psi(u_\delta) < 0.$$

Next, we observe that

$$J_\lambda(u) = \lambda \left(\frac{\Phi(u)}{\lambda} - \Psi(u) \right),$$

for every $u \in X$ and fix $0 < \lambda_1 < \lambda_2 < \lambda_\gamma^\star$. Set

$$m_{\lambda_1} := \left(\frac{\Phi(u_{0,\lambda_1})}{\lambda_1} - \Psi(u_{0,\lambda_1}) \right) = \inf_{u \in \Phi^{-1}(]0, r_2[)} \left(\frac{\Phi(u)}{\lambda_1} - \Psi(u) \right),$$

and

$$m_{\lambda_2} := \left(\frac{\Phi(u_{0,\lambda_2})}{\lambda_2} - \Psi(u_{0,\lambda_2}) \right) = \inf_{u \in \Phi^{-1}(]0,r_2[)} \left(\frac{\Phi(u)}{\lambda_2} - \Psi(u) \right).$$

Clearly, as claimed before, $m_{\lambda_i} < 0$ (for $i = 1, 2$), and $m_{\lambda_2} \leq m_{\lambda_1}$ thanks to $\lambda_1 < \lambda_2$. Then the mapping $\lambda \mapsto J_\lambda(u_{0,\lambda})$ is strictly decreasing in $]0, \lambda_\gamma^\star[$ owing to

$$J_{\lambda_2}(u_{0,\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = J_{\lambda_1}(u_{0,\lambda_1}).$$

This concludes the proof of our claim.

Remark 3.4 Roughly speaking, Theorem 3.2 ensures that if f has the global growth given by (1.2) and the asymptotic condition at zero (3.21) is verified then, for every parameter λ belonging to the real interval $\Lambda_\Omega :=]0, \lambda^\star[$, where

$$\lambda^\star := \frac{q \|\alpha\|_{L^1(\Omega)}}{p} \sup_{\gamma > 0} \frac{\gamma^p}{A(\gamma)},$$

the problem (1.1) admits at least one non-zero solution $u_{0,\lambda} \in W^{1,p}(\Omega)$. Moreover $\|u_{0,\lambda}\| \to 0$, as $\lambda \to 0^+$. Furthermore, from a direct computation, it follows that

$$\lambda^\star = \begin{cases} +\infty & \text{if} \quad 1 < q < p \\ \dfrac{1}{\kappa_p^p a_2} & \text{if} \quad q = p \\ \dfrac{q \|\alpha\|_{L^1(\Omega)}^{(p-1)/p} \overline{\gamma}_{\max}^{p-1}}{p(q\kappa_1 a_1 + \|\alpha\|_{L^1(\Omega)}^{(q-1)/p} \kappa_q^q a_2 \overline{\gamma}_{\max}^{q-1})} & \text{if} \quad q \in]p, p^*[, \end{cases} \tag{3.24}$$

where

$$\overline{\gamma}_{\max} := \frac{1}{\|\alpha\|_{L^1(\Omega)}^{1/p}} \left(q \frac{\kappa_1 a_1}{\kappa_q^q a_2} \left(\frac{p-1}{q-p} \right) \right)^{1/(q-1)}.$$

We also note that, in the case $q \in]p, p^*[$, one has $\|u_{0,\lambda}\| < \|\alpha\|_{L^1(\Omega)}^{1/p} \overline{\gamma}_{\max}$, uniformly for every $\lambda \in \Lambda_\Omega$.

Remark 3.5 We emphasize that, in particular, if f is $(p-1)$-sublinear at infinity, Theorem 3.2 ensures that, problem (1.1) admits at least one non-zero weak solution for each positive parameter λ. It is worth noticing that, in our case, the attained solution, as affirmed, is non-zero, while the classical direct method approach, that can be adopted in this setting, ensures the existence of at least one solution that may be zero.

Remark 3.6 For completeness we observe that if f is a non-negative function our results guarantee that the attained weak solution is non-negative. To this end, let u_0 be a weak solution of problem (1.1). Arguing by contradiction, assume that the set $A := \{x \in \Omega : u_0(x) < 0\}$ has positive Lebesgue measure. Put $\overline{v}(x) = \min\{0, u_0(x)\}$ for all $x \in \Omega$. Clearly, $\overline{v} \in W^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u_0(x)|^{p-2} \nabla u_0(x) \cdot \nabla \overline{v}(x) dx + \int_\Omega \alpha(x)|u_0(x)|^{p-2} u_0(x) \overline{v}(x) dx$$

$$-\lambda \int_\Omega f(x, u_0(x)) \overline{v}(x) dx = 0,$$

that is,

$$\int_A |\nabla u_0(x)|^p dx + \int_A \alpha(x)|u_0(x)|^p dx = \lambda \int_A f(x, u_0(x))u_0(x)dx \leq 0.$$

Hence

$$\int_A |\nabla u_0(x)|^p dx + \int_A \alpha(x)|u_0(x)|^p dx = 0.$$

Then, $u_0 = 0$ almost everywhere in A. This is not possible by the definition of A, so it follows that u_0 is non-negative.

Remark 3.7 A careful analysis of the proof of Theorem 3.2 ensures that the result still remains true after replacing condition (3.21) with the more general assumption

$$\limsup_{\xi \to 0^+} \frac{\int_\Omega F(x, \xi)dx}{\xi^p} = +\infty. \tag{3.25}$$

Moreover, in the autonomous case, the previous asymptotic condition at zero assume the form

$$\limsup_{\xi \to 0^+} \frac{F(\xi)}{\xi^p} = +\infty. \tag{3.26}$$

Remark 3.8 We just observe that Theorem 1.1 in Introduction is a simple consequence of Theorem 3.2 and Remark 3.3. Indeed, if

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = +\infty \tag{3.27}$$

holds, then hypothesis (3.26) is automatically verified. Moreover, hypothesis

$$\lim_{|t| \to \infty} \frac{f(t)}{|t|^s} = 0,$$

where $0 \leq s < p^* - 1$, ensures that f has a subcritical growth.

In the sequel, there is an example of application of our results.

Example 3.1 Let Ω be a bounded open subset of $\mathbb{R}^N$ with $1 < p < N$ and consider the following problem

$$\begin{cases} -\Delta_p u + |u|^{p-2}u = \lambda[\eta(x)|u|^{r-2}u + \beta(x)|u|^{s-2}u] & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.28}$$

where $\eta, \beta : \Omega \to \mathbb{R}$ are two continuous positive and bounded functions, $1 < r < p$ and $p < s < p^*$. Then, for every

$$\lambda \in \left]0, \frac{s \operatorname{meas}(\Omega)^{(p-1)/p}\overline{\gamma}_{\max}^{p-1}}{2p \max\{\|\eta\|_\infty, \|\beta\|_\infty\}(s\kappa_1 + \operatorname{meas}(\Omega)^{(s-1)/p}\kappa_s^s\overline{\gamma}_{\max}^{s-1})}\right[,$$

where

$$\overline{\gamma}_{\max} := \frac{1}{\operatorname{meas}(\Omega)^{1/p}}\left(s\frac{\kappa_1}{\kappa_s^s}\left(\frac{p-1}{s-p}\right)\right)^{1/(s-1)}$$

the problem (3.28) admits at least one non-zero weak solution $u_{0,\lambda} \in W^{1,p}(\Omega)$ such that

$$\|u_{0,\lambda}\| < \left(s\frac{\kappa_1}{\kappa_s^s}\left(\frac{p-1}{s-p}\right)\right)^{1/(s-1)},$$

and $\lim_{\lambda \to 0^+} \|u_{0,\lambda}\| = 0$. To prove this, we can apply Theorem 3.2 with

$$f(x,t) := \eta(x)|t|^{r-2}t + \beta(x)|t|^{s-2}t,$$

for every $(x,t) \in \Omega \times \mathbb{R}$. Indeed, it is easy to verify that $f \in \mathfrak{F}_s$ since

$$|f(x,t)| \leq 2 \max\{\|\eta\|_\infty, \|\beta\|_\infty\}(1 + |t|^{s-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Moreover, a direct computation shows that

$$\lim_{\xi \to 0^+} \frac{\displaystyle\int_\Omega F(x,\xi)dx}{\xi^p} \geq \frac{\mathrm{meas}(\Omega)\inf_{x\in\Omega}\eta(x)}{r}\lim_{\xi\to0^+}\frac{1}{\xi^{p-r}} = +\infty.$$

Hence, all the assumptions of Theorem 3.2 are verified and the conclusion follows.

Remark 3.9 A similar version of Theorem 3.2 in the low-dimensional case has been studied in [10] and [11]. In their setting, taking advantage of the compact embedding of $W^{1,p}(\Omega)$ into $C^0(\bar{\Omega})$, condition (1.2) can be removed and the authors are able to treat also cases with exponential growth; see, for instance, [10, Example 3.1]. Furthermore, a direct comparison with Theorems 1 and 2 of [37] can be find in [10, Remark 3.6]. In conclusion, taking into account Remarks 3.6 and 3.7, we emphasize that, Theorem 3.2 represents an extension of [10, Theorem 3.3] and [11, Theorem 4.1] to the case $p \leq N$. Moreover, in the low-dimensional case, the additional presence of the growth assumption (1.2) may determine, in some case, an improvement of the intervals of parameters for which our approach guarantee the existence of at least one non-zero weak solution. Indeed, let $p > N$ and assume $f : \Omega \times \mathbb{R} \to \mathbb{R}$ to be L^1-Carathéodory. As [11, Theorem 3.4] shows, the existence of at least one non-zero weak solution to Neumann problem is obtained for every $\lambda \in\,]0, \lambda_f[$, where

$$\lambda_f := \frac{1}{pc^p}\sup_{\gamma>0}\frac{\gamma^p}{\displaystyle\int_\Omega \max_{|\xi|\leq\gamma}F(x,\xi)dx},$$

and

$$c := \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}}\frac{\sup_{x\in\bar\Omega}|u(x)|}{\left(\displaystyle\int_\Omega |\nabla u(x)|^p dx + \int_\Omega \alpha(x)|u(x)|^p dx\right)^{1/p}} < +\infty.$$

If the function f satisfies also the growth condition (1.2) the necessity of to compare the intervals $]0, \lambda^\star[$ and $]0, \lambda_f[$ naturally arises. We exhibit a concrete example in which $]0, \lambda_f[\subset]0, \lambda^\star[$. For our goal, let us take a positive constant d and set $\Omega =]0, 2d[$. At this point, consider the ordinary non-autonomous Neumann problem

$$\begin{cases} -u'' + \dfrac{ku}{2} = \lambda\beta(x)(1 + |u|^{q-1}) & \text{in }]0, 2d[\\ u'(0) = u'(2d) = 0, \end{cases} \tag{3.29}$$

where $\beta : [0, 2d] \to \mathbb{R}$ is a positive and continuous function, $q > 2$ and

$$0 < k < \frac{\|\beta\|_{L^1(0,2d)}}{2\|\beta\|_\infty(cd)^2}. \tag{3.30}$$

Set

$$\mathcal{N}(c,q) := \frac{1}{2c^2}\left(\frac{q-2}{q-1}\right)\left(\frac{q}{q-2}\right)^{1/(q-1)},$$

and

$$G(\beta, q) := \left(\frac{d}{2\|\beta\|_{L^1(0,2d)}}\right)\left(\frac{q-2}{q-1}\right)\left(\frac{q}{q-2}\right)^{1/(q-1)}.$$

Hence, bearing in mind the above results, problem (3.29) admits a non-zero classical solution for every $\lambda \in]0, \lambda_f[$ as well as for every $\lambda \in]0, \lambda^\star[$. Now, taking into account that $1/c^2 \leq kd$, simple computations yield

$$\lambda_f = \frac{q}{2c^2\|\beta\|_{L^1(0,2d)}} \sup_{\gamma>0}\left(\frac{\gamma}{q + \gamma^{q-1}}\right) \leq k\left(\frac{qd}{2\|\beta\|_{L^1(0,2d)}}\right)\sup_{\gamma>0}\left(\frac{\gamma}{q + \gamma^{q-1}}\right).$$

Then, since

$$\sup_{\gamma>0}\left(\frac{\gamma}{q + \gamma^{q-1}}\right) = \frac{1}{q}\left(\frac{q-2}{q-1}\right)\left(\frac{q}{q-2}\right)^{1/(q-1)},$$

one has

$$\lambda_f \leq k\left(\frac{d}{2\|\beta\|_{L^1}}\right)\left(\frac{q-2}{q-1}\right)\left(\frac{q}{q-2}\right)^{1/(q-1)} = kG(\beta, q). \tag{3.31}$$

On the other hand, thanks to $\kappa_s \leq (2d)^{1/s}c$, for every $s \geq 1$, it follows that

$$\lambda^\star = \frac{q}{\|\beta\|_\infty} \sup_{\gamma>0}\left(\frac{\gamma}{q\sqrt{2}\kappa_1 + 2^{q/2}\kappa_q^q\gamma^{q-1}}\right) \geq \frac{q}{2d\|\beta\|_\infty} \sup_{\gamma>0}\left(\frac{\gamma}{q\sqrt{2}c + 2^{q/2}c^q\gamma^{q-1}}\right).$$

Hence, since

$$\sup_{\gamma>0}\left(\frac{\gamma}{q\sqrt{2}c + 2^{q/2}c^q\gamma^{q-1}}\right) = \frac{1}{2c^2q}\left(\frac{q-2}{q-1}\right)\left(\frac{q}{q-2}\right)^{1/(q-1)},$$

one has

$$\lambda^\star \geq \left(\frac{1}{2d\|\beta\|_\infty}\right)\mathcal{N}(c, q). \tag{3.32}$$

Then, inequalities (3.31) and (3.32) together with condition (3.30) yield

$$\lambda_f \leq kG(\beta, q) < \left(\frac{1}{2d\|\beta\|_\infty}\right)\mathcal{N}(c, q) \leq \lambda^\star.$$

In conclusion, the expected strict inclusion $]0, \lambda_f[\subset]0, \lambda^\star[$ holds.

In the sequel we prove how the previous results can be employed in order to pass from the existence of at least one nontrivial solution to the existence of at lest two nontrivial solutions. This goal will be achieved making use of the particular nature of the first solution found, namely it is a local minimum. This information will be used to assure the existence of a second solution as a critical point of mountain pass type. In this direction, we begin with the following theorem, where the celebrated Ambrosetti-Rabinowitz condition is required. As usual, this assumption plays a crucial role in proving that every Palais-Smale sequence is bounded, as well as that the so called 'mountain pass geometry' is satisfied.

Corollary 3.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|f(t)| \le a_1 + a_2|t|^{q-1}, \quad \forall\, t \in \mathbb{R}, \tag{3.33}$$

for some non-negative constants a_1, a_2, where $q \in\,]p, p^*[$. Furthermore, assume that condition (3.26) holds in addition to

(AR) *there are constants $\mu > p$ and $r > 0$ such that, for all $|\xi| \ge r$, one has*

$$0 < \mu F(\xi) \le \xi f(\xi).$$

Then, for each $\lambda \in \Lambda_\Omega$, problem (1.1) admits at least two weak solutions. If, in addition to the above assumptions, one has $f(0) \ne 0$, the attained solutions are non-zero.

Proof. Fix $\lambda \in \Lambda_\Omega$. Owing to (3.33) and (3.26), Theorem 3.2 ensures that the problem (1.1) admits at least one weak non-zero solution u_1 which is a local minimum of the functional J_λ as defined in the proof of Theorem 3.1. Now, we can assume that u_1 is a strict local minimum for J_λ in X. Therefore, there is $\rho > 0$ such that

$$\inf_{\|u-u_1\|=\rho} J_\lambda(u) > J_\lambda(u_1).$$

Furthermore, from (AR)-condition, by standard computations, one has that J_λ is unbounded from below. So, there is u_2 such that $J_\lambda(u_2) < J_\lambda(u_1)$, for which J_λ satisfies mountain pass geometry. At this point, again exploiting (AR), it follows that the functional J_λ satisfies the (PS)-condition. Hence, the celebrated Ambrosetti-Rabinowitz theorem ensures the existence of a critical point $\tilde{u}$ of J_λ such that $J_\lambda(\tilde{u}) > J_\lambda(u_1)$. So, u_1 and $\tilde{u}$ are two distinct weak solutions of (1.1) and the proof is complete.

Example 3.2 Let Ω be a bounded open subset of $\mathbb{R}^N$ with $1 < p < N$ and consider the following problem

$$\begin{cases} -\Delta_p u + \alpha(x)|u|^{p-2}u = \lambda(1 + u^{2m-1}) & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.34}$$

where m is a positive integer such that $1 < p < 2m < p^*$. Then, for every

$$\lambda \in \left]0, \frac{2m-p}{p\kappa_1(2m-1)}\left(\frac{2m\kappa_1(p-1)}{\kappa_{2m}^{2m}(2m-p)}\right)^{\frac{p-1}{2m-1}}\right[,$$

problem (3.34) admits at least two non-zero ($f(0) \ne 0$) weak solutions. To prove this, we can apply Corollary 3.2 with

$$f(t) := 1 + t^{2m-1},$$

for every $t \in \mathbb{R}$. Indeed, clearly the function f satisfies (3.33) and, since

$$\lim_{t \to 0^+} \frac{f(t)}{t^{p-1}} = +\infty,$$

also condition (3.26) holds true. Moreover, taking into account that

$$\lim_{|\xi| \to \infty} \frac{\xi f(\xi)}{F(\xi)} = 2m \lim_{|\xi| \to \infty} \frac{\xi^{2m} + \xi}{\xi^{2m} + 2m\xi} = 2m > p,$$

there exist $\mu > p$ and $r > 0$ such that

$$0 < \mu F(\xi) \le \xi f(\xi),$$

for every $|\xi| > r$. Hence, all the assumptions of Corollary 3.2 are verified and the conclusion follows.

Remark 3.10 Applying the mountain pass theorem to nonlinear differential problems, involving the Laplace operator $-\Delta u$, it is usual to require that $\lim_{t \to 0^+} \dfrac{f(t)}{t} = 0$; see, for instance, condition (p_3) in [30, Theorem 2.15]. On the contrary, in Example 3.2, the above assumption at zero is not verified. We also note that, very recently, Neumann problems have been studied without (AR)-condition by using a Cerami-type condition; see [18] and references therein, as well as Remark 3.11 below. Even in this case, hypothesis $\lim_{t \to 0^+} \dfrac{f(t)}{t} < +\infty$ is requested. Hence, [18, Theorem 3.6] cannot be applied to problem (3.34) in Example 3.2.

Remark 3.11 The (AR)-condition, adopted in Corollary 3.2, implies that the energy functional J_λ is unbounded from below and satisfies the classical (PS)-condition, so that the classical mountain pass theorem can be applied. Therefore, instead of assumption (AR), a Cerami-type condition could be exploited in Corollary 3.2 in order to obtain the second solution. We refer, for instance, to [4, 18, 29] for more details on related topics.

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