Research Article

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Global well-posedness analysis for the nonlinear extensible beam equations in a class of modified Woinowsky-Krieger models

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Abstract: For studying the evolution of the transverse deflection of an extensible beam derived from the connection mechanics, we investigate the initial boundary value problem of nonlinear extensible beam equation with linear strong damping term, nonlinear weak damping term, and nonlinear source term. The key idea of our analysis is to describe the invariant manifold via Nehari manifold. To establish the results of global well-posedness of solution, we consider the problem at three different initial energy levels, i.e., subcritical initial energy level, critical initial energy level, and arbitrarily high initial energy level. We first obtain the local existence of the solution by using the contraction mapping principle. Then, in the framework of potential well, we obtain global existence, nonexistence, and asymptotic behavior of solution for both subcritical initial energy level and critical initial energy level. In the end, we establish the global nonexistence of solution for the problem with linear weak damping and strong damping at the arbitrarily high initial energy level.

Keywords: extensible beam equation, global existence and nonexistence, nonlinear weak damping, strong damping

MSC 2020: 35L05, 35L35, 35L75

1 Introduction

In this study, we consider the initial boundary value problem (IBVP) of nonlinear extensible beam equations with linear strong damping term, nonlinear weak damping term and nonlinear source term

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \Delta u_t + |u_t|^{r-1}u_t = |u|^{p-1}u \quad \text{in } \Omega_T, \tag{1.1}$$

$$u(x,0) = u_0(x),\, u_t(x,0) = u_1(x) \quad \text{in } \Omega, \tag{1.2}$$

$$u(x,t) = u_t(x,t) = 0 \quad \text{on } \Gamma, \tag{1.3}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ $(n > 1)$ with smooth boundary $\partial\Omega$, $\Omega_T := \Omega \times (0, T)$, $\Gamma := \partial\Omega \times (0, T)$, $r \geq 1$, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator,

$$M(s) = 1 + \beta s^\gamma, \gamma \geq 0, \beta \geq 0, \ s \geq 0, \tag{1.4}$$

β is related to the forces applied on the system, and the exponent p satisfies

$$1 < 2\gamma + 1 < p < \infty, \quad n \leq 2; \quad 1 < 2\gamma + 1 < p \leq \frac{n}{n-2}, \ n \geq 3. \tag{1.5}$$

The extensible beam model describes the evolution of the transverse deflection of an extensible beam obeying continuous dynamics. A key feature that the extensible beam model affords a description during the vibrations is the dynamic buckling of a hinged extensible beam under an axial force, while it often depends on the fixing manner and distance of the two ends of the beam. The original version of extensible beam equation, proposed by Woinowsky-Krieger [49] in 1950, reads

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial^4 u}{\partial x^4} - \left(\beta + k\int_0^L u_x^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.6}$$

where β represents an initial axial displacement measured from the unstressed state and $u(x, t)$ represents the transverse deflection of an extensible beam of natural length l whose ends are held a fixed distance apart. Considering its wide applications in connection mechanics, the model was also discussed by Ball [1,2] when both the ends of the beam are hinged or clamped (or built-in). The multidimensional form, on the other hand, was established by Berger [4] as follows:

$$u_{tt} + \Delta^2 u - \left(Q + \int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = p(u, u_t, x), \tag{1.7}$$

which is called the Berger plate model [12], as a generalization of the Woinowsky-Krieger model to describe the large deflection of the plate, where the parameter Q describes in-plane forces applied to the plate and p represents transverse loads which may depend on the displacement u and the velocity u_t. In the hinged end case with free and forced nonlinear vibration, the extensible beam equation (1.7) has been investigated by Eisley [18], while related experimental results have been given by Burgreen [8]. As one of the typical models found in physics, the extensible beam model attracted more attention, especially in connection with industrial applications and the relevant fields such as the vibration of railway track structures [11], micro-machined beams [20,60], and microbridges [27].

In (1.1), the nonlinear term $M(\|\nabla u\|_2^2)\Delta u$ represents the extensibility effects on the beam, the dissipative terms Δu_t and $|u_t|^{m-1}u_t$ represent the friction force, and the nonlinear source term $|u|^{p-1}u$ represents the external load distribution. The aim of the present paper is to tackle this problem in the frame of variational argument to reveal the influences of the initial data on the global well-posedness of the solution; thus, the combination of these complex nonlinear structures in equation (1.1) undoubtedly increases the challenge for the construction of the variational structure. Fortunately, if the tool of the Nehari flow is used, its advantage is to make it possible to consider the equation of the following form:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u + g(u_t) = f(u) \tag{1.8}$$

under the same variational framework. In addition, the full use of the Nehari tools allows us to learn more about the qualitative properties of the solution, for example, in proving the existence of at least one minimizer of some variational problem [13] or in the search for nodal solutions to the elliptic equations [14]. It seems impossible to list the works by exploiting the advanced Nehari manifold; hence, we just recommend [15] and [47] and the references herein as examples.

Equations such as (1.8) pose a general form of the multidimensional Woinowsky-Krieger model, and the study of well-posedness theory attracts a lot of interest. See [7,17,23,34,40,51] and below for detail.

Without damping. For equation (1.8) without damping term and nonlinear source term, i.e., $g(u_t) = 0$ and $f(u) = 0$, Mederiors [34] obtained the existence and uniqueness of regular solutions. For the case

$g(u_t) = 0$ and $f(u) = |u|^{p-2}u$, Bainov and Minchev [3] proved sufficient conditions for the nonexistence of smooth solutions with negative initial energy ($E(0) \le 0$) and obtained an upper bound of the maximal time of existence. Esquivel-Avila [19] proved the blow-up and global properties of the solutions with $E(0) \le d$ (initial energy not above the mountain pass level). Later, Wu and Tsai [51] showed the local existence and global existence of solutions by combining the contraction mapping principle and the continuity arguments, and they also proved the blow-up of solution with negative initial energy ($E(0) < 0$) and subcritical positive initial energy ($0 \le E(0) < a_0$), where a_0 is a constant depending on initial data $\{u_0, u_1\}$ only, but not necessarily the depth of the potential well.

Weak damping: linear or nonlinear. For the linear weak damping case, i.e., $g(u_t) = \delta u_t$, $\delta > 0$, de Brito [17] and Biler [7] obtained decay estimates for solutions to Cauchy problems. For the IBVP, Guedda and Labani [23] derived the nonexistence of global solutions with dynamic boundary conditions. For the nonlinear weak damping case without nonlinear term, i.e., $f(u) = 0$, Patcheu [40] proved the existence and decay of global solutions. For equation (1.8) with both suitable growth assumptions on the nonlinear damping term and general source term, the existence and nonexistence of global solutions, also some properties of the solutions that were obtained in [9,12,33,46,57].

Strong damping. For a damping of fractionary order $g(u_t) = (-\Delta)^\beta u_t$, $0 < \beta \le 1$, Biazutti and Crippa in [6] derived the existence of a global attractor in an abstract setting, and then it was improved later in [16]. We also notice that without the fourth-order term $\Delta^2 u$, equation (1.8) becomes the well-known Kirchhoff equation introduced to describe the nonlinear vibrations of an elastic string [24]. The nonexistence of the global solutions of Kirchhoff equations with strong damping term Δu_t was investigated by many authors [36–38,41,50,58,59]. For $f(u) = 0$, the global existence and lower decay estimates of the solutions were derived in [35,36]. For $f(u) = |u|^{p-2}u$, Park and Bae in [41] proved the existence, uniqueness, and uniform convergence of solutions. Later, Wu and Tai [50] showed that the local solutions blow up in a finite time with positive subcritical initial energy ($E(0) < a_3$) by applying the energy method, where a_3 is a constant depending on initial data, but not necessarily the depth of the potential well. They improved the results obtained by Ono in [37,38] by considering a degenerate Kirchhoff type model ($M(s) = s^\gamma$). Later, the following more general problem was studied in [58,59]

$$u_{tt} - M(\|\nabla u\|^2)\Delta u - \Delta u_t + h(u_t) = F(x, u), \tag{1.9}$$

where F represents forcing terms. For the IBVP of equation (1.9), Yang and Li [58] proved the existence of finite-dimensional global attractor. For the Cauchy problem of (1.9) with $h(u_t) = u_t$, they discussed the longtime dynamics in [59].

Motivated by the above works, we consider problem (1.1)–(1.3) to discuss the global well-posedness of the solutions and the dynamical behaviors as time approaches infinity. By recalling the established results, the main contributions of the present paper can be summarized in the following two aspects.

(i) We conduct a comprehensive study on the global well-posedness of solution at three initial energy levels, i.e., subcritical initial energy level $E(0) < d$, critical initial energy level $E(0) = d$ and arbitrarily high initial energy level $E(0) > 0$. For the subcritical initial energy level, i.e., $E(0) < d$, we shall prove the global existence, asymptotic behavior of such global solution, and finite time blow-up of the local solution. This point can be considered as a kind of extension of the established results requiring the small initial data by pointing out how small the initial data should be in order to ensure global existence. We also point out that the negative initial energy ($E(0) < 0$) blow-up will be a special case of the subcritical energy case ($E(0) < d$). For the critical initial energy level, i.e., $E(0) = d$, we try to extend all the results for the subcritical case $E(0) < d$ to those for this critical case, i.e., $E(0) = d$. For the arbitrarily high initial energy level, i.e., $E(0) > 0$, we shall establish the global nonexistence theory, but we can only take care of the case $r = 1$, that is, the case of linear weak damping instead of nonlinear weak damping, and the problem with nonlinear weak damping will be open even after this work.

(ii) The present paper deals with the problem with both the nonlinear weak damping term and the linear strong damping term. Considering that there are a lot of established results [28,29,53,54] about the model with the single linear weak damping term or the single nonlinear strong damping term or sometimes the combinations of two of them, in the present paper, we shall consider a more general case with both the nonlinear weak damping term and the linear strong damping term, which will be an extension of the known results. But when we discuss the high energy problem, i.e., the case $E(0) > 0$, we can only tackle the problem with the linear weak damping term, and the high energy case with the nonlinear weak damping term is still an open problem.

For the subcritical initial energy case ($E(0) < d$), we prove the global existence of solutions in the framework of the potential well method [42], and then by utilizing the method in [5], we prove the exponential decay result. The potential well method is one of the important applications of the minimax theory [43]. Indeed, there are a great number of related works, including Robin problems [44], Neumann problems [45], and some other intriguing problems [55,56,61,62], which are direct or indirect applications of the minimax theory. By means of [48] and the so-called concavity method [25,26], we show a blow-up result. For the critical initial energy case ($E(0) = d$), by utilizing the method of [32,52], we prove the global existence, finite time blow-up, as well as asymptotic behavior of solutions. For the high initial energy case ($E(0) > 0$), we derive some sufficient conditions on the initial data such that certain solution blows up in a finite time with the aid of the technique of [21]. The organization of this paper is as follows. In Section 2, we introduce some notations, functionals, assumptions, and lemmas for proving the main theorem. In Section 3, we prove the local existence of solutions by using the contraction mapping principle. In Section 4, we discuss global existence, asymptotic behavior, and nonexistence of the solution for problem (1.1)–(1.3) for the subcritical initial energy $E(0) < d$. In Section 5, we discuss global existence, asymptotic behavior, and the nonexistence of the solution for problem (1.1)–(1.3) for the critical initial energy level $E(0) = d$. In Section 6, we prove global nonexistence of solutions for the high initial energy level $E(0) > 0$ when $r = 1$. Some open problems are also listed.

2 Preliminaries

In this section, we show some notations, assumptions, and preliminary results, which will be used later. Throughout the present paper, we denote the Sobolev space $H_0^1(\Omega)$ norm by $\|\cdot\|_{H_0^1}^2 = \|\cdot\|^2 + \|\nabla\cdot\|^2$ and $L^p(\Omega)$ $(2 \leqslant p < \infty)$ denotes the usual space of all L^p-functions on Ω with norm

$$\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)} \quad \text{and} \quad \|\cdot\| = \|\cdot\|_{L^2(\Omega)}.$$

Let $(u, v) = \int_\Omega uv\mathrm{d}x$ denote the L^2-inner product, and we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For problem (1.1)–(1.3), we introduce the following notations [19,31],

$$H = \left\{u \in H^2(\Omega) \cap H_0^1(\Omega) \,\middle|\, \frac{\partial u}{\partial \nu} = 0 \text{ or } \Delta u = 0 \text{ on } \partial\Omega\right\}, \tag{2.1}$$

where ν is the outward unit normal vector on $\partial\Omega$, and

$$\|u\|_H^2 = \|\nabla u\|^2 + \|\Delta u\|^2. \tag{2.2}$$

In this paper, for simplicity, we denote by $C > 0$ a generic constant, which may vary from line to line even within the same formula.

We use the following lemmas throughout this paper.

Lemma 2.1. (Sobolev-Poincaré inequality). *Let* $0 < q \le \frac{2n}{n-2m}$ *if* $n > 2m$ $(0 < q < +\infty$ *if* $n = 2m)$. *Then, the inequality*

$$\|v\|_q \le C\|(-\Delta)^{\frac{m}{2}}v\| \quad \text{for } v \in H_0^m(\Omega)$$

with some positive constant C holds.

For the initial boundary value problems (1.1)–(1.3), we introduce the total energy functional

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{\beta}{2\gamma+2}\|\nabla u\|^{2\gamma+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \tag{2.3}$$

the potential functional

$$\begin{aligned} J(t) = J(u) &= \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{\beta}{2\gamma+2}\|\nabla u\|^{2\gamma+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \frac{1}{2}\|u\|_H^2 + \frac{\beta}{2\gamma+2}\|\nabla u\|^{2\gamma+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \end{aligned} \tag{2.4}$$

and the Nehari functional

$$I(t) = I(u) = \|\Delta u\|^2 + \|\nabla u\|^2 + \beta\|\nabla u\|^{2\gamma+2} - \|u\|_{p+1}^{p+1} = \|u\|_H^2 + \beta\|\nabla u\|^{2\gamma+2} - \|u\|_{p+1}^{p+1}. \tag{2.5}$$

From (2.3)–(2.5), we can easily see that

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u) = \frac{1}{2}\|u_t\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 + \left(\frac{1}{2\gamma+2} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} + \frac{1}{p+1}I(u). \tag{2.6}$$

By $I(u)$, we introduce the potential well (stable set)

$$W = \{u \in H | I(u) > 0\} \cup \{0\}, \tag{2.7}$$

the outer space of the potential well (unstable set)

$$V = \{u \in H | I(u) < 0\}, \tag{2.8}$$

and the depth of the potential well (the so-called mountain pass level)

$$d := \inf_{u \in \mathcal{N}} J(u) = \inf_{u \in H\backslash\{0\}} \sup_{\lambda \geq 0} J(\lambda u), \tag{2.9}$$

where the Nehari manifold is

$$\mathcal{N} = \{u \in H\backslash\{O\} | I(u) = 0\}.$$

Next, we give some properties of the aforementioned manifolds and functionals as follows.

Lemma 2.2. *Let $u(x) \in H$, and $\|u\|_H \neq 0$. Then*
(i) $\lim_{\lambda \to 0} J(\lambda u) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$;
(ii) *On the interval $0 < \lambda < \infty$, there exists a unique $\lambda^* = \lambda^*(u)$, such that*

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u)\right|_{\lambda = \lambda^*} = 0;$$

(iii) *$J(\lambda u)$ is increasing on $0 \leq \lambda \leq \lambda^*$, decreasing on $\lambda^* < \lambda < \infty$ and takes the maximum at $\lambda = \lambda^*$;*
(iv) *$I(\lambda u) > 0$ for $0 < \lambda < \lambda^*$, $I(\lambda u) < 0$ for $\lambda^* < \lambda < \infty$, and $I(\lambda^* u) = 0$.*

Proof.
(i) This conclusion comes directly from

$$J(\lambda u) = \frac{\lambda^2}{2}\|u\|_H^2 + \frac{\lambda^{2\gamma+2}}{2\gamma+2}\beta\|\nabla u\|^{2\gamma+2} - \frac{\lambda^{p+1}}{p+1}\|u\|_{p+1}^{p+1}. \tag{2.10}$$

(ii) An easy calculation shows that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda u) = \lambda\|u\|_H^2 + \lambda^{2\gamma+1}\beta\|\nabla u\|^{2\gamma+2} - \lambda^p\|u\|_{p+1}^{p+1}, \tag{2.11}$$

which leads to the conclusion.

(iii) By a direct calculation, relation (2.11) gives

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) > 0 \quad \text{for } 0 < \lambda < \lambda^*$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u) < 0 \quad \text{for } \lambda^* < \lambda < \infty,$$

which proves the conclusion of (iii).

(iv) The conclusion follows from

$$I(\lambda u) = \lambda^2\|u\|_H^2 + \lambda^{2\gamma+2}\beta\|\nabla u\|^{2\gamma+2} - \lambda^{p+1}\|u\|_{p+1}^{p+1} = \lambda\frac{\mathrm{d}}{\mathrm{d}\lambda}J(\lambda u).$$

This completes the proof. □

For the depth of the potential well d, we have the following knowledge.

Lemma 2.3. (Depth of the potential well). *The potential well depth is*

$$d = \frac{p-1}{2(p+1)}\left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}}, \tag{2.12}$$

where

$$C = \sup_{u\in H, u\neq 0}\frac{\|u\|_{p+1}}{\|u\|_H} \tag{2.13}$$

is the imbedding constant form H into $L^{p+1}(\Omega)$.

Proof. By the definition of d, for $u \in \mathcal{N}$, we have $I(u) = 0$, i.e.,

$$\|u\|_H^2 + \beta\|\nabla u\|^{2\gamma+2} = \|u\|_{p+1}^{p+1}.$$

Noting $\beta \geq 0$ and (2.13), we obtain

$$\|u\|_H^2 \leq \|u\|_{p+1}^{p+1} \leq C^{p+1}\|u\|_H^{p+1}, \tag{2.14}$$

then

$$\|u\|_H^2 \geq \left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}}. \tag{2.15}$$

And, from (2.4) and (2.5), it follows that

$$\begin{aligned} J(u) &= \frac{1}{2}\|u\|_H^2 + \frac{\beta}{2\gamma+2}\|\nabla u\|^{2\gamma+2} - \frac{1}{p+1}\|u\|_{p+1}^{p+1} \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 + \left(\frac{1}{2\gamma+2} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} + \frac{1}{p+1}I(u). \end{aligned} \tag{2.16}$$

Combining (2.16) with $I(u) = 0, \beta \geq 0$ and (2.15), (2.16) gives

$$\begin{aligned} J(u) &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 + \left(\frac{1}{2\gamma+2} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 \\ &\geq \left(\frac{1}{2} - \frac{1}{p+1}\right)\left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}}. \end{aligned} \tag{2.17}$$

In turn, this implies that

$$d = \frac{p-1}{2(p+1)}\left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}}. \tag{2.18}$$

Here, we need to ensure that the equal sign of (2.18) is true, that is, there exists an extremal of the variational problem (2.9). The proof of this part is standard and similar to that in [42]; hence, we give this proof in the Appendix. □

Lemma 2.4. (Nonincreasing energy). *Let $u(x,t)$ be a solution to problem* (1.1)–(1.3), *then the energy functional $E(t)$ of problem* (1.1)–(1.3) *is nonincreasing with respect to t, i.e.*,

$$E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} < 0. \tag{2.19}$$

Proof. The energy identity can be obtained by testing (1.1) with u_t and integrating with respect to t, that is,

$$E(t) + \int_0^t (\|\nabla u_t\|^2 + \|u_\tau\|_{r+1}^{r+1})\mathrm{d}\tau = E(0), \tag{2.20}$$

which proves (2.19). □

We give the definition of the weak solution for problem (1.1)–(1.3).

Definition 2.1. (Weak solution). Function $u(x,t)$ is called a weak solution to problem (1.1)–(1.3) on Ω_T, provided

$$u(x,t) \in C([0,T];H) \cap C^2([0,T],H^{-1}(\Omega)),$$
$$u_t \in L^2([0,T];H_0^1(\Omega)) \cap L^\infty([0,T];L^{r+1}(\Omega)),$$
$$u(0) = u_0 \in H, \quad u_t(0) = u_1 \in H_0^1(\Omega)$$

and

$$\langle u_{tt},\omega\rangle + (\Delta u,\Delta\omega) + (\nabla u,\nabla\omega) + \beta\|\nabla u\|^{2\gamma}(\nabla u,\nabla\omega) + (\nabla u_t,\nabla\omega) = -(|u_\tau|^{r-1}u_\tau,\omega) + (|u|^{p-1}u,\omega)$$

for all test functions $\omega \in H$ and almost all $t \in [0,T]$.

3 Local existence

To prove the local existence of the solution to problem (1.1)–(1.3), we first give the following lemma, which can be proved by the Banach fixed point theory (see [10,22,38,39]).

Lemma 3.1. *Assume $u_0(x) \in H$, $u_1(x) \in H_0^1(\Omega)$, then for any $T > 0$, $u(x,t) \in C([0,T];H)$ and $u_t \in L^2([0,T];H_0^1(\Omega)) \cap L^{r+1}([0,T];L^{r+1}(\Omega))$, and there exists a unique solution $v(x,t) \in C([0,T];H)$ and $v_t \in L^2([0,T];H_0^1(\Omega)) \cap L^{r+1}([0,T];L^{r+1}(\Omega))$ to the following problem, which linearizes the inhomogeneous term of* (1.1)–(1.3),

$$\begin{cases} v_{tt} + \Delta^2 v - M(\|\nabla u\|^2)\Delta v - \Delta v_t + |v_t|^{r-1}v_t = |u|^{p-1}u & \textit{in } \Omega_T, \\ v(x,0) = u_0(x), v_t(x,0) = u_1(x) & \textit{in } \Omega, \\ v(x,t) = v_t(x,t) = 0 & \textit{on } \Gamma. \end{cases} \tag{3.1}$$

Next, we establish the local existence and uniqueness for solutions of problems (1.1)–(1.3) by using the Banach fixed point theorem.

Theorem 3.1. (Local existence) *Assume $u_0(x) \in H$ and $u_1(x) \in H_0^1(\Omega)$. Then, problem* (1.1)–(1.3) *admits a unique local solution $u(x,t)$ defined on a maximal time interval $[0,T]$ with*

$$u \in C([0,T],H)$$

and

$$u_t \in L^2([0,T];\ H_0^1(\Omega)) \cap L^{r+1}([0,T];\ L^{r+1}(\Omega)).$$

Proof. The method to prove the local existence theorem is similar to the method of [39]. Throughout the proof, we denote by $C > 0$ various embedding constants from line to line within the same formula. For some real numbers $R > 0$ and $T > 0$, which will be decided later, we consider the following space:

$$\begin{aligned}\mathcal{M}_T := \{u \in C([0,T],H),\quad & u_t \in L^2([0,T];\ H_0^1(\Omega)) \cap L^{r+1}([0,T];\ L^{r+1}(\Omega))| \\ u(0) = u_0,\quad u_t(0) = u_1 \quad & \text{and} \quad d(u(t)) \le R^2 \quad \text{for all } t \in [0,T]\},\end{aligned}$$

where

$$d(u(t)) = \|u_t\|^2 + \|\Delta u\|^2.$$

Then $\mathcal{M}_T$ is a complete metric space with the distance

$$d(u,v) := \sup_{t\in[0,T]} d(u(t)-v(t)) = \sup_{t\in[0,T]} (\|u_t - v_t\|^2 + \|\Delta u - \Delta v\|^2).$$

From Lemma 3.1, we consider the map Φ by $v := \Phi(u)$, which is the unique solution to problem (3.1) for $u \in \mathcal{M}_T$. For some $T > 0$ and $R > 0$, we would like to show that Φ is a contraction mapping from $\mathcal{M}_T$ into itself with respect to the metric $d(\cdot,\cdot)$ by the following two steps.

Step I: Φ maps $\mathcal{M}_T$ into itself ($\Phi(\mathcal{M}_T) \subseteq \mathcal{M}_T$).

We multiply (3.1) by v_t and integrate over Ω to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|v_t\|^2 + \|\Delta v\|^2 + M(\|\nabla u\|^2)\|\nabla v\|^2) + 2\|\nabla v_t\|^2 + 2\|v_t\|_{r+1}^{r+1} = \left(\frac{\mathrm{d}}{\mathrm{d}t}M(\|\nabla u\|^2)\right)\|\nabla v\|^2 + 2\int_\Omega |u|^{p-1}uv_t\mathrm{d}x. \tag{3.2}$$

Next, we estimate the first term in the second line of (3.2) as follows:

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}M(\|\nabla u\|^2)\right|\|\nabla v\|^2 &= 2\left|M'(\|\nabla u\|^2)\int_\Omega \nabla u\nabla u_t\mathrm{d}x\right|\|\nabla v\|^2 \\ &= 2\left|\gamma\beta\|\nabla u\|^{2\gamma-2}\int_\Omega \Delta u u_t\mathrm{d}x\right|\|\nabla v\|^2 \\ &\le 2\gamma\beta\|\nabla u\|^{2\gamma-2}\|\Delta u\|\|u_t\|\|\nabla v\|^2 \\ &\le 2\gamma\beta C^{2\gamma-2}\|\Delta u\|^{2\gamma-1}\|u_t\|\|\nabla v\|^2 \\ &\le 2\gamma\beta C^{2\gamma-2}R^{2\gamma}\|\nabla v\|^2.\end{aligned} \tag{3.3}$$

By Hölder's inequality and the Sobolev-Poincaré inequality, for the second term in the second line of (3.2), we obtain

$$2\int_\Omega |u|^{p-1}uv_t\mathrm{d}x \le 2\|u\|_{2p}^p\|v_t\| \le 2C^p\|\Delta u\|^p\|v_t\| \le 2C^pR^p\|v_t\|. \tag{3.4}$$

Thus, form (3.3) and (3.4), (3.2) gives

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|v_t\|^2+\|\Delta v\|^2+M(\|\nabla u\|^2)\|\nabla v\|^2)+2\|\nabla v_t\|^2+2\|v_t\|_{r+1}^{r+1}\\ &\leq 2\gamma\beta C^{2\gamma-2}R^{2\gamma}\|\nabla v\|^2+2C^pR^p\|v_t\|\\ &\leq 2\gamma\beta C^{2\gamma}R^{2\gamma}\|\Delta v\|^2+C^{2p}R^{2p}+\|v_t\|^2\\ &\leq C^{2p}R^{2p}+(2\gamma\beta C^{2\gamma}R^{2\gamma}+1)(\|v_t\|^2+\|\Delta v\|^2+M(\|\nabla u\|^2)\|\nabla v\|^2).\end{aligned}\tag{3.5}$$

Integrating (3.5) over $[0, t]$, we have

$$g(v(t))+2\int_0^t(\|\nabla v_\tau\|^2+\|v_\tau\|_{r+1}^{r+1})\mathrm{d}\tau\leq g(v_0)+C^{2p}R^{2p}T+\int_0^t(2\gamma\beta C^{2\gamma}R^{2\gamma}+1)g(v(\tau))\mathrm{d}\tau,$$

where $g(v(t))=\|v_t\|^2+\|\Delta v\|^2+M(\|\nabla u\|^2)\|\nabla v\|^2$.

By the Gronwall inequality, we obtain

$$g(v(t))+2\int_0^T(\|\nabla v_\tau\|^2+\|v_\tau\|_{r+1}^{r+1})\mathrm{d}\tau\leq(g(v_0)+C^{2p}R^{2p}T)e^{(2\gamma\beta C^{2\gamma}R^{2\gamma}+1)T}.$$

Then, from the definition of $d(u(t))$, we have

$$d(v(t))+2\int_0^T(\|\nabla v_\tau\|^2+\|v_\tau\|_{r+1}^{r+1})\mathrm{d}\tau\leq(g(v_0)+C^{2p}R^{2p}T)e^{(2\gamma\beta C^{2\gamma}R^{2\gamma}+1)T}.\tag{3.6}$$

Form (3.6), we have

$$v\in L^\infty([0,T],H)$$

and

$$v_t\in L^2([0,T],H_0^1(\Omega))\cap L^{r+1}([0,T];L^{r+1}(\Omega)),$$

which implies $v\in C([0,T];H)$. Choosing T and R to satisfy

$$(g(v_0)+C^{2p}R^{2p}T)e^{(2\gamma\beta C^{2\gamma}R^{2\gamma}+1)T}\leq R^2,\tag{3.7}$$

then, from (3.6) and (3.7), we have

$$d(v(t))+2\int_0^T(\|\nabla v_\tau\|^2+\|v_\tau\|_{r+1}^{r+1})\mathrm{d}\tau\leq R^2,\tag{3.8}$$

which shows $v\in\mathcal{M}_T$, that is, $\Phi(\mathcal{M}_T)\subseteq\mathcal{M}_T$.

Step II: Φ is a contraction with respect to the metric $d(\cdot,\cdot)$.

Now, by taking u_1 and u_2 in $\mathcal{M}_T$, subtracting the two equations (3.1) for $v_1=\Phi(u_1)$ and $v_2=\Phi(u_2)$, and setting $V=v_1-v_2$, we obtain for all $\eta\in\mathcal{M}_T$ and a.e. $t\in[0,T]$ that

$$\begin{cases}\langle V_{tt},\eta\rangle+(\Delta V,\Delta\eta)+M(\|\nabla u_1\|^2)(\nabla V,\nabla\eta)+(\nabla V_t,\nabla\eta)\\ \quad=-(|v_{1t}|^{r-1}v_{1t}-|v_{2t}|^{r-1}v_{2t},\eta)+(|u_1|^{p-1}u_1-|u_2|^{p-1}u_2,\eta)\\ \qquad+(M(\|\nabla u_1\|^2)-M(\|\nabla u_2\|^2))(\Delta v_2,\eta) & \text{in }\Omega_T,\\ V(x,0)=V_t(x,0)=0 & \text{in }\Omega,\\ V=0 & \text{on }\Gamma.\end{cases}\tag{3.9}$$

By setting $\eta=V_t$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|V_t\|^2+\|\Delta V\|^2+M(\|\nabla u_1\|^2)\|\nabla V\|^2)+2\|\nabla V_t\|^2+2\int_\Omega(|u_{1t}|^{r-1}u_{1t}-|u_{2t}|^{r-1}u_{2t})V_t\mathrm{d}x\\ &=2(M(\|\nabla u_1\|^2)-M(\|\nabla u_2\|^2))\int_\Omega\Delta v_2V_t\mathrm{d}x+\left(\frac{\mathrm{d}}{\mathrm{d}t}M(\|\nabla u_1\|^2)\right)\|\nabla V\|^2+2\int_\Omega(|u_1|^{p-1}u_1-|u_2|^{p-1}u_2)V_t\mathrm{d}x.\end{aligned}\tag{3.10}$$

Next, we estimate the right-hand side of (3.10). First, from the mean value theorem and definition of $M(s)$, we obtain

$$\begin{aligned} M(\|\nabla u_1\|^2) - M(\|\nabla u_2\|^2) &= M'(\xi_1)(\|\nabla u_1\|^2 - \|\nabla u_2\|^2) \\ &= \beta\gamma\xi_1^{\gamma-1}(\|\nabla u_1\| + \|\nabla u_2\|)(\|\nabla u_1\| - \|\nabla u_2\|) \\ &\le \beta\gamma\xi_1^{\gamma-1}(\|\nabla u_1\| + \|\nabla u_2\|)(\|\nabla u_1 - \nabla u_2\|) \\ &\le \beta\gamma(\|\nabla u_1\|^2 + \|\nabla u_2\|^2)^{\gamma-1}(\|\nabla u_1\| + \|\nabla u_2\|)(\|\nabla u_1 - \nabla u_2\|) \\ &\le \beta\gamma(\|\nabla u_1\| + \|\nabla u_2\|)^{2\gamma-2}(\|\nabla u_1\| + \|\nabla u_2\|)(\|\nabla u_1 - \nabla u_2\|) \\ &= \beta\gamma(\|\nabla u_1\| + \|\nabla u_2\|)^{2\gamma-1}(\|\nabla u_1 - \nabla u_2\|), \end{aligned} \tag{3.11}$$

where

$$\xi_1 := \theta\|\nabla u_1\|^2 + (1-\theta)\|\nabla u_2\|^2, \quad 0 < \theta < 1.$$

Thus, for $v = \Phi(u)$ and $\Phi(\mathcal{M}_T) \subseteq \mathcal{M}_T$ (in Step I), we see that

$$\|\Delta v\|^2 \le \|v_t\|^2 + \|\Delta v\|^2 = d(v(t)) \le R^2. \tag{3.12}$$

Then thanks to Hölder's inequality, (3.11), the Sobolev-Poincaré inequality and (3.12), we have

$$\begin{aligned} 2(M(\|\nabla u_1\|^2) - M(\|\nabla u_2\|^2))\int_\Omega \Delta v_2 V_t \mathrm{d}x &\le 2(M(\|\nabla u_1\|^2) - M(\|\nabla u_2\|^2))\|\Delta v_2\|\|V_t\| \\ &\le 2\beta\gamma(\|\nabla u_1\| + \|\nabla u_2\|)^{2\gamma-1}(\|\nabla u_1 - \nabla u_2\|)\|\Delta v_2\|\|V_t\| \\ &\le 2\beta\gamma(C\|\Delta u_1\| + C\|\Delta u_2\|)^{2\gamma-1}(C\|\Delta u_1 - \Delta u_2\|)\|\Delta v_2\|\|V_t\| \\ &\le 2\beta C^{2\gamma}(\|\Delta u_1\| + \|\Delta u_2\|)^{2\gamma-1}(\|\Delta u_1 - \Delta u_2\|)\|\Delta v_2\|\|V_t\| \\ &\le 2^{2\gamma-1}\beta\gamma C^{2\gamma}R^{2\gamma}(\|\Delta u_1\| - \|\Delta u_2\|)\|V_t\| \\ &\le 2^{2\gamma-1}\beta C^{2\gamma}R^{2\gamma}\sqrt{d(u_1-u_2)}\sqrt{d(V)}. \end{aligned} \tag{3.13}$$

Next, from Hölder's inequality and the Sobolev-Poincaré inequality, we obtain

$$\begin{aligned} \left|\frac{\mathrm{d}}{\mathrm{d}t}M(\|\nabla u_1\|^2)\right|\|\nabla V\|^2 &= 2\left|M'(\|\nabla u_1\|^2)\int_\Omega \nabla u_1\nabla u_{1t}\mathrm{d}x\right|\|\nabla V\|^2 \\ &= 2\left|M'(\|\nabla u_1\|^2)\int_\Omega \Delta u_1 u_{1t}\mathrm{d}x\right|\|\nabla V\|^2 \\ &\le 2\gamma\beta\|\nabla u_1\|^{2\gamma-2}\|\Delta u_1\|\|u_{1t}\|\ \|\nabla V\|^2 \\ &\le 2\gamma\beta C^{2\gamma-2}\|\Delta u_1\|^{2\gamma-1}\|u_{1t}\|\ C^2\|\Delta V\|^2 \\ &\le 2\gamma\beta C^{2\gamma}R^{2\gamma}d(V). \end{aligned} \tag{3.14}$$

Then, by the mean value theorem and the Hölder, Minkowski, and Sobolev inequalities, we have

$$\begin{aligned} 2\int_\Omega (|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2)V_t\mathrm{d}x &= 2\int_\Omega p|\xi_2|^{p-1}(u_1-u_2)V_t\mathrm{d}x \\ &\le 2\int_\Omega p|u_1+u_2|^{p-1}|u_1-u_2||V_t|\mathrm{d}x \\ &\le 2p\||u_1+u_2|^{p-1}\|_n\|u_1-u_2\|_{\frac{2n}{n-2}}\|V_t\| \\ &= 2p\||u_1+u_2|\|_{n(p-1)}^{p-1}\|u_1-u_2\|_{\frac{2n}{n-2}}\|V_t\| \\ &\le 2p(\|u_1\|_{n(p-1)}^{p-1} + \|u_2\|_{n(p-1)}^{p-1})\|u_1-u_2\|_{\frac{2n}{n-2}}\|V_t\| \\ &\le 2pC^p(\|\Delta u_1\|^{p-1} + \|\Delta u_2\|^{p-1})\|\Delta(u_1-u_2)\|\ \|V_t\| \\ &\le 4pC^pR^{p-1}\sqrt{d(u_1-u_2)}\sqrt{d(V)}, \end{aligned} \tag{3.15}$$

where

$$\xi_2 = \theta u_1 + (1-\theta)u_2, \quad 0 < \theta < 1,$$

$$\|u_1\|_{n(p-1)} \le C\|\Delta u_1\|,$$

$$\|u_2\|_{n(p-1)} \le C\|\Delta u_2\|$$

and

$$\|u_1 - u_2\|_{\frac{2n}{n-2}} \le C\|\Delta(u_1 - u_2)\|.$$

Hence, combining inequalities (3.13)–(3.15), (3.10) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\psi(v) + 2\|\nabla V_t\|^2 + 2\int_\Omega (|v_{1t}|^{r-1}v_{1t} - |u_{2t}|^{r-1}v_{2t})V_t\mathrm{d}x\\&\le (2^{2\gamma-1}\beta c_m^{2\gamma}R^{2\gamma} + 4pC^pR^{p-1})\sqrt{d(u_1-u_2)}\sqrt{d(V)} + 2\gamma\beta c_m^{2\gamma}R^{2\gamma}d(V)\\&\le (2^{2\gamma-2}\beta c_m^{2\gamma}R^{2\gamma} + 2pC^pR^{p-1})^2 d(u_1-u_2) + (2\gamma\beta C^{2\gamma}R^{2\gamma}+1)d(V)\\&\le (2^{2\gamma-2}\beta c_m^{2\gamma}R^{2\gamma} + 2pC^pR^{p-1})^2 d(u_1-u_2) + (2\gamma\beta C^{2\gamma}R^{2\gamma}+1)\psi(v),\end{aligned} \tag{3.16}$$

where

$$\psi(v) = \|V_t\|^2 + \|\Delta V\|^2 + M(\|\nabla u_1\|^2)\|\nabla V\|^2.$$

Then, integrating (3.16) over $[0, t]$ and using the initial conditions $V(0) = V_t(0) = 0$, namely,

$$\psi(v(0)) = \|V_t(0)\|^2 + \|\Delta V(0)\|^2 + M(\|\nabla u_1\|^2)\|\nabla V(0)\|^2 = 0,$$

we obtain

$$\begin{aligned}&\psi(v) + 2\int_0^t\|\nabla V_\tau\|^2\mathrm{d}\tau + 2\int_0^t\int_\Omega(|v_{1\tau}|^{r-1}v_{1\tau} - |v_{2\tau}|^{r-1}v_{2\tau})V_\tau\mathrm{d}x\mathrm{d}\tau\\&\le C_1(R)\int_0^t d(u_1-u_2)\mathrm{d}\tau + C_2(R)\int_0^t\psi(v)\mathrm{d}\tau\\&\le C_1(R)T\sup_{0\le t\le T} d(u_1-u_2) + C_2(R)\int_0^t\psi(v)\mathrm{d}\tau,\end{aligned}$$

where $C_1(R) = (2^{2\gamma-2}\beta c_m^{2\gamma}R^{2\gamma} + 2pC^pR^{p-1})^2$ and $C_2(R) = (2\gamma\beta C^{2\gamma}R^{2\gamma}+1)$ are constants dependent on R. Then from Gronwall's inequality, we obtain

$$\psi(v) \le C_1(R)Te^{C_2(R)T}\sup_{0\le t\le T} d(u_1-u_2).$$

Hence,

$$d(v_1 - v_2) = d(V) \le \psi(v) \le C_1(R)Te^{C_2(R)T}\sup_{0\le t\le T} d(u_1-u_2).$$

Thus, by the definition of $d(u, v)$, we obtain

$$d(v_1, v_2) \le C_1(R)Te^{C_2(R)T}d(u_1, u_2).$$

Hence, if

$$C_1(R)Te^{C_2(R)T} = (2^{2\gamma-2}\beta c_m^{2\gamma}R^{2\gamma} + 2pC^pR^{p-1})^2Te^{(2\gamma\beta C^{2\gamma}R^{2\gamma}+1)T} < 1, \tag{3.17}$$

and then Φ is a contraction mapping, which can be satisfied by choosing R sufficiently large and T sufficiently small in (3.7) and (3.17).

By using the Banach fixed point theorem, we prove the local existence result. □

4 Global existence, asymptotic behavior, and blowup of solutions at subcritical initial energy level $E(0) < d$

4.1 Global existence for the subcritical initial energy $E(0) < d$

In this subsection, we consider the global existence of weak solution to problem (1.1)–(1.3). First, we discuss the invariance of the stable set W, but we omit the proof, which can be found in [32].

Lemma 4.1. (Invariant stable set W). *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$ be given functions. Then all solutions of problem* (1.1)–(1.3) *with $E(0) < d$ belong to W, provided $u_0 \in W$.*

Now, we prove the global existence of a solution to problem (1.1)–(1.3) for $E(0) < d$.

Theorem 4.1. (Global existence when $E(0) < d$). *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$. Assume that $E(0) < d$ and $u_0 \in W$. Then the solution to problem* (1.1)–(1.3) *exists globally.*

Proof. Let $\{\omega_j(x)\}_{j=1}^{\infty}$ be a system of the base functions in H. Construct the approximate solutions to problem (1.1)–(1.3)

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t)\omega_j(x), \quad m = 1, 2, \ldots,$$

satisfying

$$\begin{aligned} &\langle u_{mtt}, w_s\rangle + (\Delta u_m, \Delta\omega_s) + (1 + \beta\|\nabla u_m\|^{2\gamma})(\nabla u_m, \nabla\omega_s) + (\nabla u_{mt}, \nabla w_s) \\ &= -(|u_{mt}|^{r-1}u_{mt}, w_s) + (|u_m|^{p-1}u_m, w_s), \quad s = 1, 2, \cdots, m, \end{aligned} \tag{4.1}$$

$$u_m(x,0) = \sum_{j=1}^{m} g_{jm}(0)\omega_j(x) \to u_0(x) \in H, \tag{4.2}$$

$$u_{mt}(x,0) = \sum_{j=1}^{m} g'_{jm}(0)\omega_j(x) \to u_1(x) \in H_0^1(\Omega). \tag{4.3}$$

Multiplying (4.1) by $g'_{sm}(t)$ and summing for s, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\|u_{mt}\|^2 + \frac{1}{2}\|u_m\|_H^2 + \frac{\beta}{2(\gamma+1)}\|\nabla u_m\|^{2\gamma+2} - \frac{1}{p+1}\|u_m\|_{p+1}^{p+1}\right) = -\|\nabla u_{mt}\|^2 - \|u_{mt}\|_{r+1}^{r+1}. \tag{4.4}$$

Integrating (4.4) over $[0, t]$, from (2.3) and (2.4), we can obtain

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \int_0^t (\|\nabla u_{m\tau}\|^2 + \|u_{m\tau}\|_{r+1}^{r+1})\mathrm{d}\tau = E_m(t) + \int_0^t (\|\nabla u_{m\tau}\|^2 + \|u_{m\tau}\|_{r+1}^{r+1})\mathrm{d}\tau = E_m(0). \tag{4.5}$$

From (4.2) and (4.3), we derive

$$\lim_{m\to\infty} E_m(0) = E(0).$$

Hence, combining (4.5) and $E(0) < d$, for sufficiently large m, we have

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \int_0^t (\|\nabla u_{m\tau}\|^2 + \|u_{m\tau}\|_{r+1}^{r+1})\mathrm{d}\tau < d. \tag{4.6}$$

Further, (2.16) and (4.6) allow us to see that

$$\begin{aligned}&\frac{1}{2}\|u_{mt}\|^2+\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_m\|_H^2+\left(\frac{1}{2\gamma+2}-\frac{1}{p+1}\right)\beta\|\nabla u_m\|^{2\gamma+2}+\frac{1}{p+1}I(u_m)\\&+\int_0^t(\|\nabla u_{m\tau}\|^2+\|u_{m\tau}\|_{r+1}^{r+1})\mathrm{d}\tau<d.\end{aligned}\tag{4.7}$$

From $u_0\in W$ and (4.2), we can obtain $u_m(0)\in W$ for sufficiently large m. By (4.6) and an argument similar to the proof of Lemma 4.1, we can prove that $u_m(t)\in W$ for $0\le t<\infty$. Hence, for sufficiently large m, we know that $I(u_m)>0$, which gives

$$\|u_m\|_{p+1}^{p+1}\le\|u_m\|_H+\beta\|\nabla u_m\|^{2\gamma+2}.\tag{4.8}$$

Again using $I(u_m)>0$, then (4.7) gives

$$\begin{aligned}&\frac{1}{2}\|u_{mt}\|^2+\left(\frac{1}{2}-\frac{1}{p+1}\right)\|u_m\|_H^2+\left(\frac{1}{2\gamma+2}-\frac{1}{p+1}\right)\beta\|\nabla u_m\|^{2\gamma+2}+\int_0^t(\|\nabla u_{m\tau}\|^2+\|u_{m\tau}\|_{r+1}^{r+1})\mathrm{d}\tau<d,\\&0\le t<\infty.\end{aligned}\tag{4.9}$$

From inequalities (4.8) and (4.9), we arrive at

$$u_m\ \text{ is bounded in }\ L^\infty(0,\infty;H);\tag{4.10}$$

$$u_{mt}\ \text{ is bounded in }\ L^2(0,\infty;H_0^1(\Omega));\tag{4.11}$$

$$|u_{mt}|^{r-1}u_{mt}\ \text{ is bounded in }\ L^{\frac{r+1}{r}}(0,\infty;L^{q_1}(\Omega)),\quad\text{where } q_1=\frac{r+1}{r};\tag{4.12}$$

$$|u_m|^{p-1}u_m\ \text{ is bounded in }\ L^\infty(0,\infty;L^{q_2}(\Omega)),\quad\text{where } q_2=\frac{p+1}{p}.\tag{4.13}$$

Then, integrating (4.1) with respect to t, we see that

$$\begin{aligned}&(u_{mt},w_s)+\int_0^t(\Delta u_m,\Delta w_s)\mathrm{d}\tau+\int_0^t(1+\beta\|\nabla u\|^{2\gamma})(\nabla u_m,\nabla\omega_s)\mathrm{d}\tau+(\nabla u_m,\nabla w_s)+\int_0^t(|u_{m\tau}|^{r-1}u_{m\tau},w_s)\mathrm{d}\tau\\&=\int_0^t(|u_m|^{p-1}u_m,w_s)\mathrm{d}\tau+(u_{m1},w_s)+(\nabla u_{m0},\nabla w_s),\quad 0\le t<\infty.\end{aligned}\tag{4.14}$$

Therefore, on the one hand, up to a subsequence, from (4.10) to (4.13), we may pass to the limit in (4.14) and obtain a weak solution $u(x,t)$ to problem (1.1)–(1.3) with the aforementioned regularity and (4.1). On the other hand, from (4.2) and (4.3), we obtain $u(x,0)=u_0\in H$ and $u_t(x,0)=u_1\in H_0^1(\Omega)$. □

4.2 Asymptotic behavior for the subcritical initial energy $E(0)<d$

In this section, we state the asymptotic behavior of solutions to problem (1.1)–(1.3) emanating from the initial data satisfying the conditions required by that for the global solutions in Theorem 4.1.

Theorem 4.2. (Asymptotic behavior when $E(0)<d$). *Suppose all the assumptions in* Theorem 4.1 *hold. Then there exist positive constants K and k such that the global solution of* (1.1)–(1.3) *satisfies*

$$E(t)\le Ke^{-kt},\quad t\ge0.$$

Proof. To prove Theorem 4.2, we consider $G(t):[0,T]\to\mathbb{R}^+$ defined by

$$G(t) := E(t) + \varepsilon\left((u,u_t) + \frac{1}{2}\|\nabla u\|^2\right), \tag{4.15}$$

for $\varepsilon > 0$ so small such that

$$\alpha_1 E(t) \le G(t) \le \alpha_2 E(t), \tag{4.16}$$

for two positive constants $\alpha_1, \alpha_2 > 0$. Moreover, by means of the Lions-Magenes theorem [30], and due to equation (1.1), we obtain

$$\langle u_{tt}, u\rangle + \|u\|_H^2 + \beta\|\nabla u\|^{2(\gamma+1)} + (\nabla u, \nabla u_t) + (|u_t|^{r-1}u_t, u) = (|u|^{p-1}u, u). \tag{4.17}$$

Using (4.15), (2.19), and (4.17), we have

$$\begin{aligned} G'(t) &= E'(t) + \varepsilon(\|u_t\|^2 + \langle u_{tt}, u\rangle + (\nabla u, \nabla u_t)) \\ &= -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} + \varepsilon(\|u_t\|^2 + \langle u_{tt}, u\rangle + (\nabla u, \nabla u_t)) \\ &= -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} + \varepsilon\left(\|u_t\|^2 + \|u\|_{p+1}^{p+1} - \|u\|_H^2 - \beta\|\nabla u\|^{2(\gamma+1)} - \int_\Omega |u_t|^{r-1}u_t u\,dx\right). \end{aligned} \tag{4.18}$$

Next, we will estimate the right-hand side of (4.18). For the term $\|u\|_H^2$, note that Lemma 4.1 tells $I(u) > 0$, then (2.16) gives

$$J(u(t)) > \frac{p-1}{2(p+1)}\|u(t)\|_H^2. \tag{4.19}$$

From (4.19), (2.3), (2.4), and Lemma 2.4, we obtain

$$\|u\|_H^2 \le \frac{2(p+1)}{p-1}J(t) \le \frac{2(p+1)}{p-1}E(t) \le \frac{2(p+1)}{p-1}E(0). \tag{4.20}$$

For the term $\|u\|_{p+1}^{p+1}$, using Sobolev's inequality and (4.20), we have

$$\|u\|_{p+1}^{p+1} \le C^{p+1}\|u\|_H^{p+1} \le C^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}\|u\|_H^2 = \xi\|u\|_H^2, \tag{4.21}$$

where C is the best Sobolev constant for the embedding $H \hookrightarrow L^{p+1}(\Omega)$ and

$$\xi = C^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}}.$$

Moreover, by $E(0) < d$ and Lemma 2.3 (the definition of d), we have $\xi < 1$. Using (4.21) and (2.3), we obtain

$$\begin{aligned} \|u\|_{p+1}^{p+1} &= (1-a)\|u\|_{p+1}^{p+1} + a\|u\|_{p+1}^{p+1} \\ &\le \xi(1-a)\|u\|_H^2 + a\|u\|_{p+1}^{p+1} \\ &= \xi(1-a)\|u\|_H^2 - a(p+1)E(t) + \frac{a(p+1)}{2}\left(\|u_t\|^2 + \|u\|_H^2 + \frac{\beta}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right), \end{aligned} \tag{4.22}$$

where $a \in (0,1)$. Consequently, substituting (4.22) into (4.18) yields

$$\begin{aligned} G'(t) &= -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} + \varepsilon\Bigg(\|u_t\|^2 + \xi(1-a)\|u\|_H^2 - a(p+1)E(t) \\ &\quad + \frac{a(p+1)}{2}\left(\|u_t\|^2 + \|u\|_H^2 + \frac{\beta}{\gamma+1}\|\nabla u\|^{2(\gamma+1)}\right) - \|u\|_H^2 - \beta\|\nabla u\|^{2(\gamma+1)} - \int_\Omega |u_t|^{r-1}u_t u\,dx\Bigg) \\ &= -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} + \varepsilon\Bigg(\left(1 + \frac{a(p+1)}{2}\right)\|u_t\|^2 - a(p+1)E(t) + \left(\frac{a(p+1)}{2} + \xi(1-a) - 1\right)\|u\|_H^2 \\ &\quad + \left(\frac{a(p+1)}{2(\gamma+1)} - 1\right)\beta\|\nabla u\|^{2(\gamma+1)} - \int_\Omega |u_t|^{r-1}u_t u\,dx\Bigg). \end{aligned} \tag{4.23}$$

Applying Hölder and Young inequalities

$$XY \le \frac{\delta^k}{k}X^k + \frac{\delta^{-q}}{q}Y^q \quad \text{for } X, Y \ge 0, \quad \delta > 0, \frac{1}{k} + \frac{1}{q} = 1,$$

to estimate $\int_\Omega |u_t|^{r-1}u_t u dx$, we take $k = \frac{r+1}{r}$ and $q = r+1$ to obtain

$$\left| \int_\Omega |u_t|^{r-1}u_t u dx \right| \le \delta_1 \|u_t(t)\|_{r+1}^{r+1} + C(\delta_1)\|u(t)\|_{r+1}^{r+1}, \tag{4.24}$$

where

$$\delta_1 = \frac{r}{r+1}\delta^{\frac{r+1}{r}}, \quad C(\delta_1) = \frac{1}{(r+1)\delta^{r+1}}.$$

Hence, from (4.24), (4.23) gives

$$\begin{aligned} G'(t) \le -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} + \varepsilon\Bigg(\left(1 + \frac{a(p+1)}{2}\right)\|u_t\|^2 - a(p+1)E(t) + \left(\frac{a(p-1)}{2} - (1-\xi)(1-a)\right)\|u\|_H^2 \\ + \left(\frac{a(p+1)}{2(\gamma+1)} - 1\right)\beta\|\nabla u\|^{2(\gamma+1)} + \delta_1\|u_t(t)\|_{r+1}^{r+1} + C(\delta_1)\|u(t)\|_{r+1}^{r+1}\Bigg). \end{aligned} \tag{4.25}$$

Moreover, from Poincaré inequality $\|u_t\| \le C\|\nabla u_t\|$, $C > 0$, we conclude that

$$\begin{aligned} G'(t) \le \left(\varepsilon C^2\left(1 + \frac{a(p+1)}{2}\right) - 1\right)\|\nabla u_t\|^2 + (\delta_1\varepsilon - 1)\|u_t\|_{r+1}^{r+1} + \varepsilon\left(\frac{a(p-1)}{2} - (1-\xi)(1-a)\right)\|u\|_H^2 \\ - \varepsilon a(p+1)E(t) + \varepsilon\left(\frac{a(p+1)}{2(\gamma+1)} - 1\right)\beta\|\nabla u\|^{2(\gamma+1)} + \varepsilon C(\delta_1)\|u(t)\|_{r+1}^{r+1}. \end{aligned} \tag{4.26}$$

Note that

$$0 < \frac{1-\xi}{p-1+2(1-\xi)} < \frac{1}{p+1} < \frac{\gamma+1}{p+1} < \frac{1}{2},$$

where $0 < \xi < 1$, $p > 2\gamma + 1$ and $\gamma > 0$. Hence, we can choose a constant

$$a \in \left(\frac{2(1-\xi)}{p-1+2(1-\xi)}, \frac{2(\gamma+1)}{p+1}\right) \subset (0,1),$$

to make

$$\frac{a(p-1)}{2} - (1-\xi)(1-a) \ge 0 \tag{4.27}$$

and

$$\frac{a(p+1)}{2(\gamma+1)} - 1 < 0. \tag{4.28}$$

Then from (4.27) and (4.20), inequality (4.26) gives

$$\begin{aligned} G'(t) &\le \left(\varepsilon C^2\left(1 + \frac{a(p+1)}{2}\right) - 1\right)\|\nabla u_t\|^2 + (\delta_1\varepsilon - 1)\|u_t\|_{r+1}^{r+1} \\ &\quad + \varepsilon\left(\frac{a(p-1)}{2} - (1-\xi)(1-a)\right)\frac{2(p+1)}{p-1}E(t) - \varepsilon a(p+1)E(t) + \varepsilon\left(\frac{a(p+1)}{2(\gamma+1)} - 1\right)\beta\|\nabla u\|^{2(\gamma+1)} \\ &\quad + \varepsilon C(\delta_1)\|u(t)\|_{r+1}^{r+1} \\ &= \left(\varepsilon C^2\left(1 + \frac{a(p+1)}{2}\right) - 1\right)\|\nabla u_t\|^2 + (\delta_1\varepsilon - 1)\|u_t\|_{r+1}^{r+1} - \varepsilon(1-\xi)(1-a)\frac{2(p+1)}{p-1}E(t) \\ &\quad + \varepsilon\left(\frac{a(p+1)}{2(\gamma+1)} - 1\right)\beta\|\nabla u\|^{2(\gamma+1)} + \varepsilon C(\delta_1)\|u(t)\|_{r+1}^{r+1}. \end{aligned} \tag{4.29}$$

For the term $\|u(t)\|_{r+1}^{r+1}$ in (4.29), using the Sobolev embedding inequality and (4.20), we have

$$\begin{aligned}\|u(t)\|_{r+1}^{r+1} &\le C\|u\|_H^{r+1}\\ &= C(\|u\|_H^2)^{\frac{r-1}{2}}\|u\|_H^2\\ &\le C\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{r-1}{2}}\|u\|_H^2\\ &\le C\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{r-1}{2}}\frac{2(p+1)}{p-1}E(t).\end{aligned} \tag{4.30}$$

From (4.30), (4.29) gives

$$\begin{aligned}G'(t) \le &\left(\varepsilon C^2\left(1+\frac{a(p+1)}{2}\right)-1\right)\|\nabla u_t\|^2 + (\delta_1\varepsilon - 1)\|u_t\|_{r+1}^{r+1} + \varepsilon\left(\frac{a(p+1)}{2(\gamma+1)}-1\right)\beta\|\nabla u\|^{2(\gamma+1)}\\ &- \varepsilon(1-\xi)(1-a)\frac{2(p+1)}{p-1}E(t) + \varepsilon C(\delta_1)C\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{r-1}{2}}\frac{2(p+1)}{p-1}E(t)\\ = &\lambda_1\|\nabla u_t\|^2 + \lambda_2\|u_t\|_{r+1}^{r+1} + \lambda_3\|\nabla u\|^{2(\gamma+1)} - \frac{2\varepsilon(p+1)}{p-1}\lambda_4 E(t),\end{aligned} \tag{4.31}$$

where

$$\lambda_1 = \varepsilon C^2\left(1+\frac{a(p+1)}{2}\right)-1,$$

$$\lambda_2 = \varepsilon\delta_1 - 1,$$

$$\lambda_3 = \varepsilon\beta\left(\frac{a(p+1)}{2(\gamma+1)}-1\right),$$

and

$$\lambda_4 = (1-\xi)(1-a) - C(\delta_1)C\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{r-1}{2}}.$$

Since $C(\delta_1) = \frac{1}{(r+1)\delta^{r+1}}$ and $\xi < 1$, we can choose δ large enough so that $\lambda_4 > 0$. Once δ is fixed (hence, δ_1 is also fixed), we can pick ε so small such that

$$\lambda_1 = \varepsilon C^2\left(1+\frac{a(p+1)}{2}\right)-1 < 0$$

and

$$\lambda_2 = \varepsilon\delta_1 - 1 < 0.$$

From (4.28), we easily have $\lambda_3 < 0$. Consequently noting (4.16), $\lambda_1 < 0, \lambda_2 < 0, \lambda_3 < 0$, and $\lambda_4 > 0$, (4.31) becomes

$$G'(t) \le \lambda_1\|\nabla u_t\|^2 + \lambda_2\|u_t\|_{r+1}^{r+1} + \lambda_3\|\nabla u\|^{2(\gamma+1)} - \frac{2\varepsilon(p+1)}{p-1}\lambda_4 E(t) \le -\frac{2\varepsilon(p+1)}{\alpha_2(p-1)}\lambda_4 G(t). \tag{4.32}$$

Then integration of (4.32) (Gronwall inequality) leads to

$$G(t) \le G(0)e^{-kt},$$

where $k = \frac{2\varepsilon(p+1)}{\alpha_2(p-1)}\lambda_4$. And then from (4.16), we can obtain

$$E(t) \le Ke^{-kt},$$

where $K = \frac{\alpha_2 E(0)}{\alpha_1}$. This completes the proof. □

4.3 Global nonexistence for the subcritical initial energy $E(0) < d$

In this subsection, we prove the finite time blowup of solutions to problem (1.1)–(1.3). First, by the same argument as Lemma 4.1, we can obtain the following invariance of the unstable set V with respect to time, but we omit the proof (see [32] for details).

Lemma 4.2. (Invariant unstable set V). *Let $u_0 \in H$, $u_1 \in H_0^1(\Omega)$ be given functions. Then all solutions to problem* (1.1)–(1.3) *with $E(0) < d$ belong to V, provided $u_0 \in V$.*

Now, we give some relations of the depth of the potential well d and norm $\|u\|_H^2$, which can be easily derived from Lemma 4.2.

Corollary 4.1. *If $u \in V$, then*

$$d < \frac{p-1}{2(p+1)}\|u\|_H^2. \tag{4.33}$$

Proof. The condition $u \in V$ implies $I(u) < 0$. By the definition of $I(u)$, i.e., (2.5), we have

$$\|u\|_H^2 + \beta\|\nabla u\|^{2\gamma+2} < \|u\|_{p+1}^{p+1}.$$

Noting $\beta \geq 0$ and the definition of the best embedding constant, i.e., (2.13), we obtain

$$\|u\|_H^2 < \|u\|_{p+1}^{p+1} \leq C^{p+1}\|u\|_H^{p+1},$$

i.e.,

$$\|u\|_H^2 > \left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p-1}}.$$

By the expression of potential well depth, i.e., (2.12), we obtain (4.33). □

Theorem 4.3. (Global nonexistence when $E(0) < d$). *Let $u_0 \in H$ and $u_1(x) \in H_0^1(\Omega)$. Assume that $E(0) < d$ and $u_0 \in V$. Then, the solution to problems* (1.1)–(1.3) *blows up in finite time.*

Proof. As we need different strategies for the cases $r > 1$ and $r = 1$, the proof will consider the following two distinct situations.

Case I: $r > 1$. Let $u(t)$ be any solution to problem (1.1)–(1.3) with $E(0) < d$ and $I(u_0) < 0$. Let T be the maximum existence time of $u(t)$, we aim to prove $T < \infty$. Arguing by contradiction, we suppose that $T = +\infty$. Then for any $T_0 > 0$, we define the following functions:

$$\theta(t) := \|u\|^2$$

and

$$\varphi(t) := \|u\|_{p+1}^{p+1}, \quad t \in [0, T_0].$$

Then

$$\theta'(t) = 2(u, u_t). \tag{4.34}$$

Testing equation (1.1) by u, we can reach that

$$\theta''(t) = 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle = 2\|u_t\|^2 - 2\|u\|_H^2 - 2\beta\|\nabla u\|^{2\gamma+2} + 2\|u\|_{p+1}^{p+1} - 2(\nabla u_t, \nabla u) - 2(|u_t|^{r-1}u_t, u). \tag{4.35}$$

Next, we estimate the last two terms of (4.35). By Lemma 4.2, we obtain

$$\|u\|^2 \leq C\|u\|_H^2 < C\|u\|_{p+1}^{p+1} = C\varphi(t). \tag{4.36}$$

Again using Hölder's inequality, interpolation inequality, and (4.36), we have

$$|(|u_t|^{r-1}u_t, u)| \le \|u\|_{r+1}\|u_t\|_{r+1}^r \le \|u\|^\delta \|u\|_{p+1}^{1-\delta}\|u_t\|_{r+1}^r \le C\varphi(t)^{\frac{1}{r+1}}\varphi(t)^{\frac{1-\delta}{p+1}-\frac{1}{r+1}+\frac{\delta}{2}}\|u_t\|_{r+1}^r, \tag{4.37}$$

where

$$\delta = \left(\frac{1}{r+1} - \frac{1}{p+1}\right) \Big/ \left(\frac{1}{2} - \frac{1}{p+1}\right).$$

Note that

$$\frac{1-\delta}{p+1} - \frac{1}{r+1} + \frac{\delta}{2} = 0,$$

and hence,

$$\varphi(t)^{\frac{1-\delta}{p+1}-\frac{1}{r+1}+\frac{\delta}{2}} = 1.$$

Using Hölder and Young inequalities,

$$XY \le \frac{\delta^k}{k}X^k + \frac{\delta^{-q}}{q}Y^q \quad \text{for } X, Y \ge 0, \quad \delta > 0, \quad \frac{1}{k} + \frac{1}{q} = 1,$$

where we set $k = r + 1$ and $q = \frac{r+1}{r}$, and we estimate (4.37) as follows:

$$|(|u_t|^{r-1}u_t, u)| \le C\varphi(t)^{1/(r+1)}\|u_t\|_{r+1}^r \le \eta_1\varphi(t) + \eta_2\|u_t\|_{r+1}^{r+1}, \tag{4.38}$$

where

$$\eta_1 = \frac{C\delta^{r+1}}{r+1} \quad \text{and} \quad \eta_2 = \frac{r}{r+1}\delta^{\frac{-(r+1)}{r}}.$$

We set $k = q = 2$, then

$$|(\nabla u, \nabla u_t)| \le \eta_3\|\nabla u\|^2 + \eta_4\|\nabla u_t\|^2, \tag{4.39}$$

where

$$\eta_3 = \frac{\delta^2}{2} \quad \text{and} \quad \eta_4 = \frac{1}{2\delta^2}.$$

From (2.5), (4.38), and (4.39), (4.35) becomes

$$\begin{aligned} \theta''(t) &= 2\|u_t\|^2 - 2I(u) - 2(\nabla u, \nabla u_t) - 2(|u_t|^{r-1}u_t, u) \\ &\ge 2\|u_t\|^2 - 2I(u) - 2\eta_1\varphi(t) - 2\eta_2\|u_t\|_{r+1}^{r+1} - 2\eta_3\|\nabla u\|^2 - 2\eta_4\|\nabla u_t\|^2. \end{aligned} \tag{4.40}$$

By Lemma 2.4, (2.3), and (2.5), we obtain

$$\begin{aligned} I(t) &\le I(t) + \sigma(E(0) - E(t)) \\ &= -\frac{\sigma}{2}\|u_t\|^2 + \left(1 - \frac{\sigma}{2}\right)\|u\|_H^2 + \left(1 - \frac{\sigma}{2\gamma+2}\right)\beta\|\nabla u\|^{2\gamma+2} + \left(\frac{\sigma}{p+1} - 1\right)\varphi(t) + \sigma E(0), \end{aligned} \tag{4.41}$$

where the constant $\sigma > 2(\gamma + 1) > 0$ will be chosen later. Substituting (4.41) into (4.40), we have

$$\begin{aligned} \theta''(t) + 2\eta_2\|u_t\|_{r+1}^{r+1} + 2\eta_4\|\nabla u_t\|^2 &\ge 2\left(1 - \frac{\sigma}{p+1} - \eta_1\right)\varphi(t) - 2\eta_3\|\nabla u\|^2 + (\sigma - 2)\|u\|_H^2 - 2\sigma E(0) \\ &\quad + 2\left(\frac{\sigma}{2\gamma+2} - 1\right)\beta\|\nabla u\|^{2\gamma+2} + (2 + \sigma)\|u_t\|^2 \\ &\ge 2\left(1 - \frac{\sigma}{p+1} - \eta_1\right)\varphi(t) - 2\eta_3\|\nabla u\|^2 + (\sigma - 2)\|u\|_H^2 - 2\sigma E(0) \\ &\quad + 2\left(\frac{\sigma}{2\gamma+2} - 1\right)\beta\|\nabla u\|^{2\gamma+2}. \end{aligned} \tag{4.42}$$

We now estimate the right-hand side of (4.42). From Lemma 4.2, we obtain

$$\|\nabla u\|^2 < \|u\|_H^2 + \beta\|\nabla u\|^{2\gamma+2} < \|u\|_{p+1}^{p+1} = \varphi(t). \tag{4.43}$$

Then by (4.43) and (4.42), we have

$$\begin{aligned}\theta''(t) + 2\eta_2\|u_t\|_{r+1}^{r+1} + 2\eta_4\|\nabla u_t\|^2 \geq 2\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right)\varphi(t) + (\sigma - 2)\|u\|_H^2 - 2\sigma E(0)\\ + 2\left(\frac{\sigma}{2\gamma+2} - 1\right)\beta\|\nabla u\|^{2\gamma+2}.\end{aligned} \tag{4.44}$$

As $E(0) < d$, choose the constant σ so that

$$p + 1 > \sigma > \rho, \quad \rho = \max\left\{\frac{2(p+1)d}{(p+1)d - (p-1)E(0)}, \quad 2(\gamma+1)\right\}, \tag{4.45}$$

which guarantees $\sigma > 2(\gamma + 1)$, then

$$\left(\frac{\sigma}{2\gamma+2} - 1\right)\beta\|\nabla u\|^{2\gamma+2} \geq 0. \tag{4.46}$$

Combining (4.44) and (4.46), we obtain

$$\theta''(t) + 2\eta_2\|u_t\|_{r+1}^{r+1} + 2\eta_4\|\nabla u_t\|^2 \geq 2\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right)\varphi(t) + (\sigma - 2)\|u\|_H^2 - 2\sigma E(0). \tag{4.47}$$

By Corollary 4.1 and (4.45), we have

$$(\sigma - 2)\|u\|_H^2 - 2\sigma E(0) > \frac{2(p+1)}{p-1}(\sigma - 2)d - 2\sigma E(0) = 2\left(\frac{p+1}{p-1}d - E(0)\right)\sigma - \frac{4(p+1)}{p-1}d > 0. \tag{4.48}$$

Then, from (4.48) and (4.47), we arrive at

$$\theta''(t) + 2\eta_2\|u_t\|_{r+1}^{r+1} + 2\eta_4\|\nabla u_t\|^2 \geq 2\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right)\varphi(t). \tag{4.49}$$

Due to

$$\eta_1 = \frac{C\delta^{r+1}}{r+1}, \eta_3 = \frac{\delta^2}{2}$$

and $\sigma < p + 1$, we can choose δ small enough such that

$$1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3 > 0.$$

Therefore, by (4.43) and Corollary 4.1, (4.49) becomes

$$\theta''(t) + 2\eta_2\|u_t\|_{r+1}^{r+1} + 2\eta_4\|\nabla u_t\|^2 > 2\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right)\|u\|_H^2 > 4\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right)\frac{p+1}{p-1}d > 0. \tag{4.50}$$

Integrating (4.50) over the time interval $[0, t]$ gives

$$\theta'(t) + \int_0^t (2\eta_2\|u_\tau\|_{r+1}^{r+1} + 2\eta_4\|\nabla u_\tau\|^2)d\tau > \frac{4d(p+1)}{p-1}\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right)t + \theta'(0). \tag{4.51}$$

Note that (2.20) gives

$$\int_0^t (\|u_\tau\|_{r+1}^{r+1} + \|\nabla u_\tau\|^2)d\tau = E(0) - E(t) < C. \tag{4.52}$$

Then by (4.52), (4.51) becomes

$$\theta'(t) > \eta t + \theta'(0) - c_\eta C, \tag{4.53}$$

where

$$\eta = \frac{4d(p+1)}{p-1}\left(1 - \frac{\sigma}{p+1} - \eta_1 - \eta_3\right) \quad \text{and} \quad c_\eta = \max\{2\eta_2, 2\eta_4\}.$$

Integrating (4.53) over the time interval $[0, t]$,

$$\theta(t) > \frac{\eta}{2}t^2 + (\theta'(0) - c_\eta C)t + \theta(0), \quad t > 0. \tag{4.54}$$

Thus, the norm $\theta(t)$ has the quadratic growth as $t \to \infty$.

On the other side, we estimate $\|u(t)\|^2$. By the regularity of $u(t)$ in $L^2(\Omega)$ and the Schwarz inequality, we have

$$\begin{aligned}
\int_\Omega |u(t)|^2 \mathrm{d}x &= \int_\Omega \left(u_0 + \int_0^t u_\tau(\tau)\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&= \|u_0\|^2 + 2\int_\Omega u_0 \int_0^t u_\tau \mathrm{d}\tau \mathrm{d}x + \int_\Omega \left(\int_0^t |u_\tau| \mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\le 2\|u_0\|^2 + 2\int_\Omega \left(\int_0^t |u_\tau|\mathrm{d}\tau\right)^2 \mathrm{d}x \\
&\le 2\|u_0\|^2 + 2\int_\Omega \left(\int_0^t 1\mathrm{d}\tau \int_0^t |u_\tau|^2 d\tau\right)\mathrm{d}x \\
&= 2\|u_0\|^2 + 2t\int_\Omega \int_0^t |u_\tau|^2 \mathrm{d}\tau \mathrm{d}x.
\end{aligned} \tag{4.55}$$

To estimate $\int_\Omega \int_0^t |u_\tau|^2 \mathrm{d}\tau \mathrm{d}x$, we again use the Hölder inequality for the integral with respect to t. Exploiting the inequality $\|u\| \le C\|u\|_{r+1} (r > 1)$, we have

$$\begin{aligned}
2t\int_\Omega \int_0^t |u_\tau|^2 \mathrm{d}\tau \mathrm{d}x &= 2t\int_0^t \|u_\tau\|^2 \mathrm{d}\tau \\
&\le 2t\left(\int_0^t 1\mathrm{d}\tau\right)^{\frac{r-1}{r+1}} \left(\int_0^t \|u_\tau\|^{r+1} \mathrm{d}\tau\right)^{\frac{2}{r+1}} \\
&= 2t^{1+\frac{r-1}{r+1}} \left(\int_0^t \|u_\tau\|^{r+1} \mathrm{d}\tau\right)^{\frac{2}{r+1}} \\
&\le 2t^{1+\frac{r-1}{r+1}} \left(\int_0^t C^{r+1}\|u_\tau\|_{r+1}^{r+1} \mathrm{d}\tau\right)^{\frac{2}{r+1}} \\
&= 2C^2 t^{1+\frac{r-1}{r+1}} \left(\int_0^t \|u_\tau\|_{r+1}^{r+1} \mathrm{d}\tau\right)^{\frac{2}{r+1}}.
\end{aligned} \tag{4.56}$$

From (4.56) and (4.55), we obtain

$$\theta(t) = \int_\Omega |u(t)|^2 \mathrm{d}x \le 2\|u_0\|^2 + 2C^2 t^{\frac{2r}{r+1}} \left(\int_0^t \|u_\tau\|_{r+1}^{r+1} \mathrm{d}\tau \right)^{\frac{2}{r+1}}. \tag{4.57}$$

From (4.52) and (4.57), we have

$$\theta(t) \le 2\|u_0\|^2 + 2C^2 t^{\frac{2r}{r+1}} d^{\frac{2}{r+1}}, \tag{4.58}$$

where $\frac{2r}{r+1} < 2$, which tells that (4.58) contradicts (4.54). Hence, the solution does not exist over the whole interval $[0, \infty)$. This completes the proof of Case I.

Case II: $r = 1$. Let $u(t)$ be any weak solution to problem (1.1)–(1.3) with $E(0) < d$ and $I(u_0) < 0$. Now, we prove the solution to problem (1.1)–(1.3) blows up in finite time. Arguing by contradiction, we suppose that the solution $u(x, t)$ is global. Then, for any $T_0 > 0$, we may consider $F : [0, T_0] \to \mathbb{R}^+$ defined by

$$F(t) := \|u\|^2 + \int_0^t \|u(x,\tau)\|_{H_0^1}^2 \mathrm{d}\tau + (T_0 - t)\|u_0\|_{H_0^1}^2. \tag{4.59}$$

It is clear that $F(t) > 0$ for all $t \in [0, T_0]$. As $F(t)$ is continuous on $[0, T_0]$, there exists a $\rho > 0$ (independent of the choice of T_0) such that

$$F(t) \ge \rho, \quad t \in [0, T_0]. \tag{4.60}$$

Hence, we have

$$F'(t) = 2(u, u_t) + \left(\|u\|_{H_0^1}^2 - \|u_0\|_{H_0^1}^2\right) = 2(u, u_t) + 2\int_0^t (u, u_\tau)\mathrm{d}\tau + 2\int_0^t (\nabla u, \nabla u_\tau)\mathrm{d}\tau \tag{4.61}$$

and

$$F''(t) = 2\|u_t\|^2 + 2\langle u_{tt}, u\rangle + 2(u, u_t) + 2(\nabla u, \nabla u_t) = 2\|u_t\|^2 - 2I(u). \tag{4.62}$$

From (4.61), we have

$$(F'(t))^2 = 4\left((u, u_t)^2 + 2(u, u_t)\int_0^t ((u, u_\tau) + (\nabla u, \nabla u_\tau))\mathrm{d}\tau \right) + 4\left(\int_0^t ((u, u_\tau) + (\nabla u, \nabla u_\tau))\mathrm{d}\tau \right)^2. \tag{4.63}$$

Using the Schwarz inequality, we estimate each term in (4.63) as follows

$$(u, u_t)^2 \le \|u\|^2 \|u_t\|^2,$$

$$\left(\int_0^t (u, u_\tau) + (\nabla u, \nabla u_\tau)\mathrm{d}\tau \right)^2 \le \int_0^t \|u(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau,$$

and

$$\begin{aligned} 2(u, u_t)\int_0^t ((u, u_\tau) + (\nabla u, \nabla u_\tau))\mathrm{d}\tau &\le 2\|u\|\|u_t\| \left(\int_0^t \|u(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \right)^{1/2} \left(\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \right)^{1/2} \\ &\le \|u\|^2 \int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau + \|u_t\|^2 \int_0^t \|u(\tau)\|_{H_0^1}^2 \mathrm{d}\tau. \end{aligned}$$

Then, (4.63) becomes

$$\begin{aligned} (F'(t))^2 &\le 4\left(\|u\|^2 + \int_0^t \|u(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \right)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \right) \\ &\le 4F(t)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau \right). \end{aligned} \tag{4.64}$$

By (4.62) and (4.64), we obtain

$$F''(t)F(t) - \frac{p+3}{4}(F'(t))^2 \geq F(t)\left(F''(t) - (p+3)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|^2_{H_0^1} d\tau\right)\right)$$
$$\geq F(t)\left(2\|u_t\|^2 - 2I(u) - (p+3)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|^2_{H_0^1} d\tau\right)\right).$$

Setting

$$\xi_1(t) := 2\|u_t\|^2 - 2I(u) - (p+3)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|^2_{H_0^1} d\tau\right). \tag{4.65}$$

Note that

$$E(0) = E(t) + \int_0^t \|u_\tau\|^2_{H_0^1} d\tau$$
$$= \frac{1}{2}\|u_t\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 + \left(\frac{1}{2(\gamma+1)} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} + \frac{1}{p+1}I(u) + \int_0^t \|u_\tau\|^2_{H_0^1} d\tau,$$

then

$$\frac{1}{p+1}I(u) = E(0) - \frac{1}{2}\|u_t\|^2 - \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_H^2 - \left(\frac{1}{2(\gamma+1)} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2} - \int_0^t \|u_\tau\|^2_{H_0^1} d\tau.$$

Hence, we obtain

$$\xi_1(t) = (p-1)\|u\|_H^2 - 2(p+1)E(0) + (p-1)\int_0^t \|u_\tau(\tau)\|^2_{H_0^1} d\tau + \left(\frac{1}{2(\gamma+1)} - \frac{1}{p+1}\right)\beta\|\nabla u\|^{2\gamma+2}. \tag{4.66}$$

Now, we set

$$\phi(t) = (p-1)\|u\|_H^2 - 2(p+1)E(0). \tag{4.67}$$

Moreover, by Lemma 4.2 and Corollary 4.1, we conclude that there exists $\sigma_1 > 0$ (independent of T_0) such that

$$\phi(t) = (p-1)\|u\|_H^2 - 2(p+1)d + 2(p+1)d - 2(p+1)E(0) \geq \sigma_1 > 0,$$

hence,

$$F''(t)F(t) - \frac{p+3}{4}F'(t)^2 \geq F(t)\xi_1(t) \geq \rho\sigma_1 > 0, \quad t \in [0, T_0].$$

Let $y(t) := F(t)^{-\frac{p-1}{4}}$, then

$$y''(t) \leq -\frac{p-1}{4}\rho\sigma_1 y(t)^{\frac{p+6}{p-2}}, \quad t \in [0, T_0],$$

which tells that

$$\lim_{t\to T_*} y(t) = 0.$$

This proves that $y(t)$ reaches 0 at finite time, say as $t \to T_*$. Since T_* is independent of the initial choice T_0, we may assume that $T_* < T_0$. In other words, $\lim_{t\to T_*} F(t) = +\infty$. This completes the proof of Case II. ☐

5 Global existence, asymptotic behavior, and blowup of solutions at critical initial energy level $E(0) = d$

5.1 Global existence for the critical initial energy level $E(0) = d$

This subsection proves the global existence of weak solution to problem (1.1)–(1.3) for the critical initial energy level $E(0) = d$. Due to the strong connection with the subcritical initial energy case, we do not need to rebuild the whole proof of the global existence conclusion for the critical initial energy, and we only need to find a path from the subcritical case to the critical case. So, roughly speaking, the following proof is to find a sequence of initial data from the subcritical energy range to approach the critical situation.

Theorem 5.1. (Global existence when $E(0) = d$). *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$ be given functions. Assume that $E(0) = d$ and $u_0 \in W$. Then the solution to problem* (1.1)–(1.3) *exists globally.*

Proof. Let $\lambda_m = 1 - \frac{1}{m}$, $u_{0m} = \lambda_m u_0$, $u_{1m} = \lambda_m u_1$, $m = 2, 3, \cdots$. Consider problems (1.1) and (1.3) corresponding to the initial conditions:

$$u(x, 0) = u_{0m}(x), \quad u_t(x, 0) = u_{1m}(x). \tag{5.1}$$

As $u_0 \in W$, i.e., $u_0 = 0$ or $u_0 \neq 0$, we consider the following two cases.

Case I. $u_0 = 0$. Then $J(u_0) = 0$, which can be included in the subcritical initial energy case.

Case II. $u_0 \neq 0$. Then $I(u_0) > 0$, from (iv) of Lemma 2.2, we obtain $\lambda^* = \lambda^*(u_0) > 1$. Thus, we have $1 - \frac{1}{m} < 1 < \lambda^*$. Again from (iv) of Lemma 2.2, we obtain $I(u_{0m}) > 0$. Moreover, by $1 - \frac{1}{m} < 1 < \lambda^*$ and (iii) of Lemma 2.2, we have $J(u_{0m}) < J(u_0)$.

Hence, for the aforementioned two cases, we have

$$\begin{aligned} E_m(0) &= \frac{1}{2}\|u_{1m}\|^2 + J(u_{0m}) \\ &\le \frac{1}{2}\|u_{1m}\|^2 + J(u_0) \\ &= \frac{1}{2}\lambda_m^2\|u_1\|^2 + J(u_0) \\ &< \frac{1}{2}\|u_1\|^2 + J(u_0) = E(0) = d. \end{aligned}$$

Therefore, it follows from Theorem 4.1 that for sufficiently large m problem (1.1), (5.1) and (1.3) admits a global weak solution $u_m(t) \in C([0, \infty), H)$, $u_{mt} \in L^2([0, \infty), H_0^1(\Omega)) \cap L^{r+1}([0, \infty), L^{r+1}(\Omega))$ and $u_m \in W$ for $0 \le t \le \infty$. Similar as the proof of Theorem 4.1, we have (4.1) and (4.2). Moreover we have $u_{mt}(x, 0) \to u_1$ and $u_m(x, 0) \to u_0$. Therefore, $u(x, t)$ is the global weak solution to problem (1.1)–(1.3). Hence, we complete this theorem. □

5.2 Asymptotic behavior for the critical initial energy level $E(0) = d$

This subsection shows the asymptotic behavior of the global solution to problem (1.1)–(1.3) for the critical initial energy level $E(0) = d$. First, we show the following lemma, which helps connect the subcritical case with the critical case by decaying the subcritical initial energy to the lower level, which may pass the critical level but only for the nonsteady state solution.

Lemma 5.1. *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$ be given functions. Assume that $u(t)$ is the global solution (but not steady-state solution) to problem* (1.1)–(1.3) *and $T_{\max}$ is the existence time of $u(t)$. Then there exists a $t_0 \in (0, T_{\max})$ such that*

$$\int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{k+1}^{k+1}) d\tau > 0. \tag{5.2}$$

Proof. Let $u(t)$ be any solution (but not steady-state solution) to problem (1.1)–(1.3), and $T_{\max}$ be the existence time of $u(t)$. We prove that there exists a $t_0 \in (0, T_{\max})$ such that (5.2) holds. Arguing by contradiction, we suppose that for any $t_0 \in [0, T_{\max})$,

$$\int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{k+1}^{k+1}) d\tau \equiv 0,$$

which gives $\|\nabla u_t\|^2 + \|u_t\|_{k+1}^{k+1} = 0$ for $0 \le t < T_{\max}$. Hence, we have

$$\frac{du}{dt} = \frac{d\nabla u}{dt} = 0, \quad x \in \Omega, \quad t_0 \in (0, T_{\max}),$$

which gives $u(t) \equiv u_0$, $\nabla u(t) = \nabla u_0$, i.e., $u(t)$ is a steady-state solution of problem (1.1)–(1.3). □

Next, we prove the asymptotic behavior for the case $E(0) = d$ by Theorems 4.1, 5.1, and Lemma 5.1.

Theorem 5.2. (Asymptotic behavior when $E(0) = d$). *Under the assumptions of* Theorem 5.1, *for the global solution* (*but not steady-state solution*) *to problem* (1.1)–(1.3), *we have*

$$E(t) \le Ce^{-kt}, \quad 0 \le t < \infty \tag{5.3}$$

for some positive constants C and k.

Proof. First, Theorem 5.1 gives the existence of the global solution $u(t)$ to problem (1.1)–(1.3) for the case $E(0) = d$. Furthermore, from Lemma 5.1, it follows that there exists a $t_0 \in (0, T_{\max})$ such that

$$\int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{k+1}^{k+1}) d\tau > 0.$$

Hence, from (2.20), it follows that

$$E(t_0) = E(0) - \int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{k+1}^{k+1}) d\tau = d - \int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{k+1}^{k+1}) d\tau < d.$$

From Theorem 5.1, we have $u(t_0) \in W$. Thus, Theorem 4.2 directly gives

$$E(t) \le Ke^{-k(t-t_0)} = Ke^{kt_0}e^{-kt}, \quad 0 \le t < \infty,$$

then

$$E(t) \le Ce^{-kt},$$

where $C = Ke^{kt_0}$. □

5.3 Global nonexistence for the critical initial energy level $E(0) = d$

In this section, to prove the finite time blowup result for the case $E(0) = d$, we give the invariance of the unstable set V under the flow of problem (1.1)–(1.3) first. Regarding the finite time blowup for the critical initial energy case, it seems impossible to approach from the inside of the potential well, so we need to retrieve the corresponding invariant method by decaying the energy.

Lemma 5.2. *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$ be given functions. Then all solutions to problem* (1.1)–(1.3) *with $E(0) = d$ belong to V, provided $u_0 \in V$.*

Proof. Let u be any weak solution to problem (1.1)–(1.3) with $E(0) = d$, $u_0 \in V$ and T be the maximum existence time of u. We prove that $(u, v) \in V$ for $0 < t < T$. Arguing by contradiction, we suppose that there exists a first time $t_0 \in (0, T)$ such that $I(u(t_0)) = 0$ and $I(u) < 0$ for $0 \le t < t_0$. By the definition of d, we obtain $J(u(t_0)) \ge d$. Recalling (2.20), (2.3), and (2.4), we have

$$\frac{1}{2}\|u_t\|^2 + J(u) + \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|_{r+1}^{r+1})\mathrm{d}\tau = E(t) + \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|_{r+1}^{r+1})\mathrm{d}\tau = E(0) = d.$$

Therefore, we obtain

$$2\|u_t(t_0)\|^2 + \int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{r+1}^{r+1})\mathrm{d}\tau = 0,$$

which implies $\frac{\mathrm{d}u}{\mathrm{d}t} = 0$ for $x \in \Omega$, $0 \le t \le t_0$ and $u(x, t) = u_0(x)$. Hence, we have $I(u(t_0)) = I(u_0) > 0$, which contradicts $I(u(t_0)) = 0$. Hence, we complete this lemma. □

Next, we display the finite time blowup result at the critical initial energy level $E(0) = d$.

Theorem 5.3. (Global nonexistence when $E(0) = d$). *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$ be given functions. Assume that $E(0) = d$ and $u_0 \in V$. Then the solution to problem* (1.1)–(1.3) *blows up in finite time.*

Proof. First, Theorem 3.1 gives the existence and uniqueness of the local solution u. We prove that if u is not a steady-state solution to problem (1.1)–(1.3), which can be excluded by the aforementioned arguments, then $T_{\max} < \infty$, where $T_{\max}$ is the maximum existence time of u. In fact, from Lemma 5.1, it follows that there exists a $t_0 \in (0, T_{\max})$ such that

$$\int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{r+1}^{r+1})\mathrm{d}\tau > 0.$$

From (2.20) and $E(0) = d$, we obtain

$$E(t_0) = d - \int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|_{r+1}^{r+1})\mathrm{d}\tau < d.$$

In addition, from Lemma 5.2, it follows $u(t_0) \in V$. So from Theorem 4.3, it follows that the maximum existence time of $u(t)$ is finite. Hence, we complete this theorem. □

6 Global nonexistence of solutions for the high arbitrarily initial energy $E(0) > 0$ when $r = 1$

In the following, we show the global nonexistence of solution to problem (1.1)–(1.3) with strong and linear weak damping ($r = 1$) at the arbitrarily high initial energy level $E(0) > 0$.

First, we present the following two lemmas to prove the global nonexistence stated in Theorem 6.1.

Lemma 6.1. (Increasing map). *Let $u_0 \in H$, $u_1 \in H_0^1(\Omega)$, $r = 1$ and u be the solution to problem* (1.1)–(1.3). *Then the function*

$$B(t) := \|\nabla u\|^2 + \|u\|^2 + 2(u, u_t)$$

is strictly increasing as long as $I(t) - \|u_t\|^2 < 0$.

Proof. Noting

$$B'(t) = 2(\nabla u, \nabla u_t) + 2(u, u_t) + 2\langle u, u_{tt}\rangle + 2\|u_t\|^2.$$

Testing equation (1.1) by u, we obtain

$$\langle u, u_{tt}\rangle + (\nabla u, \nabla u_t) + (u, u_t) = -\|\Delta u\|^2 - M(\|\nabla u\|^2)\|\nabla u\|^2 + \|u\|_{p+1}^{p+1}, \tag{6.1}$$

then

$$B'(t) = 2\|u_t\|^2 - 2(\|\Delta u\|^2 + M(\|\nabla u\|^2)\|\nabla u\|^2 - \|u\|_{p+1}^{p+1}) = 2\|u_t\|^2 - 2I(t).$$

Hence, by $I(t) < \|u_t\|^2$, we have $B'(t) > 0$. Therefore, we can obtain that the function $B(t) = \|\nabla u\|^2 + \|u\|^2 + 2(u, u_t)$ is strictly increasing. □

Lemma 6.2. (Invariant sign of $I(t) - \|u_t\|^2$). *Let $u_0 \in H$, $u_1 \in H_0^1(\Omega)$ and $r = 1$ hold. Assume that the initial data satisfy*

$$\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) > \frac{4(p+1)}{(p-1)\tilde{c}}E(0), \tag{6.2}$$

where $\tilde{c} = \min\{1, C\} > 0$ and C is the optimal Sobolev constant from H into $H_0^1(\Omega)$. Then all solutions to problem (1.1)–(1.3) *with $E(0) > 0$ satisfy $I(t) - \|u_t\|^2 < 0$, provided $I(0) - \|u_1\|^2 < 0$ for all $t \in (0, T_{\max})$, where $T_{\max}$ is the maximum existence time.*

Proof. We prove $I(t) - \|u_t\|^2 < 0$ for all $t \in (0, T_{\max})$, where $T_{\max}$ is the maximum existence time of u. Arguing by contradiction, by the continuity of $I(t) - \|u_t\|^2$ in t, we suppose that $t_0 \in (0, T_{\max})$ is the first time such that

$$I(t_0) - \|u_t(t_0)\|^2 = 0 \tag{6.3}$$

and

$$I(t) - \|u_t\|^2 < 0, \quad t \in [0, t_0).$$

From Lemma 6.1, we obtain that function $\|\nabla u\|^2 + \|u\|^2 + 2(u, u_t)$ is strictly increasing on the interval $[0, t_0)$, which together with (6.2) gives

$$\|\nabla u\|^2 + \|u\|^2 + 2(u, u_t) > \|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) > \frac{4(p+1)}{(p-1)\tilde{c}}E(0), \quad t \in [0, t_0).$$

Moreover, from the continuity of $u(t)$ and $u_t(t)$ in t, we can obtain

$$\|\nabla u(t_0)\|^2 + \|u(t_0)\|^2 + 2(u(t_0), u_t(t_0)) > \frac{4(p+1)}{(p-1)\tilde{c}}E(0). \tag{6.4}$$

On the other hand, by (2.19), (2.6), and (6.3), we can obtain

$$\begin{aligned}
E(0) &= E(t_0) + \int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\mathrm{d}\tau \\
&= \frac{1}{2}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|u(t_0)\|_H^2 + \frac{1}{p+1}I(t_0) \\
&\quad + \frac{p-2\gamma-1}{2(\gamma+1)(p+1)}\beta\|\nabla u(t_0)\|^{2\gamma+2} + \int_0^{t_0} (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\mathrm{d}\tau \\
&\geq \frac{1}{2}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|u(t_0)\|_H^2 + \frac{1}{p+1}I(t_0) \\
&\geq \frac{p+3}{2(p+1)}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|u(t_0)\|_H^2 \\
&\geq \frac{p-1}{2(p+1)}(\|u_t(t_0)\|^2 + \|u(t_0)\|_H^2).
\end{aligned} \tag{6.5}$$

Using Sobolev, Young, and Cauchy-Schwarz inequalities, we have

$$\begin{aligned}\|u_t(t_0)\|^2 + \|u(t_0)\|_H^2 &\geq \|u_t(t_0)\|^2 + C(\|\nabla u(t_0)\|^2 + \|u(t_0)\|^2)\\ &\geq \tilde{c}(\|u_t(t_0)\|^2 + \|\nabla u(t_0)\|^2 + \|u(t_0)\|^2)\\ &\geq \frac{\tilde{c}}{2}(\|u_t(t_0)\|^2 + \|\nabla u(t_0)\|^2 + 2\|u(t_0)\|^2)\\ &\geq \frac{\tilde{c}}{2}(2\|u(t_0)\|\|u_t(t_0)\| + \|\nabla u(t_0)\|^2 + \|u(t_0)\|^2)\\ &\geq \frac{\tilde{c}}{2}(2(u(t_0), u_t(t_0)) + \|\nabla u(t_0)\|^2 + \|u(t_0)\|^2).\end{aligned} \tag{6.6}$$

Then by (6.5) and (6.6), we obtain

$$E(0) > \frac{(p-1)\tilde{c}}{4(p+1)}(2(u(t_0), u_t(t_0)) + \|\nabla u(t_0)\|^2 + \|u(t_0)\|^2), \tag{6.7}$$

which contradicts (6.4). Hence, this lemma is proved. □

Next, we show the blowup result with arbitrarily positive initial energy $E(0) > 0$.

Theorem 6.1. (Global nonexistence when $E(0) > 0$ and $r = 1$). *Let $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$ hold. Assume that $E(0) > 0$, $I(u_0) - \|u_1\|^2 < 0$ and (6.2) hold. Then the solution to problem (1.1)–(1.3) with $r = 1$ blows up in finite time.*

Proof. Suppose by contradiction that the solution $u(x, t)$ is global. Then for any $T_0 > 0$, we define the auxiliary function $F(t)$ as (4.59) mentioned earlier, then we have (4.60)–(4.64). By (4.62) and (4.64), we obtain

$$F''(t)F(t) - \frac{\alpha+3}{4}(F'(t))^2 \geq F(t)\left(F''(t) - (\alpha+3)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau\right)\right) = F(t)\xi_2(t),$$

where

$$\xi_2(t) := 2\|u_t\|^2 - 2I(u) - (\alpha+3)\left(\|u_t\|^2 + \int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau\right), \tag{6.8}$$

and the constant $\alpha > 0$ will be chosen later. To proceed the estimation of function $\xi_2(t)$, we first observe from (2.6) that

$$-2I(u) = (p+1)\|u_t\|^2 + (p-1)\|u\|_H^2 + \frac{p-2\gamma-1}{\gamma+1}\beta\|\nabla u\|^{2\gamma+2} + 2(p+1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau - 2(p+1)E(0). \tag{6.9}$$

Substituting (6.9) into $\xi_2(t)$, we obtain

$$\begin{aligned}\xi_2(t) = (p-\alpha)\|u_t\|^2 + (p-1)\|u\|_H^2 + \frac{p-2\gamma-1}{\gamma+1}\beta\|\nabla u\|^{2\gamma+2} + (2p-1-\alpha)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau\\ - 2(p+1)E(0).\end{aligned} \tag{6.10}$$

Set $\alpha = \frac{p+1}{2}$, since $p > 1$, then we have $p > \alpha > 1$. Thus, (6.10) becomes

$$\xi_2(t) = \frac{p-1}{2}\|u_t\|^2 + (p-1)\|u\|_H^2 - 2(p+1)E(0) + \frac{p-2\gamma-1}{\gamma+1}\beta\|\nabla u\|^{2\gamma+2} + \frac{3}{2}(p-1)\int_0^t \|u_\tau(\tau)\|_{H_0^1}^2 \mathrm{d}\tau,$$

which together with $p > 2\gamma + 1 > 1$ and the Cauchy-Schwarz inequality gives

$$\begin{aligned}\xi_2(t) &\geq \frac{p-1}{2}\|u_t\|^2 + (p-1)\|u\|_H^2 - 2(p+1)E(0)\\ &\geq \frac{p-1}{2}(\|u_t\|^2 + 2C(\|\nabla u\|^2 + \|u\|^2)) - 2(p+1)E(0)\\ &\geq \frac{p-1}{2}(\|u_t\|^2 + C(\|\nabla u\|^2 + 2\|u\|^2)) - 2(p+1)E(0)\\ &\geq \frac{(p-1)\tilde{c}}{2}(\|u_t\|^2 + \|\nabla u\|^2 + 2\|u\|^2) - 2(p+1)E(0)\\ &\geq \frac{(p-1)\tilde{c}}{2}(2(u,u_t) + \|\nabla u\|^2 + \|u\|^2) - 2(p+1)E(0).\end{aligned}$$

Then by Lemmas 6.2, 6.1 and (6.2), we have

$$\begin{aligned}\xi_2(t) &\geq \frac{(p-1)\tilde{c}}{2}(2(u,u_t) + \|\nabla u\|^2 + \|u\|^2) - 2(p+1)E(0)\\ &\geq \frac{(p-1)\tilde{c}}{2}(2(u_0,u_1) + \|\nabla u_0\|^2 + \|u_0\|^2) - 2(p+1)E(0) > 0,\end{aligned}$$

which means there exists a constant $\sigma_2 > 0$ such that

$$\xi_2(t) \geq \sigma_2 > 0.$$

Hence,

$$F''(t)F(t) - \frac{p+3}{8}F'(t)^2 \geq \rho\sigma_2 > 0, \quad t \in [0, T_0].$$

Letting $y(t) := F(t)^{-\frac{p-1}{8}}$, then

$$y''(t) \leq -\frac{p-1}{8}\rho\sigma_2 y(t)^{\frac{p+13}{p-3}}, \quad t \in [0, T_0],$$

which tells

$$\lim_{t\to T_*} y(t) = 0.$$

It proves that $y(t)$ reaches 0 in finite time, say as $t \to T_*$. Since T_* is independent of the initial choice of T_0, we may assume that $T_* < T_0$. In other words, $\lim_{t\to T_*}F(t) = +\infty$. □

Remark 6.1. Here, we give an example to show that the set of the initial data satisfying the following condition (Theorem 6.1):
(i) $E(0) > 0$;
(ii) $I(0) - \|u_1\|^2 < 0$;
(iii) $\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) > \frac{4(p+1)}{(p-1)\tilde{c}}E(0)$,

is not empty.

For the initial data $u_0 \in H$ and $u_1 \in H_0^1(\Omega)$, we let

$$u_0 = \lambda\phi, \ \ \phi \in H$$

and

$$u_1 = \mu\varphi, \ \ \varphi \in H_0^1(\Omega),$$

where λ and μ are positive constants satisfying certain conditions to be stated later.

Taking any $\phi \in H$ and $\varphi \in H_0^1(\Omega)$ such that $(\phi, \varphi) > 0$ and letting $(u_0, u_1) = \lambda\mu(\phi, \varphi) > 0$ for any constants $\lambda > 0$ and $\mu > 0$, we show that we can choose suitably positive constants λ and μ such that u_0 and u_1 conform the conditions (i)–(iii).

Note that $\lambda^2 < \lambda^{2\gamma+2} < \lambda^{p+1}$ for $\lambda > 1$, hence, we can choose λ large enough such that

$$J(0) = \frac{1}{2}\lambda^2\|\phi\|_H^2 + \frac{\beta}{2\gamma+2}\lambda^{2\gamma+2}\|\nabla\phi\|^{2\gamma+2} - \frac{1}{p+1}\lambda^{p+1}\|\phi\|_{p+1}^{p+1} < 0 \tag{6.11}$$

and

$$I(0) = \lambda^2\|\phi\|_H^2 + \beta\lambda^{2\gamma+2}\|\nabla\phi\|^{2\gamma+2} - \lambda^{p+1}\|\phi\|_{p+1}^{p+1} < 0. \tag{6.12}$$

Once λ is fixed, then by (6.11), we can choose constant $\mu > 0$ such that

$$0 < -J(0) < \frac{1}{2}\mu^2\|\varphi\|^2 < \frac{(p-1)\tilde{c}}{4(p+1)}(\lambda^2\|\nabla\phi\|^2 + \lambda^2\|\phi\|^2) - J(0). \tag{6.13}$$

For condition (i), from (6.13), we see that

$$E(0) = \frac{1}{2}\mu^2\|\varphi\|^2 + J(0) > 0.$$

For condition (ii), from (6.12), for the aforementioned μ, we have

$$I(0) < 0 < \mu^2\|\varphi\|^2,$$

which tells that $I(0) - \|u_1\|^2 < 0$.

For condition (iii), from (6.13), we obtain

$$\lambda^2\|\nabla\phi\|^2 + \lambda^2\|\phi\|^2 > \frac{4(p+1)}{(p-1)\tilde{c}}\left(\frac{1}{2}\mu^2\|\varphi\|^2 + J(0)\right) = \frac{4(p+1)}{(p-1)\tilde{c}}E(0). \tag{6.14}$$

Then, from (6.14) and $\lambda\mu(\phi,\varphi) > 0$, we have

$$\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) = \lambda^2\|\nabla\phi\|^2 + \lambda^2\|\phi\|^2 + 2\lambda\mu(\phi,\varphi) > \lambda^2\|\nabla\phi\|^2 + \lambda^2\|\phi\|^2 > \frac{4(p+1)}{(p-1)\tilde{c}}E(0).$$

So this example shows one of such initial data satisfying the conditions required in Theorem 6.1, which makes the solution blow up.

Open problems. For the high energy case, i.e., $E(0) > 0$, we only established global nonexistence for the linear damping case, and hence, the problem with nonlinear damping term like $|u_t|^{r-1}u_t$, $r > 1$ is still open. Moreover, global existence and asymptotic behavior of solution for the high initial energy level are also open problems.

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References

[1] J. M. Ball, *Initial-boundary value problems for an extensible beam*, J. Math. Anal. Appl. **42** (1973), no. 1, 61–90.

[2] J. M. Ball, *Stability theory for an extensible beam*, J. Differential Equations **14** (1973), no. 3, 399–418.

[3] D. Bainov and E. Minchev, *Upper estimate of the interval of existence of solutions of a nonlinear Timoshenko equation*, Georgian Math. J. **4** (1997), no. 3, 219–222.

[4] H. M. Berger, *A new approach to the analysis of large deflections of plates*, J. Appl. Mech. **22** (1955), no. 4, 465–472.
[5] A. Benaissa and S. A. Messaoudi, *Exponential decay of solutions of a nonlinearly damped wave equation*, NoDEA Nonlinear Differential Equations Appl. **12** (2005), no. 4, 391–399.
[6] A. C. Biazutti and H. R. Crippa, *Global attractor and inertial set for the beam equation*, Anal. Appl. **55** (1994), no. 1–2, 61–78.
[7] P. Biler, *Remark on the decay for damped string and beam equations*, Nonlinear Anal. **10** (1986), no. 9, 839–842.
[8] D. Burgreen, *Free vibrations of a pin-ended column with constant distance between pin ends*, J. Appl. Mech. **18** (1951), no. 2, 135–139.
[9] M. M. Cavalcanti, V. N. DomingosCavalcanti, and J. A. Soriano, *Global existence and asymptotic stability for the nonlinear and generalized damped extensible plate equation*, Commun. Contemp. Math. **6** (2004), no. 5, 705–731.
[10] H. Chen and G. Liu, *Global existence, uniform decay and exponential growth for a class of semi-linear wave equation with strong damping*, Acta Math. Sci. Ser. B Engl. Ed. **33** (2013), no. 1, 41–58.
[11] Y. S. Cheng and Y. K. Cheung, *Vibration analysis of bridges under moving vehicles and trains: An overview*, Prog. Struct. Engng. Mater. **3** (2001), 299–304.
[12] I. Chueshov and I. Lasiecka, *Long-time behavior of second order evolution equations with nonlinear damping*, Mem. Amer. Math. Soc. **195** (2008), no. 912, 8.
[13] M. Conti, S. Terracini, and G. Verzini, *A variational problem for the spatial segregation of reaction-diffusion systems*, Indiana Univ. Math. J. **54** (2005), no. 3, 779–816.
[14] M. Conti, S. Terracini, and G. Verzini, *An optimal partition problem related to nonlinear eigenvalues*, J. Funct. Anal. **198** (2003), no. 1, 160–196.
[15] M. Conti, S. Terracini, and G. Verzini, *Neharias problem and competing species systems*, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), no. 6, 871–888.
[16] M. A. J. DaSilva and V. Narciso, *Attractors and their properties for a class of nonlocal extensible beams*, Discrete Contin. Dyn. Syst. **35** (2015), no. 3, 985–1008.
[17] E. H. de Brito, *Decay estimates for the generalized damped extensible string and beam equations*, Nonlinear Anal. **8** (1984), no. 12, 1489–1496.
[18] J. G. Eisley, *Nonlinear vibration of beams and rectangular plates*, Z. Angew. Math. Phys. **15** (1964), no. 2, 167–175.
[19] J. A. Esquivel-Avila, *Dynamics around the ground state of a nonlinear evolution equation*, Nonlinear Anal. **63** (2005), no. 5–7, 331–343.
[20] W. Fang and J. A. Wickert, *Post-buckling of micromachined beams*, J. Micromach. Microeng. **4** (1994), 116–122.
[21] F. Gazzola and M. Squassina, *Global solutions and finite time blow-up for damped semilinear wave equations*. Ann. Inst. H. Poincaré Anal. Non Linéaire **23** (2006), no. 2, 185–207.
[22] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations **109** (1994), no. 2, 295–308.
[23] M. Guedda and H. V. Labani, *Nonexistence of global solutions to a class of nonlinear wave equations with dynamic boundary conditions*, Bull. Belg. Math. Soc. Simon Stevin **9** (2002), no. 1, 39–46.
[24] G. Kirchhoff, *Vorlesungenüber Mathematische Physik: Mechanik*, Teubner, Leipzig, 1883.
[25] H. A. Levine, *Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au. F(u)$*, Trans. Amer. Math. Soc. **192** (1974), 1–21.
[26] H. A. Levine, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*, SIAM J. Math. Anal. **5** (1974), no. 1, 138–146.
[27] N. Lobontiu and E. Garcia, *Mechanics of microelectromechanical systems*, Kluwer, New York, 2005.
[28] W. Lian and R. Xu, *Global well-posedness of nonlinear wave equation with weak and strong damping terms and logarithmic source term*, Adv. Nonlinear Anal. **9** (2020), no. 1, 613–632.
[29] W. Lian, R. Xu, V. D. Radulescu, Y. Yang, and N. Zhao, *Global well-posedness for a class of fourth order nonlinear strongly damped wave equations*, Adv. Calc. Var. **14** (2021), no. 4, 589–611.
[30] J. L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problem and Applications I*, Springer-Verleag, 143 New York-Heidelberg-Berlin, 1972.
[31] Y. Liu and R. Xu, *Fourth order wave equations with nonlinear strain and source terms*, J. Math. Anal. Appl. **331** (2007), no. 1, 585–607.
[32] Y. Liu and J. Zhao, *On potential wells and applications to semilinear hyperbolic equations and parabolic equations*, Nonlinear Anal. **64** (2006), no. 12, 2665–2687.
[33] T. F. Ma and V. Narciso, *Global attractor for a model of extensible beam with nonlinear damping and source terms*, Nonlinear Anal. **73** (2010), no. 10, 3402–3412.
[34] L. A. Medeiros, *On a new class of nonlinear wave equations*, J. Math. Anal. Appl. **69** (1979), no. 1, 252–262.
[35] K. Nishihara, *Decay properties of solutions of some quasilinear hyperbolic equations with strong damping*, Nonlinear Anal. **21** (1993), no. 1, 17–21.
[36] K. Ono, *On decay properties of solutions for degenerate strongly damped wave equations of Kirchhoff type*, J. Math. Anal. Appl. **381** (2011), no. 1, 229–239.

[37] K. Ono, *Blowing up and global existence of solutions for some degenerate nonlinear wave equations with some dissipation*, Nonlinear Anal. **30** (1997), no. 7, 4449–4457.

[38] K. Ono, *On global existence, asymptotic stability and blowing up of solutions for some degenerate non-linear wave equations of Kirchhoff type with a strong dissipation*, Math. Methods Appl. Sci. **20** (1997), no. 2, 151–177.

[39] K. Ono, *On global solutions and blow-up of solutions of nonlinear Kirchhoff string with nonlinear dissipation*, J. Math. Anal. Appl. **216** (1997), no. 1, 321–342.

[40] S. K. Patcheu, *On a global solution and asymptotic behaviour for the generalized damped extensible beam equation*, J. Differential Equations **135** (1997), no. 2, 229–314.

[41] J. Y. Park and J. J. Bae, *On the existence of solutions of strongly damped nonlinear wave equations*, Int. J. Math. **23** (2000), no. 6, 369–382.

[42] L. E. Payne and D. H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), no. 3-4, 273–303.

[43] N. S. Papageorgiou, V. D. Rădulescu, and D. D. Repovs, *Nonlinear Analysis - Theory and Methods, Springer Monographs in Mathematics*, Springer, Cham, 2019.

[44] N. S. Papageorgiou and V. D. Rădulescu, *Multiple solutions with precise sign for nonlinear parametric Robin problems*, J. Differential Equations **256** (2014), no. 7, 2449–2479.

[45] N. S. Papageorgiou and V. D. Rădulescu, *Multiplicity of solutions for resonant Neumann problems with an indefinite and unbounded potential*, Trans. Amer. Math. Soc. **367** (2015), no. 12, 8723–8756.

[46] T. Taniguchi, *Existence and asymptotic behaviour of solutions to weakly damped wave equations of Kirchhoff type with nonlinear damping and source terms*, J. Math. Anal. Appl. **361** (2010), no. 2, 566–578.

[47] S. Terracini and G. Verzini, *Oscillating solutions to second-order ODEs with indefinite superlinear nonlinearities*, Nonlinearity **13** (2000), no. 5, 1501–1514.

[48] G. Todorova, *Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms*, J. Math. Anal. Appl. **239** (1999), no. 2, 213–226.

[49] S. Woinowsky-Krieger, *The effect of axial force on the vibration of hinged bars*, J. Appl. Mech. **17** (1950), no. 1, 35–36.

[50] S. Wu and L. Tai, *Blow up of solutions for some non-linear wave equations of Kirchhoff type with some dissipation*, Nonlinear Anal. **65** (2006), no. 2, 243–264.

[51] S. Wu and L. Tsai, *Existence and nonexistence of global solutions for a nonlinear wave equation*, Taiwanese J. Math. **13** (2009), no. 6B, 2069–2091.

[52] R. Xu, *Initial boundary value problem for semilinear hyperbolic equations and parabolic equations with critical initial data*, Quart. Appl. Math. **68** (2010), no. 3, 459–468.

[53] R. Xu, X. Wang, Y. Yang, and S. Chen, *Global solutions and finite time blow-up for fourth order nonlinear damped wave equation*, J. Math. Phys. **56** (2018), no. 6, 061503.

[54] R. Xu, M. Zhang, S. Chen, Y. Yang, and J. Shen, *The initial-boundary value problems for a class of six order nonlinear wave equation*, Discrete Contin. Dyn. Syst. **37** (2017), no. 11, 5631–5649.

[55] R. Xu and J. Su, *Global existence and finite time blow-up for a class of semilinear pseudo-parabolic equations*, J. Funct. Anal. **264** (2013), no. 12, 2732–2763.

[56] R. Xu, W. Lian, and Y. Niu, *Global well-posedness of coupled parabolic systems*. Sci. China Math. **63** (2020), no. 2, 321–356.

[57] Z. Yang, *On an extensible beam equation with nonlinear damping and source terms*, J. Differential Equations **254** (2013), no. 9, 3903–3927.

[58] Z. Yang and X. Li, *Finite-dimensional attractors for the Kirchhoff equation with a strong dissipation*, J. Math. Anal. Appl. **375** (2011), no. 2, 579–593.

[59] Z. Yang and P. Ding, *Longtime dynamics of the Kirchhoff equation with strong damping and critical nonlinearity on $\mathbb{R}^N$*, J. Math. Anal. Appl. **434** (2016), no. 2, 1826–1851.

[60] M. I. Younis and A. H. Nayfeh, *A study of the nonlinear response of a resonant microbeam to an electric actuation*, Nonlinear Dynam. **31** (2003), no. 1, 91–117.

[61] Q. Zhang and V. D. Rădulescu, *Double phase anisotropic variational problems and combined effects of reaction and absorption terms*, J. Math. Pures Appl. **118** (2018), no. 9, 159–203.

[62] J. Zhang and S. H. Zhu, *Sharp energy criteria and singularity of blow-up solutions for the Davey-Stewartson system*, Commun. Math. Sci. **17** (2019), no. 3, 653–667.

Appendix

In this section, by following the ideas in [42], we prove that there exists an extremal of the variational problem (2.9), which was used to prove Lemma 2.3.

Proof. Let $\{u_n\}$ be a minimizing sequence, then

$$u_n \in H, \quad \|u_n\|_H \neq 0, \quad I(u_n) = 0, \quad \lim_{n\to\infty} J(u_n) = d.$$

The space H, the norm $\|\cdot\|_H$, and the functionals J, I are defined in (2.1), (2.2), (2.4), and (2.5), respectively. From (2.17), we obtain that $\{u_n\}$ is bounded in H. Therefore, there exists a subsequence that converges strongly in $L^{p+1}(\Omega)$, where p satisfies (1.5). We still use the notation $\{u_n\}$ to represent the subsequence and denote its limit function by w. Hence, we have

$$\lim_{n\to\infty} \|u_n - w\|_{p+1} = 0,$$

and

$$\|w\|_H \geq \varliminf_{n\to\infty} \|u_n\|_H = \lim_{n\to\infty} \|u_n\|_H.$$

From (2.14) and (2.15), we know that

$$\|w\|_{p+1} = \lim_{n\to\infty} \|u_n\|_{p+1} \geq \left(\frac{1}{C^{p+1}}\right)^{\frac{2}{p^2-1}}.$$

Therefore, $\|w\|_{p+1} \neq 0$.

If $\|w\|_H = \lim_{n\to\infty}\|u_n\|_H$, then $I(w) = 0$ and $J(w) = d$, and we obtain the existence of the extremal of (2.9).

If $\|w\|_H < \lim_{n\to\infty}\|u_n\|_H$, then $I(w) < 0$ and $J(w) < d$, and we will prove this case is not true. By (iv) of Lemma 2.2, for sufficient small λ, we have $I(\lambda u) > 0$ for any $u \in H,\ u \neq 0$, and then there exists a $0 < \bar{\lambda} < 1$ such that $I(\bar{\lambda} w) = 0$. For this choice of $\bar{\lambda}$, we obtain

$$\begin{aligned}
J(\bar{\lambda} w) &= \frac{\bar{\lambda}^2}{2}\|w\|_H^2 + \frac{\beta\bar{\lambda}^{2\gamma+2}}{2\gamma+2}\|\nabla w\|^{2\gamma+2} - \int_\Omega F(\bar{\lambda} w)\mathrm{d}x \\
&< \frac{\bar{\lambda}^2}{2}\lim_{n\to\infty}\|u_n\|_H^2 + \frac{\beta\bar{\lambda}^{2\gamma+2}}{2\gamma+2}\|\nabla w\|^{2\gamma+2} - \int_\Omega F(\bar{\lambda} w)\mathrm{d}x \\
&= \frac{\bar{\lambda}^2}{2}\lim_{n\to\infty}\|u_n\|_H^2 + \frac{\beta\bar{\lambda}^{2\gamma+2}}{2\gamma+2}\|\nabla w\|^{2\gamma+2} - \int_\Omega F(\bar{\lambda} w)\mathrm{d}x + d - \lim_{n\to\infty} J(u_n) \\
&< d - \left(\frac{1-\bar{\lambda}^{2\gamma+2}}{2\gamma+2} + \frac{\bar{\lambda}^2 - 1}{2}\right)\beta\|\nabla w\|^{2\gamma+2} + \int_\Omega \left(F(w) - F(\bar{\lambda} w) - \frac{1}{2}(1-\bar{\lambda}^2) w f(w)\right)\mathrm{d}x,
\end{aligned}$$

where $f(w) = |w|^{p-1}w$, $F(w) = \int_0^w f(s)\mathrm{d}s$, β and γ satisfy (1.4) and (1.5). Let

$$Q(\lambda) := -\left(\frac{1-\lambda^{2\gamma+2}}{2\gamma+2} + \frac{\lambda^2-1}{2}\right)\beta\|\nabla w\|^{2\gamma+2} + \int_\Omega \left(F(w) - F(\lambda w) - \frac{1}{2}(1-\lambda^2) w f(w)\right)\mathrm{d}x. \tag{A1}$$

For $\lambda = 1$, we obtain

$$Q(1) = 0.$$

For $\lambda = 0$, we have

$$Q(0) = \left(\frac{1}{2} - \frac{1}{2\gamma+1}\right)\beta\|\nabla w\|^{2\gamma+2} - \left(\frac{1}{2} - \frac{1}{p+1}\right)\|w\|_{p+1}^{p+1}.$$

From $I(w) < 0$, we know that

$$\beta\|\nabla w\|^{2\gamma+2} < \|w\|_{p+1}^{p+1}, \tag{A2}$$

which combining (1.5) gives

$$Q(0) < 0.$$

In addition,

$$Q'(\lambda) = \beta\|\nabla w\|^{2\gamma+2}(\lambda^{2\gamma+1} - \lambda) + (\lambda - \lambda^p)\|w\|_{p+1}^{p+1},$$

which combining (A2) and (1.5) gives

$$Q'(\lambda) > 0 \quad \text{for } 0 < \lambda < 1.$$

Hence,

$$Q(\lambda) < 0 \quad \text{for } 0 < \lambda < 1.$$

Then, we obtain,

$$J(\bar{\lambda}w) < d \quad \text{while } I(\bar{\lambda}w) = 0,$$

which contradicts the definition of the potential well depth d, i.e., (2.9). The proof is completed. □