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Mingqi Xiang, Binlin Zhang*, and Vicenţiu D. Rădulescu

Superlinear Schrödinger–Kirchhoff type problems involving the fractional p–Laplacian and critical exponent

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Abstract: This paper concerns the existence and multiplicity of solutions for the Schrődinger–Kirchhoff type problems involving the fractional p–Laplacian and critical exponent. As a particular case, we study the following degenerate Kirchhoff-type nonlocal problem:

$$\|u\|_{\lambda}^{(\theta-1)p}[\lambda(-\Delta)_p^s u + V(x)|u|^{p-2}u] = |u|^{p_s^\star-2}u + f(x,u) \text{ in } \mathbb{R}^N,$$
$$\|u\|_\lambda = \left(\lambda \iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}dxdy + \int\limits_{\mathbb{R}^N} V(x)|u|^p dx\right)^{1/p}$$

where $(-\Delta)_p^s$ is the fractional p–Laplacian with $0 < s < 1 < p < N/s$, $p_s^\star = Np/(N-ps)$ is the critical fractional Sobolev exponent, $\lambda > 0$ is a real parameter, $1 < \theta \leq p_s^\star/p$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying superlinear growth conditions. For $\theta \in (1, p_s^\star/p)$, by using the concentration compactness principle in fractional Sobolev spaces, we show that if $f(x,t)$ is odd with respect to t, for any $m \in \mathbb{N}^+$ there exists a $\Lambda_m > 0$ such that the above problem has m pairs of solutions for all $\lambda \in (0, \Lambda_m]$. For $\theta = p_s^\star/p$, by using Krasnoselskii's genus theory, we get the existence of infinitely many solutions for the above problem for λ large enough. The main features, as well as the main difficulties, of this paper are the facts that the Kirchhoff function is zero at zero and the potential function satisfies the critical frequency $\inf_{x\in\mathbb{R}} V(x) = 0$. In particular, we also consider that the Kirchhoff term satisfies the critical assumption and the nonlinear term satisfies critical and superlinear growth conditions. To the best of our knowledge, our results are new even in p–Laplacian case.

Keywords: Schrödinger–Kirchhoff problem; Fractional p–Laplacian; Multiple solutions; Critical exponent; Principle of concentration compactness

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Mingqi Xiang, College of Science, Civil Aviation University of China, Tianjin, 300300, P.R. China,
E-mail: xiangmingqi_hit@163.com
***Corresponding Author: Binlin Zhang,** College of Mathematics and System Science, Shandong University of Science and Technology, Qingdao, 266590, P.R. China, E-mail: zhangbinlin2012@163.com
Vicenţiu D. Rădulescu, Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland and Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania,
E-mail: vicentiu.radulescu@imar.ro

1 Introduction

In this article we concern with existence and multiplicity of solutions for critical Kirchhoff–type problems involving the fractional p–Laplacian. More precisely, we consider

$$M(\|u\|_\lambda^p)[\lambda(-\Delta)_p^s u + V(x)|u|^{p-2}u] = |u|^{p_s^\star-2}u + f(x,u) \text{ in } \mathbb{R}^N, \tag{1.1}$$

$$\|u\|_\lambda = \left(\lambda \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u|^p dx\right)^{1/p},$$

where $p_s^\star = Np/(N-sp)$, $N > sp$ with $s \in (0,1)$, $(-\Delta)_p^s$ is the fractional p–Laplacian which (up to normalization factors) may be defined for any $x \in \mathbb{R}^N$ as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\delta\to 0} \int_{\mathbb{R}^N\setminus B_\delta(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dy$$

for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, where $B_\delta(x)$ denotes the ball in $\mathbb{R}^N$ centered at x with radius δ. For a simple introduction about the fractional p–Laplacian, we refer to [1] and the references therein.

Furthermore, we always assume M, V and f satisfy the following assumptions:

(M) $M \in C(\mathbb{R}, \mathbb{R})$ and there exist $\theta \in (1, p_s^\star/p)$ and $0 < m_0 \le m_1$ such that

$$m_0 t^{\theta-1} \le M(t) \le m_1 t^{\theta-1} \text{ for all } t \in \mathbb{R}_0^+;$$

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$, $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$ and there exists a constant $h > 0$ such that the Lebesgue measure of set $V^h = \{x \in \mathbb{R}^N : V(x) < h\}$ is finite; there is $\varrho > 0$ such that $\lim_{|y|\to\infty} \text{meas}(\{x \in B_\varrho(y) : V(x) < c\}) = 0$ for any $c \in \mathbb{R}^+$;

(f_1) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $q \in (\theta p, p_s^\star)$ such that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ and

$$|f(x,\xi)| \le \theta p \varepsilon |\xi|^{\theta p-1} + qC_\varepsilon |\xi|^{q-1} \text{ for a.e. } x \in \mathbb{R}^N \text{ and all } \xi \in \mathbb{R};$$

(f_2) There exists $q_1 > m_1\theta p/m_0$ such that

$$q_1 F(x,\xi) \le f(x,\xi)\xi \text{ for all } (x,\xi) \in \mathbb{R}^N \times \mathbb{R},$$

where $F(x,\xi) = \int_0^\xi f(x,\tau)d\tau$, m_0 and m_1 are the numbers given in (M);

(f_3) There exists $q_2 \in (\theta p, p_s^\star)$ such that $F(x,\xi) \ge a_0|\xi|^{q_2}$ for a.e. $x \in \mathbb{R}^N$ and all $\xi \in \mathbb{R}$.

Note that condition (V_2), which is weaker than the coercivity assumption: $V(x) \to \infty$ as $|x| \to \infty$, was first introduced by Bartsch and Wang in [2] to conquer the lack of compactness.

In the last few years, great attention has been paid to the study of non-local fractional Laplacian problems involving critical nonlinearities. It is worth mentioning that the semilinear Laplace equation of elliptic type involving critical exponent was investigated in the crucial paper of Brézis and Nirenberg [3]. After that, many researchers dedicated to the study of several kinds of elliptic equations with critical growth in bounded domains or in the whole space. For example, by variational techniques, Servadei and Valdinoci [4] showed a Brézis–Nirenberg type result for non-local fractional Laplacian in bounded domains with homogeneous Dirichlet boundary datum, see also [5] for further discussions. In [6], Ros-Oton and Serra considered nonexistence results for nonlocal equations involving critical and supercritical nonlinearities. Autuori and Pucci [7] obtained a multiplicity result for fractional Laplacian problems in $\mathbb{R}^N$ by using the mountain pass theorem and the direct method in variational methods, in which one of two superlinear nonlinearities could be critical or even supercritical.

Indeed, the interest in the study of partial differential equations involving the non-local fractional Laplacian goes beyond the mathematical curiosity. This type of non-local operator comes to real world with many different applications in a quite natural way, such as finance, ultra-relativistic limits of quantum mechanics, materials science, water waves, phase transition phenomena, anomalous diffusion, soft thin films, minimal surfaces and game theory, see for example [1, 8, 9] and the references therein. The literature on fractional Laplace operators and their applications is quite large and interesting, here we just list a few, see [10–12] and the references therein. For the basic properties of fractional Sobolev spaces and the study of fractional Laplacian based on variational methods, we refer the readers to [1, 13]. It is worth pointing out that one of the reasons that forced the rapid expansion of the fractional Laplacian results has been the nonlinear fractional Schrödinger equation, which was proposed by Laskin [14, 15] as a result of expanding the Feynman path integral, from the Brownian–like to the Lévy–like quantum mechanical paths.

In the last decade, the existence and multiplicity of solutions for the Kirchhoff–type elliptic equations with critical exponents have attracted much interest of many scholars. For instance, we refer to [16–18] for the setting of bounded domains; we collect also some articles, see [19–21] for the context set in the whole space. In particular, Fiscella and Valdinoci [22] proposed a stationary Kirchhoff–type equation which models the *nonlocal* aspect of the tension arising from nonlocal measurements of the fractional length of the string. More precisely, they considered a model as follows:

$$\begin{cases} M\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2_s^\star-2}u & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $M(y) = \alpha + \beta y$ for all $y \geq 0$, here $\alpha > 0, \beta \geq 0$. Note that M is this type, problem (1.2) is called non-degenerate if $\alpha > 0$ and $\beta \geq 0$, while it is named degenerate if $\alpha = 0$ and $\beta > 0$, see [23] for some physical motivation about degenerate Kirchhoff problems. For more details about the physical background of the fractional Kirchhoff model, we refer to [22, Appendix A]. Afterwards, the fractional Kirchhoff–type problems have been extensively investigated, for example, we refer to [24–27] for some recent results about non-degenerate Kirchhoff–type problems.

In the following, let us recall some existence results about degenerate Kirchhoff–type fractional p–Laplacian problems. By using the mountain pass theorem and Ekeland's variational principle, Xiang *et al.* [28] obtained the existence of two solutions for a nonhomogeneous Kirchhoff type problem driven by the fractional p–Laplacian, where the nonlinearity is convex-concave, see [26] for related results obtained by the same methods. In [29], Mingqi *et al.* investigated the existence of infinitely many solutions for Kirchhoff type fractional p–Laplacian problems, in which the symmetric mountain pass theorem is applied to study the suplinear case and the Krasnoselskii's genus theory is used to consider the sublinear case. In [23], Pucci *et al.* studied the existence and multiplicity of entire solutions for a class of fractional p–Laplacian problems of Kirchhoff type via variational methods and topological degree theory. In [30], Mingqi *et al.* considered the multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p–Laplacian by using the Nehari manifold method and the symmetric mountain pass theorem. Evidently, the above works did not involve the critical case. For the critical case, with the help of Kajikiya's new version of the symmetric mountain pass lemma, the existence of infinitely many solutions for a critical problem similar to (1.1) is proved in [31], see [32, 33] for more related results.

However, there are few results in the available literature on problems like problem (1.1). In particular, there are no result on the multiplicity of solutions for problem (1.1). There is no doubt that we encounter serious difficulties because of the lack of compactness and of the nonlocal nature of the fractional p–Laplacian. To overcome the loss of compactness, Xiang *et al.* [34] extended the concentration compactness principle of Lions [35] to the setting of fractional p–Laplacian in $\mathbb{R}^N$, and used it to get the existence of solutions for the following critical p–Kirchhoff problem

$$\left[a + b\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy\right)^{(\theta-1)}\right](-\Delta)_p^s u(x) = |u(x)|^{p_s^\star-2}u(x) + \lambda f(x) \text{ in } \mathbb{R}^N,$$

where $a \geq 0$, $b > 0$, $\theta > 1$, $\lambda > 0$ is a parameter and $f \in L^{\frac{p_s^*}{p_s^*-1}}(\mathbb{R}^N)$. In [36], Fiscella and Pucci studied the following p–Kirchhoff problem involving critical Hardy-Sobolev nonlinearity

$$M(\|u\|^p)[(-\Delta)_p^s u + V(x)|u|^{p-2}u] - \frac{|u|^{p_s^*(\alpha)-2}u}{|x|^\alpha} = \lambda f(x,u) + g(x,u) \text{ in } \mathbb{R}^N,$$

where $p_s^*(\alpha) = \frac{(N-\alpha)p}{N-ps}$ is the critical Hardy-Sobolev exponent with $\alpha \in [0, ps)$, f and g are subcritical nonlinear terms, and $V \in C(\mathbb{R}^N, \mathbb{R})$ with $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$. Moreover, the existence of infinitely many solutions for problem (1.1) is investigated, assuming $\inf_{x\in\mathbb{R}^N} V(x) = 0$ and the subcritical nonlinear term f satisfies superlinear growth condition. In [37], Byeon and Wang first studied the asymptotic behavior of positive solutions to Schrodinger equations under the condition $\inf_{x\in\mathbb{R}^N} V(x) = 0$, which is called critical frequency. In [38], Cao and Noussair extended the results of Byeon and Wang, and studied multi-bump standing waves for nonlinear Schrödinger equations. In this paper, we follow the ideas of [39–41]. Although the ideas were used before for other problems, the adaptation of the procedure to our problem is not trivial because of the appearance of degenerate Kirchhoff function and the nonlocal nature of the fractional p–Laplacian. For this, we need more delicate estimates and computations.

To show our main results, we first give some notations. For $\lambda > 0$, let W_λ be the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_\lambda = \left(\lambda[u]_{s,p}^p + \|u\|_{p,V}^p\right)^{1/p},$$

where

$$[u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dxdy\right)^{1/p}$$

and

$$\|u\|_{p,V} = \left(\int_{\mathbb{R}^N} V(x)|u|^p\,dx\right)^{1/p}.$$

Then $(W_\lambda, \|\cdot\|_\lambda)$ is a uniformly convex Banach space, see [26] for the details. Moreover, under the condition (V), for each $\lambda > 0$ the embedding $W_\lambda \hookrightarrow W^{s,p}(\mathbb{R}^N)$ is continuous. Indeed, for each $u \in W_\lambda$, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^p\,dx &\leq \int_{\{\mathbb{R}^N:V(x)\geq h\}} |u|^p\,dx + \int_{\{\mathbb{R}^N:V(x)<h\}} |u|^p\,dx \\
&\leq \frac{1}{h}\int_{\{\mathbb{R}^N:V(x)\geq h\}} V(x)|u|^p\,dx + \left|\{\mathbb{R}^N : V(x) < h\}\right|^{\frac{p_s^*-p}{p_s^*}} \left(\int_{\{\mathbb{R}^N:V(x)<h\}} |u|^{p_s^*}\,dx\right)^{\frac{p}{p_s^*}} \\
&\leq \left(\frac{1}{h} + |\{\mathbb{R}^N : V(x) < h\}|^{\frac{p_s^*-p}{p_s^*}} S^{-1}\lambda^{-1}\right)\|u\|_\lambda^p,
\end{aligned}$$

where $S > 0$ is defined as follows

$$S = \inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}} \frac{[u]_{s,p}^p}{|u|_{p_s^*}^p},$$

where

$$|u|_{p_s^*} = \left(\int_{\mathbb{R}^N} |u|^{p_s^*}\,dx\right)^{1/p_s^*}.$$

Thus,

$$[u]_{s,p}^p + |u|_p^p \le \left(\frac{1}{\lambda} + \frac{1}{h} + |\{\mathbb{R}^N : V(x) < h\}|^{\frac{p_s^\star - p}{p_s^\star}} S^{-1}\lambda^{-1} \right) \|u\|_\lambda^p.$$

From this it follows that the embedding $W_\lambda \hookrightarrow W^{s,p}(\mathbb{R}^N)$ is continuous. Next we give the definition of solutions for problem (1.1).

Definition 1.1. We say that $u \in W_\lambda$ is a (weak) solution of equation (1.1), if

$$M\left(\|u\|_\lambda^p\right)\left(\lambda \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}(\varphi(x)-\varphi(y))dxdy + \int_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi dx\right)$$
$$= \int_{\mathbb{R}^N} |u|^{p_s^\star-2}u\varphi dx + \int_{\mathbb{R}^N} f(x,u)\varphi dx,$$

for any $\varphi \in W_\lambda$.

Now we are in a position to state the first result of our paper as follows:

Theorem 1.1. *Let* (M), (V) *and* (f_1)–(f_3) *hold. Then for any* $\lambda > 0$, *there exists* $\lambda^\star > 0$ *such that problem* (1.1) *has a nontrivial solution* u_λ *for any* $\lambda \in (0, \lambda^\star)$ *which satisfies*

$$\lambda \iint_{\mathbb{R}^{2N}} \frac{|u_\lambda(x)-u_\lambda(y)|^p}{|x-y|^{N+ps}}dxdy + \int_{\mathbb{R}^N} V(x)|u_\lambda|^p dx \le \left(\frac{\theta p q_1}{m_0 q_1 - m_1\theta p}\right)^{\frac{1}{\theta}} \sigma^{\frac{1}{\theta}} \lambda^{\frac{p_s^\star}{p_s^\star - \theta p}}, \tag{1.3}$$

where $\sigma = \frac{1}{q_1}\left(1 - \frac{m_1}{m_0}\right) + \frac{1}{\theta p} - \frac{1}{p_s^\star}$. *Assume additionally that* $f(x,t)$ *is odd with respect to* t, *for any* $m \in \mathbb{N}$, *there is* $\lambda_m > 0$ *such that problem* (1.1) *admits at least* m *pairs of solutions* $u_{\lambda,i}(i = 1, 2, \cdots, m)$ *which satisfy* (1.3) *whenever* $0 < \lambda \le \lambda_m$.

The proof of Theorem 1.1 is mainly based on the application of the concentration compactness lemma in fractional Sobolev spaces developed by Xiang *et al.* in [34]. We show that the energy functional I_λ associated to problem (1.1) satisfies $(PS)_c$ condition for $c > 0$ small and $\lambda > 0$ small. To get the multiplicity of solutions for problem (1.1), we find a special finite dimensional subspaces by which we construct sufficiently small minimax levels. It is worth to point out that the authors in [42] just concerned with the case that $M(t) = a + bt^{\theta-1}$ with $a, b > 0$, which just focused on the non-degenerate Kirchhoff problems, that is $M(0) > 0$. Moreover, for the nonlinear term f, our assumption (f_1) is more general than (h_1) and (h_2) in [42].

Finally, we consider the critical case $\theta = p_s^\star/p$. To this aim, we assume the subcritical term f satisfies following assumptions.

(f_4) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $q \in (p, p_s^\star)$ such that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ and

$$|f(x,t)| \le p\varepsilon|t|^{p-1} + qC_\varepsilon|t|^{q-1} \text{ for a.e. } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R};$$

(f_5) There exists $q_1 \in (p, p_s^\star)$ such that $F(x,t) \ge a_0|t|^{q_1}$ for a.e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$.

Theorem 1.2. *Assume that* M *satisfies* (M) *with* $\theta = p_s^\star/p$ *and* $2 \le p < N/s$, *and* $f(x,t)$ *is odd with respect to* t *and satisfies* (f_4) – (f_5). *Then problem* (1.1) *has infinitely many pairs distinct solutions in* W_λ *for all* $\lambda > 2^p S^{-p_s^\star/p}/m_0$.

For the critical case $\theta = p_s^\star/p$, the method used in Theorem 1.1 seems to be invalid. For this, we will use Krasnoselskii's genus theory to prove Theorem 1.2, see also [43] about the application of the same method to the multiplicity of solutions for a class of fractional Choquard-Kirchhoff equations. Furthermore, as usual for elliptic problems involving critical nonlinearities, we must pay attention to the lack of compactness. To overcome this difficulty, we fix parameter λ larger than a suitable threshold. We would like to point out that the authors in [34] just obtained the existence of two weak solutions for a variant of problem (1.1) by using Ekeland's variational principle and the mountain pass theorem. To our best knowledge, this is the first time to consider the existence of infinitely many solutions for the critical case $\theta = p_s^\star/p$ in the study of general Kirchhoff problems.

The rest of our paper is organized as follows. In Section 2, we give the proof of Theorems 1.1. In Section 3, we consider the critical case $\theta = p_s^\star/p$ and obtain the proof of the Theorem 1.2.

2 Proof of Theorem 1.1

In this section, we prove the main result of this paper. In the following, we shortly denote the norm of $L^\nu(\mathbb{R}^N)$ by $|\cdot|_\nu$.

Obviously, the energy functional $I_\lambda : W_\lambda \to \mathbb{R}$ associated with problem (1.1)

$$I_\lambda(u) = \frac{1}{p}\mathscr{M}(\|u\|_\lambda^p) - \frac{1}{p_s^\star}\int_{\mathbb{R}^N} |u|^{p_s^\star}dx - \int_{\mathbb{R}^N} F(x,u)dx$$

is well defined, where $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$. It is easy to verify that as argued in [26], $I_\lambda \in C^1(W_\lambda, \mathbb{R})$ and its critical points are solutions of (1.1).

Under our assumptions, we can show that functional has mountain pass geometry.

Lemma 2.1. *Assume that (M), (V), and (f_1) are satisfied. Then for each $\lambda \in (0,1)$ there exist $\alpha_\lambda > 0$ and $\rho_\lambda > 0$ such that $I_\lambda(u) > 0$ for $u \in B_{\rho_\lambda}\setminus\{0\}$, and $I_\lambda(u) \geq \alpha_\lambda$ for all $u \in W_\lambda$ with $\|u\|_\lambda = \rho_\lambda$. Here $B_{\rho_\lambda} = \{u \in W_\lambda : \|u\|_\lambda < \rho_\lambda\}$.*

Proof. By (f_1), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x,t)| \leq \varepsilon\theta p|t|^{\theta p-1} + C_\varepsilon q|t|^{q-1} \text{ for a.e. } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}.$$

Furthermore, we have

$$|F(x,t)| \leq \varepsilon|t|^{\theta p} + C_\varepsilon|t|^q \text{ for a.e. } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}.$$

For any $u \in W_\lambda$, by (M), Hölder's inequality and the fractional Sobolev inequality, one has

$$I_\lambda(u) \geq \frac{m_0}{\theta p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}\int_{\mathbb{R}^N}|u|^{p_s^\star}dx - \varepsilon\int_{\mathbb{R}^N}|u|^{\theta p}dx - C_\varepsilon\int_{\mathbb{R}^N}|u|^q dx.$$

Note that by the fractional Sobolev embedding theorem (see [1]), there exists $C > 0$ such that

$$|u|_{\theta p} \leq C\|u\|_\lambda \text{ and } |u|_q \leq C\|u\|_\lambda.$$

Then choosing $\varepsilon \in (0, m_0/(2\theta p C^{\theta p}))$, we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{m_0}{\theta p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}S^{-p_s^\star/p}[u]_{s,p}^{p_s^\star} - \varepsilon C^{\theta p}\|u\|_\lambda^{\theta p} - C^q C_\varepsilon\|u\|_\lambda^q \\ &\geq \left(\frac{m_0}{2\theta p} - \frac{1}{p_s^\star}S^{-p_s^\star/p}\|u\|_\lambda^{p_s^\star-\theta p} - C^q C_\varepsilon\|u\|_\lambda^{q-\theta p}\right)\|u\|_\lambda^{\theta p}. \end{aligned}$$

Let us define

$$g(t) := \frac{m_0}{2\theta p} - \frac{1}{p_s^\star}S^{-p_s^\star/p}t^{p_s^\star-\theta p} - C^q C_\varepsilon t^{q-\theta p} \text{ for all } t \geq 0.$$

Clearly, $\lim_{t\to 0^+} g(t) = m_0/(2\theta p) > 0$, since $p_s^\star > \theta p$ and $q > \theta p$. Taking $\rho_\lambda := \|u\|_\lambda$ small enough such that

$$\frac{1}{p_s^\star}S^{-p_s^\star/p}\rho_\lambda^{p_s^\star-\theta p} + C^q C_\varepsilon\rho_\lambda^{q-\theta p} < \frac{m_0}{2\theta p},$$

then we have

$$I_\lambda(u) \geq g(\rho_\lambda)\rho_\lambda^{\theta p} =: \alpha_\lambda.$$

Thus we complete the proof. □

Lemma 2.2. *Under the assumptions of Lemma 2.1, for any finite dimensional subspace $E \subset W_\lambda$,*

$$I_\lambda(u) \to -\infty \ \text{ as } u \in E, \ \|u\|_\lambda \to \infty.$$

Proof. By (M) and $F(x,t) \geq 0$ for a.e. $x \in \mathbb{R}^N$ and all $t \in \mathbb{R}$, we have

$$I_\lambda(u) \leq \frac{m_1}{\theta p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}\int_{\mathbb{R}^N} |u|^{p_s^\star} dx$$

for all $u \in E$. Note that all norms in a finite dimensional space are equivalent. Hence there exists $C_E > 0$ such that $|u|_{p_s^\star} \geq C_E\|u\|_\lambda$ for all $u \in E$. Then,

$$I_\lambda(u) \leq \frac{m_1}{\theta p}\|u\|_\lambda^{\theta p} - a_0 C_E^{p_s^\star}\|u\|_\lambda^{p_s^\star}.$$

It follows from $p_s^\star > \theta p$ that $I_\lambda(u) \to -\infty$ as $u \in E$, $\|u\|_\lambda \to \infty$. □

Definition 2.1. *A sequence $\{u_n\}_n \subset W_\lambda$ is called a $(PS)_c$ sequence, if $I_\lambda(u_n) \to c$ and $I_\lambda'(u_n) \to 0$. We say I_λ satisfies $(PS)_c$ condition if any $(PS)_c$ sequence admits a converging subsequence.*

Lemma 2.3. *Assume that (M) and (f_2) are fulfilled. If $\{u_n\}_n$ is a $(PS)_c$ sequence, then $\{u_n\}_n$ is bounded in W_λ and $c \geq 0$.*

Proof. Since $\{u_n\}_n$ is a $(PS)_c$ sequence, there exists $n_0 > 0$ such that

$$I_\lambda(u_n) - \frac{1}{q_1}\langle I_\lambda'(u_n), u_n\rangle \leq c + o(1) + o(1)\|u_n\|_\lambda \ \text{ for all } n \geq n_0, \tag{2.1}$$

Then, by (M) and (f_2), it follows that

$$I_\lambda(u_n) - \frac{1}{q_1}\langle I_\lambda'(u_n), u_n\rangle \geq \left(\frac{m_0}{\theta p} - \frac{m_1}{q_1}\right)\|u_n\|_\lambda^{\theta p}.$$

Hence, it follows from (2.1) and $\big(m_0/(\theta p) - m_1/q_1\big) > 0$ that

$$\left(\frac{m_0}{\theta p} - \frac{m_1}{q_1}\right)\|u_n\|_\lambda^{\theta p} \leq c + o(1) + o(1)\|u_n\|_\lambda. \tag{2.2}$$

This, together with $\big(m_0/(\theta p) - m_1/q_1\big) > 0$, yields that $\{u_n\}_n$ is bounded in W_λ. Then taking the limit in (2.2), we deduce that $c \geq 0$. This completes the proof of Lemma 2.3. □

Lemma 2.4. *Assume that (V), (M) and (f_1) – (f_2) hold. For any $\lambda \in (0,1)$, I_λ satisfies the $(PS)_c$ condition for all $c \in \left(0, \sigma(m_0\lambda^\theta S^\theta)^{\frac{p_s^\star}{p_s^\star - \theta p}}\right)$, where $\sigma = \left[\frac{1}{q_1}\left(1 - \frac{m_1}{m_0}\right) + \frac{1}{\theta p} - \frac{1}{p_s^\star}\right]$.*

Proof. Let $\{u_n\}_n$ be a $(PS)_c$ sequence. Then by Lemma 2.3, $\{u_n\}_n$ is bounded W_λ, up to a subsequence, there exists a nonnegative function $u \in W_\lambda$ such that $u_n \rightharpoonup u$ in W_λ, $u_n \to u$ in L^σ_{loc} for $\sigma \in [1, p_s^\star)$, and $u_n \to u$ a.e. in $\mathbb{R}^N$. By Theorem 2.2 of [34], up to a subsequence, there exists a (at most) countable set J, a non-atomic measure $\widetilde{\zeta}$, points $\{x_j\}_{j\in J} \subset \mathbb{R}^N$ and $\{\zeta_j\}_{j\in J}$, $\{\eta_j\}_{j\in J} \subset \mathbb{R}^+$ such that as $n \to \infty$

$$\int_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}} dy \rightharpoonup \zeta = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} dy + \sum_{j\in J} \zeta_j \delta_{x_j} + \widetilde{\zeta} \tag{2.3}$$

and

$$|u_n|^{p_s^\star} \rightharpoonup \eta = |u|^{p_s^\star} + \sum_{j\in J} \eta_j \delta_{x_j} \tag{2.4}$$

in the measure sense, where δ_{x_j} is the Dirac measure concentrated x_j. Moreover,

$$\eta_j \le S^{-p_s^\star/p} \zeta_j^{p_s^\star/p}, \quad \forall j \in J, \tag{2.5}$$

where $S > 0$ is the best constant of the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^\star}(\mathbb{R}^N)$.

Next we prove that $J = \emptyset$. Otherwise, suppose that $J \neq \emptyset$, then for fixed $j \in J$ and $\varepsilon > 0$, choose $\varphi_{\varepsilon,j} \in C_0^\infty(\mathbb{R}^N)$ such that

$$\varphi_{\varepsilon,j} = 1 \ \text{ for } |x - x_j| \le \varepsilon; \ \varphi_{\varepsilon,j} = 0 \ \text{ for } |x - x_j| \ge 2\varepsilon,$$

and $|\nabla\varphi_{\varepsilon,j}| \le 2/\varepsilon$. Evidently, $\varphi_{\varepsilon,j} u_n \in D^{s.p}(\mathbb{R}^N)$. Hence it follows from $\langle I_\lambda'(u_n), \varphi_{\varepsilon,j} u_n\rangle \to 0$ that

$$\begin{aligned} &M(\|u_n\|_\lambda^p)\left[\lambda\langle u_n, u_n\varphi_\varepsilon\rangle_{s,p} + \int_{\mathbb{R}^N} V(x)|u_n|^p\varphi_\varepsilon dx\right] \\ &= \int_{\mathbb{R}^N} |u_n|^{p_s^\star}\varphi_{\varepsilon,j} dx + \int_{\mathbb{R}^N} f(x, u_n)\varphi_{\varepsilon,j} u_n dx + o(1), \end{aligned} \tag{2.6}$$

where

$$\langle u_n, u_n\varphi_{\varepsilon,j}\rangle_{s,p} = \lambda \iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{\varepsilon,j}(x)u_n(x) - \varphi_{\varepsilon,j}(y)u_n(y))}{|x - y|^{N+ps}} dxdy.$$

By using Hölder's inequality and Lemma 2.3 of [34], we have

$$\begin{aligned} &\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left| M(\|u_n\|_\varepsilon^p)\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y))u_n(x)}{|x - y|^{N+ps}} dxdy\right| \\ &\le C\lim_{\varepsilon\to0}\limsup_{n\to\infty}\left(\iint_{\mathbb{R}^{2N}} \frac{|(\varphi_{\varepsilon,j}(x) - \varphi_{\varepsilon,j}(y))u_n(x)|^p}{|x - y|^{N+ps}} dxdy\right)^{1/p} = 0. \end{aligned} \tag{2.7}$$

By (2.3), (2.4) and (M), we have

$$\begin{aligned} &\lim_{\varepsilon\to0}\limsup_{n\to\infty} M(\|u_n\|_\lambda^p)\left[\lambda\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}}\varphi_{\varepsilon,j}(y) dydx + \int_{\mathbb{R}^N} V(x)|u_n|^p\varphi_{\varepsilon,j} dx\right] \\ &\ge \lim_{\varepsilon\to0}\lim_{n\to\infty}\left[m_0\left(\lambda\iint_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^p}{|x - y|^{N+ps}}\varphi_{\varepsilon,j}(y) dxdy\right)^\theta\right] = m_0(\lambda\zeta_j)^\theta, \end{aligned} \tag{2.8}$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N} |u_n|^{p_s^\star}\varphi_{\varepsilon,j} dx = \lim_{\varepsilon\to0}\int_{\mathbb{R}^N} |u|^{p_s^\star}\varphi_{\varepsilon,j} dx + \eta_j = \eta_j, \tag{2.9}$$

and

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x, u_n)\varphi_{\varepsilon,j} u_n dx = \lim_{\varepsilon\to0}\int_{\mathbb{R}^N} f(x, u)\varphi_{\varepsilon,j} u dx = 0. \tag{2.10}$$

Here we applied the fact that $W_\lambda \hookrightarrow L^\nu_{\mathrm{loc}}(\mathbb{R}^N)$ is compact for all $\nu \in [1, p_s^\star)$. Then we can deduce from (2.6)–(2.10) that

$$\eta_j \geq m_0(\lambda\zeta_j)^\theta.$$

From the above inequality, together with (2.5), it follows that

$$\eta_j \geq m_0 \lambda^\theta S^\theta \eta_j^{\theta p/p_s^\star}.$$

Hence,

$$\eta_j \geq (m_0\lambda^\theta S^\theta)^{\frac{p_s^\star}{p_s^\star - \theta p}}. \tag{2.11}$$

On the other hand, by (2.3) and (2.4), we obtain

$$\begin{aligned} c &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{q_1}\langle I'_\lambda(u_n), u_n\rangle\right) \\ &\geq \left(\frac{m_0}{\theta p} - \frac{m_1}{q_1}\right)(\lambda\zeta_j)^\theta + \left(\frac{1}{q_1} - \frac{1}{p_s^\star}\right)\eta_j \\ &\geq \left[\frac{1}{q_1}\left(1 - \frac{m_1}{m_0}\right) + \frac{1}{\theta p} - \frac{1}{p_s^\star}\right](m_0\lambda^\theta S^\theta)^{\frac{p_s^\star}{p_s^\star-\theta p}}, \end{aligned}$$

which is a contradiction. Hence the desired conclusion holds.

Letting $R > 0$, we define

$$\zeta_\infty = \lim_{R\to\infty}\limsup_{n\to\infty} \int\limits_{\{x\in\mathbb{R}^N:|x|>R\}} \int\limits_{\mathbb{R}^N} \frac{|u_n(x) - u_n(y)|^p}{|x-y|^{N+ps}}\,dy\,dx,$$

and

$$\eta_\infty = \lim_{R\to\infty}\limsup_{n\to\infty} \int\limits_{\{x\in\mathbb{R}^N:|x|>R\}} |u_n|^{p_s^\star}\,dx.$$

In view of Theorem 2.4 of [34], ζ_∞ and η_∞ are well defined and satisfy

$$\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dy\,dx = \int\limits_{\mathbb{R}^N} d\zeta + \zeta_\infty, \tag{2.12}$$

and

$$\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} u_n^{p_s^\star}\,dx = \int\limits_{\mathbb{R}^N} d\eta + \eta_\infty. \tag{2.13}$$

Assume that $\chi_R \in C^\infty(\mathbb{R}^{\mathbb{N}})$ satisfies the properties: $\chi_R \in [0,1]$ and $\chi_R(x) = 0$ for $|x| < R$, $\chi_R(x) = 1$ for $|x| > 2R$, and $|\nabla\chi_R| \leq 2/R$. By Theorem 2.4 of [34], we have

$$\zeta_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\iint\limits_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p\chi_R(x)^p}{|x-y|^{N+ps}}\,dy\,dx \tag{2.14}$$

and

$$\eta_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}|u_n(x)\chi_R(x)|^{p_s^\star}\,dx. \tag{2.15}$$

Moreover, we have

$$S\eta_\infty^{p/p_s^\star} \leq \zeta_\infty. \tag{2.16}$$

Because $\|u_n\|^p$ and $\|u_n\|_{p_s^\star}^{p_s^\star}$ are bounded, up to a subsequence, we can assume that $\|u_n\|^p$ and $\|u_n\|_{p_s^\star}^{p_s^\star}$ are both convergent. Hence by (2.12) and (2.13), we can obtain

$$\lim_{n\to\infty}\|u_n\|^p = \int_{\mathbb{R}^N} d\zeta + \zeta_\infty \tag{2.17}$$

and

$$\lim_{n\to\infty}\|u_n\|_{p_s^\star}^{p_s^\star} = \int_{\mathbb{R}^N} d\eta + \eta_\infty. \tag{2.18}$$

It follows from $\langle I'_\lambda(u_n), \chi_R u_n\rangle \to 0$ as $n \to \infty$ that

$$\begin{aligned}
M(\|u_n\|_\lambda^p)\Bigg[\lambda \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\chi_R(x)}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)|u_n|^p\chi_R\,dx \\
+ \lambda \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+ps}}\,dxdy\Bigg] \\
= \int_{\mathbb{R}^N} |u_n|^{p_s^\star}\chi_R\,dx + \int_{\mathbb{R}^N} f(x,u_n)u_n\chi_R\,dx + o(1).
\end{aligned} \tag{2.19}$$

By employing Hölder's inequality and (2.15) in [34], we get

$$\lim_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))u_n(y)(\chi_R(x)-\chi_R(y))}{|x-y|^{N+ps}}\,dxdy = 0. \tag{2.20}$$

Hence we deduce from (2.14), (2.17), (2.19) and (2.20) that

$$\begin{aligned}
&\lim_{R\to\infty}\limsup_{n\to\infty} M(\|u_n\|_\lambda^p)\Bigg[\lambda \iint_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^p\chi_R(x)}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)|u_n|^p\chi_R\,dx\Bigg] \\
&\geq m_0\lambda^\theta\Bigg(\int_{\mathbb{R}^N} d\zeta + \zeta_\infty\Bigg)^{\theta-1} \lim_{R\to\infty}\limsup_{n\to\infty}\Bigg(\int_{\{x\in\mathbb{R}^N:|x|>R\}}\int_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}\,dydx\Bigg) \\
&\geq m_0\lambda^\theta\zeta_\infty^\theta,
\end{aligned} \tag{2.21}$$

thanks to the assumption $\theta > 1$. It is easy to see that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n\chi_R\,dx = \lim_{R\to\infty}\int_{\mathbb{R}^N} f(x,u)u\chi_R\,dx = 0, \tag{2.22}$$

Thanks to the fact that the embedding $W_\lambda \to L^q_{\mathrm{loc}}(\mathbb{R}^N)$ is compact. Therefore, we conclude from (2.19)–(2.22) and (2.15) that

$$m_0\lambda^\theta\zeta_\infty^\theta \leq \eta_\infty,$$

which together with (2.16) yields

$$m_0\lambda^\theta S^\theta\eta_\infty^{\theta p/p_s^\star} \leq \eta_\infty.$$

This implies that $\eta_\infty = 0$ or

$$\eta_\infty \geq (m_0\lambda^\theta S^\theta)^{\frac{p_s^\star}{p_s^\star-\theta p}}. \tag{2.23}$$

Assume that (2.23) holds. Then

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(I_\lambda(u_n) - \frac{1}{q_1}\langle I'_\lambda(u_n), u_n\rangle\right)\\
&\geq \lim_{n\to\infty}\left[\left(\frac{m_0}{\theta p} - \frac{m_1}{q_1}\right)\left(\iint_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^p}{|x-y|^{N+ps}}dxdy\right)^\theta + \left(\frac{1}{q_1} - \frac{1}{p_s^\star}\right)\int_{\mathbb{R}^N}|u_n|^{p_s^\star}dx\right]\\
&\geq \left[\frac{1}{q_1}\left(1 - \frac{m_1}{m_0}\right) + \frac{1}{\theta p} - \frac{1}{p_s^\star}\right](m_0\lambda^\theta S^\theta)^{\frac{p_s^\star}{p_s^\star - \theta p}},
\end{aligned}$$

which is absurd. Hence, we have $\nu_\infty = 0$. In view of $J = \emptyset$ and (2.18), we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^\star}dx = \int_{\mathbb{R}^N}|u|^{p_s^\star}dx. \tag{2.24}$$

Now we show that $u_n \to u$ in W_λ. To this aim, we first assume that $d := \inf_{n\geq 1}\|u_n\|_\lambda > 0$. Since $\langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle \to 0$, we have

$$\begin{aligned}
&M(\|u_n\|_\lambda^p)\langle u_n, u_n - u\rangle - M(\|u\|_\lambda^p)\langle u, u_n - u\rangle\\
&= \int_{\mathbb{R}^N}\left(|u_n|^{p_s^\star-2}u_n - |u|^{p_s^\star-2}u\right)(u_n - u)dx + o(1),
\end{aligned}$$

where

$$\langle u_n, u_n - u\rangle := \lambda\langle u_n, u_n - u\rangle_{s,p} + \int_{\mathbb{R}^N}V(x)|u_n|^{p-2}u_n(u_n - u)dx.$$

Here we used the following fact:

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n - u)dx = 0. \tag{2.25}$$

Now we show that (2.25) is true. By Theorem 2.1 of [26], we know that the embedding $W_\lambda \hookrightarrow L^\nu(\mathbb{R}^N)$ is compact for any $\nu \in [p, p_s^\star)$. Thus, up to a subsequence, we have $u_n \to u$ in $L^\nu(\mathbb{R}^N)$ for any $\nu \in [p, p_s^\star)$. According to (f_1) and (f_2), for any $\varepsilon > 0$ we have

$$|f(x,t)| \leq \varepsilon|t|^{\theta p-1} + C_\varepsilon|t|^{q-1}$$

for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Then

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n - u)dx\right|\\
&\leq \varepsilon\int_{\mathbb{R}^N}(|u_n|^{\theta p-1} + |u|^{\theta p-1})(u_n - u)dx + C_\varepsilon\int_{\mathbb{R}^N}(|u_n|^{q-1} + |u|^{q-1})(u_n - u)dx\\
&\leq C\varepsilon + C_\varepsilon(\|u_n\|_q^{q-1} + \|u\|_q^{q-1})\|u_n - u\|_q,
\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n - u)dx = 0.$$

Thus, we obtain

$$\begin{aligned}
&M(\|u_n\|_\lambda^p)(\langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle) + M(\|u_n\|_\lambda^p)\langle u, u_n - u\rangle - M(\|u\|_\lambda^p)\langle u, u_n - u\rangle\\
&= \int_{\mathbb{R}^N}\left(|u_n|^{p_s^\star-2}u_n - |u|^{p_s^\star-2}u\right)(u_n - u)dx + o(1).
\end{aligned}$$

By the boundedness of $\{u_n\}_n$ and $u_n \rightharpoonup u$ in W_λ, we can deduce that

$$\lim_{n\to\infty} M(\|u_n\|_\lambda^p)\langle u, u_n - u\rangle = 0$$

and

$$\lim_{n\to\infty} M(\|u\|_\lambda^p)\langle u, u_n - u\rangle = 0.$$

Hence, we conclude from (2.24) that

$$\lim_{n\to\infty} M(\|u_n\|_\lambda^p)\left(\langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle\right) = 0.$$

This, together with $d := \inf_{n\geq 1}\|u_n\| > 0$, implies that

$$\lim_{n\to\infty}\left(\langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle\right) = 0. \tag{2.26}$$

Let us now recall the well-known inequalities:

$$|a-b|^p \leq \begin{cases} 2^p\left(|a|^{p-2}a - |b|^{p-2}b\right)\cdot(a-b) & \text{for} \quad p \geq 2 \\ \dfrac{1}{p-1}\left[\left(|a|^{p-2}a - |b|^{p-2}b\right)\cdot(a-b)\right]^{p/2}\left(|a|^p + |b|^p\right)^{(2-p)/2} & \text{for } 1 < p < 2, \end{cases} \tag{2.27}$$

for all $a, b \in \mathbb{R}^N$.

Similar to the proof of Lemma 6 in [26], it is easy to deduce from (2.27) that $u_n \to u$ strongly in W_λ as $n \to \infty$.

In the end, we consider the case $\inf_n \|u_n\|_\lambda = 0$. Then either 0 is an accumulation point of the sequence $\{u_n\}_n$ and so there exists a subsequence of $\{u_n\}_n$ strongly converging to $u = 0$, or 0 is an isolated point of the sequence $\{u_n\}_n$ and so there exists a subsequence, still denoted by $\{u_n\}_n$, such that $\inf_n \|u_n\| > 0$. In the first case we are done, while in the latter case we can proceed as above. □

Since the functional I_λ satisfies the $(PS)_c$ condition for small $c > 0$, we will find a special finite dimensional subspaces by which we construct sufficiently small minimax levels.

By (V), we know that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality, we assume from now on that $x_0 = 0$. By means of (M) and (f_3), we have

$$\begin{aligned} I_\lambda(u) &\leq \frac{m_1}{\theta p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}\int_{\mathbb{R}^N} |u|^{p_s^\star}dx - a_0\int_{\mathbb{R}^N} |u|^{q_2}dx \\ &\leq \frac{m_1}{\theta p}\|u\|_\lambda^{\theta p} - a_0\int_{\mathbb{R}^N} |u|^{q_2}dx \end{aligned}$$

for all $u \in W_\lambda$. Define the functional $\Phi_\lambda : W_\lambda \to \mathbb{R}$ by

$$J_\lambda(u) = \frac{m_1}{\theta p}\|u\|_\lambda^{\theta p} - a_0\int_{\mathbb{R}^N} |u|^{q_2}dx.$$

Then $I_\lambda(u) \leq J_\lambda(u)$ for all $u \in W_\lambda$. Hence it suffices to construct small minimax levels for J_λ.

For any $\delta > 0$, one can choose $\phi_\delta \in C_0^\infty(\mathbb{R}^N)$ with $\|\phi_\delta\|_{q_2} = 1$ and supp $\phi_\delta \subset B_{r_\delta}(0)$ such that $[\phi_\delta]_{s,p}^p < \delta$. Let

$$e_\lambda = \phi_\delta\left(\lambda^{-\frac{\theta p_s^\star}{N(p_s^\star - \theta p)}}x\right).$$

Then $\operatorname{supp} e_\lambda \subset B_{\lambda^{\frac{\theta p_s^\star}{N(p_s^\star-\theta p)}} r_\delta}(0)$. Thus, for $t \geq 0$, we have

$$\begin{aligned} J_\lambda(te_\lambda) &= \frac{t^{\theta p}}{\theta p}\|e_\lambda\|_\lambda^{\theta p} - a_0 t^{q_2}\int_{\mathbb{R}^N}|e_\lambda|^{q_2}\,dx \\ &= \lambda^{\frac{\theta p_s^\star}{p_s^\star-\theta p}} t^{\theta p}\frac{1}{\theta p}\Bigg[\Bigg(\iint_{\mathbb{R}^{2N}}\frac{|\phi_\delta(x)-\phi_\delta(y)|^p}{|x-y|^{N+ps}}\,dx\,dy \\ &\quad + \lambda^{(\theta-1)\frac{p_s^\star}{p_s^\star-\theta p}}\int_{\mathbb{R}^N} V\big(\lambda^{\frac{\theta p_s^\star}{N(p_s^\star-\theta p)}}x\big)|\phi_\delta|^p\,dx\Bigg)^\theta - t^{q_2}a_0\int_{\mathbb{R}^N}|\phi_\delta|^{q_2}\,dx\Bigg]. \end{aligned}$$

It follows from $\lambda \in (0,1)$ and $\theta > 1$ that

$$\begin{aligned} &J_\lambda(te_\lambda) \\ &\leq \lambda^{\frac{\theta p_s^\star}{p_s^\star-\theta p}} t^{\theta p}\frac{1}{\theta p}\Bigg[\Bigg(\iint_{\mathbb{R}^{2N}}\frac{|\phi_\delta(x)-\phi_\delta(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} V(\lambda^{\frac{\theta p_s^\star}{N(p_s^\star-\theta p)}}x)|\phi_\delta|^p\,dx\Bigg)^\theta - a_0 t^{q_2}\int_{\mathbb{R}^N}|\phi_\delta|^{q_2}\,dx\Bigg] \\ &= \lambda^{\frac{\theta p_s^\star}{p_s^\star-\theta p}}\Phi_\lambda(t\phi_\delta), \end{aligned}$$

where $\Phi_\lambda \in C^1(W_\lambda,\mathbb{R})$ defined by

$$\Phi_\lambda(u) = \frac{1}{\theta p}\Bigg(\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} V(\lambda^{\frac{\theta p_s^\star}{N(p_s^\star-\theta p)}}x)|u|^p\,dx\Bigg)^\theta - a_0\int_{\mathbb{R}^N}|u|^{q_2}\,dx$$

for all $u \in W_\lambda$. Clearly,

$$\max_{t\geq 0}\Phi_\lambda(t\phi_\delta) = \frac{q-\theta p}{\theta p q (q a_0)^{\frac{\theta p}{q-\theta p}}}\Bigg(\iint_{\mathbb{R}^{2N}}\frac{|\phi_\delta(x)-\phi_\delta(y)|^p}{|x-y|^{N+ps}}\,dx\,dy + \int_{\mathbb{R}^N} V(\lambda^{\frac{\theta p_s^\star}{N(p_s^\star-\theta p)}}x)|\phi_\delta|^p\,dx\Bigg)^{\frac{q}{q-\theta p}}.$$

Observe that $V(0) = 0$ and $V \in C(\mathbb{R}^N)$, then there exists $\Lambda_\delta > 0$ such that

$$0 \leq V\left(\lambda^{\frac{\theta p_s^\star}{N(p_s^\star-\theta p)}}x\right) \leq \frac{\delta}{\|\phi_\delta\|_p^p}$$

for all $|x| \leq r_\delta$ and $0 < \lambda \leq \Lambda_\delta$. It follows from $[\phi_\delta]_{s,p}^p < \delta$ that

$$\max_{t\geq 0}\Phi_\lambda(t\phi_\delta) \leq \frac{q_2-\theta p}{\theta p q_2 (q_2 a_0)^{\frac{\theta p}{q_2-\theta p}}}(2\delta)^{\frac{q_2}{q_2-\theta p}}.$$

Furthermore, we have

$$\max_{t\geq 0} I_\lambda(t\phi_\delta) \leq \frac{q_2-\theta p}{\theta p q_2 (q_2 a_0)^{\frac{\theta p}{q_2-\theta p}}}(2\delta)^{\frac{q_2}{q_2-\theta p}}\lambda^{\frac{\theta p_s^\star}{p_s^\star-\theta p}}, \tag{2.28}$$

for all $\lambda \in (0,\Lambda_\delta]$. In conclusion, we have the following lemma.

Lemma 2.5. *Under the assumptions of Lemma 2.1, there exists $\Lambda > 0$ such that for all $\lambda \in (0,\Lambda)$ there exists $\widetilde{e}_\lambda \in W_\lambda$ with $\|\widetilde{e}_\lambda\|_\lambda > \rho_\lambda$, $I_\lambda(\widetilde{e}_\lambda) < 0$ and*

$$\max_{t\in[0,1]} I_\lambda(t\widetilde{e}_\lambda) < \sigma\lambda^{\frac{\theta p_s^\star}{p_s^\star-\theta p}},$$

where $\sigma = \frac{1}{q_1}\left(1-\frac{m_1}{m_0}\right) + \frac{1}{\theta p} - \frac{1}{p_s^\star}$.

Proof. Let $\delta > 0$ small enough such that

$$\frac{q_2 - \theta p}{\theta p q_2 (q_2 a_0)^{\frac{\theta p}{q_2 - \theta p}}} (2\delta)^{\frac{q_2}{q_2 - \theta p}} < \sigma.$$

Taking $\Lambda = \Lambda_\delta$ and choosing $\widetilde{t}_\lambda > 0$ such that $\widetilde{t}_\lambda \|e_\lambda\|_\lambda > \rho_\lambda$ and $I_\lambda(te_\lambda) < 0$ for all $t \geq \widetilde{t}_\lambda$. The result follows by letting $\widetilde{e}_\lambda = \widetilde{t}_\lambda e_\lambda$. □

Let $m \in \mathbb{N}$, we choose m functions $\phi_\delta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp} \phi_\delta^i \bigcap \operatorname{supp} \phi_\delta^j = \emptyset$ for all $1 \leq i \neq j \leq m$, and $\|\phi_\delta\|_q = 1$ and $[\phi_\delta]_{s,p}^p < \delta$. Let $r_\delta^m > 0$ be such that $\operatorname{supp} \phi_\delta^i \subset B_{r_\delta^m}(0)$ for $i = 1, 2, \cdots, m$. Set

$$e_\lambda^i = \phi_\delta^i \left(\lambda^{-\frac{\theta p_s^\star}{N(p_s^\star - \theta p)}} x \right) \quad \text{for all } i = 1, 2, \cdots, m,$$

and

$$E_{\lambda,\delta}^m = \operatorname{span}\{e_\lambda^1, e_\lambda^2, \cdots, e_\lambda^m\}.$$

Then for each $u = \sum_{i=1}^m c_i e_\lambda^i \in E_{\lambda,\delta}^m$, we have

$$\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} dxdy = \sum_{i=1}^m |c_i|^p \iint_{\mathbb{R}^{2N}} \frac{|e_\lambda^i(x) - e_\lambda^i(y)|^p}{|x - y|^{N+ps}} dxdy,$$

$$\int_{\mathbb{R}^N} V(x)|u|^p dx = \sum_{i=1}^m |c_i|^p \int_{\mathbb{R}^N} V(x)|e_\lambda^i|^p dx,$$

$$\frac{1}{p_s^\star} \int_{\mathbb{R}^N} |u|^{p_s^\star} dx = \frac{1}{p_s^\star} \sum_{i=1}^m |c_i|^{p_s^\star} \int_{\mathbb{R}^N} |e_\lambda^i|^{p_s^\star} dx$$

and

$$\int_{\mathbb{R}^N} F(x, u) dx = \sum_{i=1}^m \int_{\mathbb{R}^N} F(x, c_i e_\lambda^i) dx.$$

Hence,

$$I_\lambda(u) = \sum_{i=1}^m I_\lambda(c_i e_\lambda^i).$$

and as above

$$I_\lambda(c_i e_\lambda^i) \leq \lambda^{\frac{\theta p_s^\star}{N(p_s^\star - \theta p)}} \Phi_\lambda(c_i e_\lambda^i).$$

Set

$$\beta_\delta := \max\{\|\phi_\delta^i\|_p^p : i = 1, 2, \cdots, m\},$$

and choose $\Lambda_{m,\delta} > 0$ such that

$$V\left(\lambda^{-\frac{\theta p_s^\star}{N(p_s^\star - \theta p)}} x \right) \leq \frac{\delta}{\beta_\delta}$$

for all $|x| \leq r_\delta^m$ and $\lambda \leq \Lambda_{m,\delta}$. As in the proof of Lemma 2.4, we can get

$$\max_{u \in E_{\lambda,\delta}^m} I_\lambda(u) \leq \frac{m(q_2 - \theta p)}{\theta p q_2 (q_2 a_0)^{\frac{\theta p}{q_2 - \theta p}}} (2\delta)^{\frac{q_2}{q_2 - \theta p}} \lambda^{\frac{\theta p_s^\star}{p_s^\star - \theta p}},$$

for all $\lambda \in (0, \Lambda_{m,\delta}]$. Then we have the following lemma.

Lemma 2.6. *Under the assumptions of Lemma 2.1, for any $m \in \mathbb{N}$ there exists $\Lambda_m > 0$ such that for all $\lambda \in (0, \Lambda_m)$ there exists m-dimensional subspace E_λ^m such that*

$$\max_{u \in E_\lambda^m} I_\lambda(u) < \sigma \lambda^{\frac{\theta p_s^\star}{p_s^\star - \theta p}}.$$

Proof. Choose $\delta > 0$ so small that

$$\frac{q_2 - \theta p}{\theta p q_2 (q_2 a_0)^{\frac{\theta p}{q_2 - \theta p}}} (2\delta)^{\frac{q_2}{q_2 - \theta p}} < \sigma$$

and take $E_\lambda^m = E_{\lambda,\delta}^m$. The result follows from (2.28) and the definition of $E_{\lambda,\delta}^m$. □

Proof of Theorem 1.1. According to Lemma 2.5, we choose $\Lambda_\sigma > 0$ and define

$$c_\lambda = \inf_{y \in \Gamma} \max_{t \in [0,1]} I_\lambda(t\widetilde{e}_\lambda),$$

where

$$\Gamma = \{y \in C([0,1], W_\lambda) : y(0) = 0 \text{ and } y(1) = \widetilde{e}_\lambda\}.$$

By Lemma 2.1, we have $\alpha_\lambda \le c_\lambda < \sigma \lambda^{\theta p_s^\star/(p_s^\star - \theta p)}$. In view of Lemma 2.4, we know that I_λ satisfies the $(PS)_{c_\lambda}$ condition, and there exists $u_\lambda \in W_\lambda$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$. Thus, u_λ is a solution of (1.1).

It follows from $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$ that

$$\begin{aligned} \sigma \lambda^{\frac{\theta p_s^\star}{p_s^\star - \theta p}} &\ge I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{q_1} \langle I'(u_\lambda), u_\lambda \rangle \\ &\ge \left(\frac{m_0}{\theta p} - \frac{m_1}{q_1} \right) \|u_\lambda\|_\lambda^\theta + \left(\frac{1}{q_1} - \frac{1}{p_s^\star} \right) \int_{\mathbb{R}^N} |u_\lambda|^{p_s^\star} dx. \end{aligned}$$

Hence (1.3) holds.

Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ, for each $Z \in \Sigma$. Let $y(Z)$ be the Krasnoselski genus and

$$i(Z) = \min_{h \in \Gamma_m} y \left(h(Z) \bigcap \partial B_{\rho_\lambda} \right),$$

where Γ_m is the set of all odd homeomorphisms $h \in C(W_\lambda, W_\lambda)$ and ρ_λ is the number from Lemma 2.1. Then i is a version of Benci's pseudo-index (see [44]). Let

$$c_{\lambda_j} := \inf_{i(Z) \ge j} \sup_{u \in Z} I_\lambda(u), \quad 1 \le j \le m.$$

Since $I_\lambda(u) \ge \alpha_\lambda$ for all $u \in \partial B_{\rho_\lambda}$ and $i(E_{\lambda,m}) = \dim E_{\lambda,m} = m$, we have

$$\alpha_\lambda \le c_{\lambda_1} \le \cdots \le c_{\lambda_m} \le \sup_{u \in E_{\lambda,m}} I_\lambda(u) < \left[\frac{1}{q_1} \left(1 - \frac{m_1}{m_0} \right) + \frac{1}{\theta p} - \frac{1}{p_s^\star} \right] \lambda^{\frac{\theta p_s^\star}{p_s^\star - \theta p}}.$$

It follows from Lemma 2.4 that I_λ satisfies the $(PS)_c$ condition at all levels c_{λ_j} $(j = 1, 2, \cdots, m)$. According to standard critical point theory (see [45]), all c_{λ_j} are critical values, and then I_λ has at least m pairs of nontrivial critical points. □

3 Proof of Theorem 1.2

In this section, we consider the existence of infinitely many solutions of problem (1.1), where the Kirchhoff function M satisfies (M) with the critical case $\theta = p_s^\star/p$. Let us first recall some basic results about Krasnoselskii's genus, which can be found in [45]. Let $\mathcal{G}$ be a real Banach space. Set

$$\Gamma = \{A \subset \mathcal{G} \backslash \{0\} : A \text{ is compact and } A = -A.\}$$

Definition 3.1. Let $A \in \Gamma$. The Krasnoselskii genus $y(A)$ of A is defined as being the least positive integer k such that there is an odd mapping $\phi \in C(A, \mathbb{R}^k)$ such that $\phi(x) \neq 0$ for all $x \in A$. If such a k does not exist we set $y(A) = \infty$. Moreover, by definition, $y(\emptyset) = 0$.

Lemma 3.1. (see [45]) *Let $\mathcal{G} = \mathbb{R}^N$ and $\partial\Omega$ be the boundary of an open, symmetric, and bounded subset $\Omega \subset \mathbb{R}^N$ with $0 \in \Omega$. Then $\gamma(\partial\Omega) = N$.*

Denote by $\mathbb{S}^{N-1}$ the surface of the unit sphere in $\mathbb{R}^N$. Then we can deduce from Lemma 3.1 that $\gamma(\mathbb{S}^{N-1}) = N$.
We shall use the following theorem to obtain the existence of infinitely many solutions for (1.1).

Theorem 3.1. (see [46]) *Let $T \in C^1(\mathcal{G}, \mathbb{R})$ be an even functional satisfying the (PS) condition. Furthermore,*
(1) *T is bounded from below and even;*
(2) *there is a compact set $E \in \Gamma$ such that $\gamma(E) = k$ and $\sup_{u\in E} T(u) < T(0)$.*
Then T has at least k pairs of distinct critical points and their corresponding critical values are less than $T(0)$.

Lemma 3.2. *Assume that $s \in (0, 1)$, $2 \le p < N/s$, $\theta = p_s^\star/p$, $p < q < p_s^\star$ and f satisfies (f_4). Then functional I_λ satisfies the $(PS)_c$ conditions in W_λ for all $\lambda > 2^p S^{-p_s^\star/p}/m_0$.*

Proof. Let $\{u_n\}_n \subset W_\lambda$ be the $(PS)_c$ sequence of functional I_λ, i.e.

$$I_\lambda(u_n) \to c, \quad I_\lambda(u_n) \to 0$$

as $n \to \infty$.
By (f_4), we have

$$|F(x,t)| \le |t|^p + C_1|t|^q \quad \text{for a.e. } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}.$$

It follows from (M), (f_4), Hölder's inequality and the fractional Sobolev inequality that

$$\begin{aligned} I_\lambda(u) &\ge \frac{m_0}{\theta p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}\int_{\mathbb{R}^N} |u|^{p_s^\star}dx - \int_{\mathbb{R}^N}|u|^p dx - C_1\int_{\mathbb{R}^N}|u|^q dx \\ &\ge \frac{m_0}{\theta p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}S^{-\frac{p_s^\star}{p}}\lambda^{-1}\|u\|_\lambda^{p_s^\star} - C\|u\|_\lambda^p - C\|u\|_\lambda^q, \end{aligned} \tag{3.1}$$

for all $u \in W_\lambda$. When $\theta = p_s^\star/p$, since $\lambda > \frac{2^p S^{-p_s^\star/p}}{m_0} > \frac{S^{-p_s^\star/p}}{p_s^\star}\frac{\theta p}{m_0}$ and $p < q < p_s^\star$, it is easy to see that I_λ is coercive and bounded from below on W_λ. Hence, $\{u_n\}_n$ is bounded in W_λ. Then there exist a subsequence of $\{u_n\}_n$ (still denoted by $\{u_n\}_n$) and $u \in W_\lambda$ such that

$$\begin{aligned} &u_n \rightharpoonup u \ \text{ in } W_\lambda \ \text{ and in } \ L^{p_s^\star}(\mathbb{R}^N), \\ &u_n \to u \ \text{ a.e. in } \mathbb{R}^N, \\ &|u_n|^{p_s^\star-2}u_n \rightharpoonup |u|^{p_s^\star-2}u \ \text{ in } \ L^{\frac{p_s^\star}{p_s^\star-1}}(\mathbb{R}^N), \end{aligned} \tag{3.2}$$

as $n \to \infty$. Similar to the discussion as in Section 2, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n - u)dx = 0. \tag{3.3}$$

Let $w_n = u_n - u$. Then by using similar arguments as in Lemma 3.2 of [4], we get

$$[u_n]_{s,p}^p = [w_n]_{s,p}^p + [u]_{s,p}^p + o(1). \tag{3.4}$$

By the celebrated Brézis–Lieb lemma, one has

$$\int_{\mathbb{R}^N} V(x)|w_n|^p dx = \int_{\mathbb{R}^N} V(x)|u_n|^p dx - \int_{\mathbb{R}^N} V(x)|u|^p dx + o(1) \tag{3.5}$$

and

$$\int_{\mathbb{R}^N} |w_n|^{p_s^\star}dx = \int_{\mathbb{R}^N} |u_n|^{p_s^\star}dx - \int_{\mathbb{R}^N} |u|^{p_s^\star}dx + o(1). \tag{3.6}$$

Let us now introduce, for simplicity, for all $v \in W_\lambda$ the linear functional $\mathcal{L}(v)$ on W_λ defined by

$$\langle \mathcal{L}(v), w\rangle = \lambda \iint_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+ps}}\,dxdy + \int_{\mathbb{R}^N} V(x)|v|^{p-2}vw\,dx$$

for all $w \in W_\lambda$. The Hölder inequality gives that

$$\begin{aligned}\big|\langle \mathcal{L}(v), w\rangle\big| &\le \lambda[v]_{s,p}^{p-1}[w]_{s,p} + \Big(\int_{\mathbb{R}^N} V(x)|v|^p dx\Big)^{(p-1)/p}\Big(\int_{\mathbb{R}^N} V(x)|w|^p dx\Big)^{1/p}\\ &\le \Bigg[\lambda[v]_{s,p}^{p-1} + \Big(\int_{\mathbb{R}^N} V(x)|v|^p dx\Big)^{(p-1)/p}\Bigg]\|w\|_\lambda.\end{aligned}$$

Thus, for each $v \in W_\lambda$, the linear functional $\mathcal{L}(v)$ is continuous on W_λ. Hence, the weak convergence of $\{u_n\}_n$ in W_λ gives that

$$\lim_{n\to\infty} \langle \mathcal{L}(u), u_n - u\rangle = 0. \tag{3.7}$$

Without loss of generality, we assume that $\lim_{n\to\infty}\|w_n\|_\lambda = \eta$. Since $\{u_n\}_n$ is a $(PS)_c$ sequence, by the boundedness of $\{u_n\}_n$, (3.3), (3.6) and (3.7), we have

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n) - I'_\lambda(u), u_n - u\rangle\\ &= M(\|u_n\|_\lambda^p)\langle \mathcal{L}(u_n), u_n-u\rangle - M(\|u\|_\lambda^p)\langle \mathcal{L}(u), u_n - u\rangle\\ &\quad - \int_{\mathbb{R}^N} [f(x,u_n) - f(x,u)](u_n-u)dx - \int_{\mathbb{R}^N}\Big[|u_n|^{p_s^\star-2}u_n - |u|^{p_s^\star-2}u\Big](u_n-u)dx\\ &= M(\|u_n\|_\lambda^p)\big[\langle \mathcal{L}(u_n), u_n-u\rangle - \langle \mathcal{L}(u), u_n-u\rangle\big] - \int_{\mathbb{R}^N}|u_n-u|^{p_s^\star}dx + o(1). \end{aligned}\tag{3.8}$$

Here we use the following fact:

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|^{p_s^\star-2}u_n u\,dx = \int_{\mathbb{R}^N}|u|^{p_s^\star}dx, \tag{3.9}$$

thanks to (3.2).

It follows from (3.4), (3.5) and (3.8) that

$$M(\|u_n-u\|_\lambda^p + \|u\|_\lambda^p)\big[\langle \mathcal{L}(u_n), u_n-u\rangle - \langle \mathcal{L}(u), u_n-u\rangle\big] - \int_{\mathbb{R}^N}|u_n-u|^{p_s^\star}dx = o(1). \tag{3.10}$$

From the definition of S, we get

$$\int_{\mathbb{R}^N}|w_n|^{p_s^\star}dx \le S^{-\frac{p_s^\star}{p}}[w_n]_{s,p}^{p_s^\star}.$$

Putting this in (3.10) and using (M) and (2.27) with the case $p \ge 2$, we arrive at the inequality

$$m_0(\|u_n-u\|_\lambda^p + \|u\|_\lambda^p)^{(\theta-1)p}\frac{1}{2^p}\|u_n-u\|_\lambda^p \le S^{-p_s^\star/p}\lambda^{-1}\|u_n-u\|_\lambda^{p_s^\star} + o(1).$$

Letting $n\to\infty$, we have

$$\frac{1}{2^p}m_0(\eta^p + \|u\|_\lambda^p)^{(\theta-1)p}\eta^p \le S^{-p_s^\star/p}\lambda^{-1}\eta^{p_s^\star}.$$

This implies that

$$m_0\eta^{\theta p} \le \lambda^{-1}2^p S^{-p_s^\star/p}\eta^{p_s^\star}. \tag{3.11}$$

Since $\theta = p_s^\star/p$ and $2^p S^{-p_s^\star/p}/m_0 < \lambda$, it follows from (3.11) that $\eta = 0$. Thus, $u_n \to u$ in W_λ. □

Remark 3.1. It seems that the method used in the proof of Lemma 3.2 could not be applied to the case $\theta > p_s^\star/p$.

Proof of Theorem 1.2. Denote by $\{e_1, e_2, \cdots\}$ a basis of W_λ, and for each $k \in \mathbb{N}$ consider $\mathcal{E}_k = \operatorname{span}\{e_1, e_2, \cdots, e_k\}$, the subspace of W_λ generated by $e_1, e_2, \cdots, e_k$. By assumption $p < q < p_s^\star$, we know that $\mathcal{E}_k$ can be continuously embedded into $L^q(\mathbb{R}^N)$. Note that all norms are equivalent on a finite dimensional Banach space. Thus there exists a positive $C(k)$ depending on k such that

$$\|u\|_\lambda^q \le C(k) \int_{\mathbb{R}^N} |u|^q dx,$$

for all $u \in \mathcal{E}_k$. Then by (M_1) and (f_5), we deduce

$$\begin{aligned} I_\lambda(u) &\le \frac{m_1}{p}\|u\|_\lambda^{\theta p} - \frac{1}{p_s^\star}\int_{\mathbb{R}^N} |u(x)|^{p_s^\star} dx - a_0 C(k)\|u\|_\lambda^q \\ &\le \left(\frac{m_1}{p}\|u\|_\lambda^{\theta p - q} - a_0 C(k)\right)\|u\|_\lambda^q, \end{aligned}$$

for all $u \in \mathcal{E}_k$. Let R be a positive constant such that

$$\frac{m_1}{p} R^{\theta p - q} < a_0 C(k).$$

Hence, for all $0 < r < R$, we get

$$I_\lambda(u) \le r^q\left(\frac{m_1}{p} r^{\theta p - q} - a_0 C(k)\right) \le R^q\left(\frac{m_1}{p} R^{\theta p - q} - a_0 C(k)\right) < 0 = \mathcal{I}(0),$$

for all $u \in \mathcal{K} := \{u \in \mathcal{E}_k : \|u\|_\lambda = r\}$. It follows that

$$\sup_{u \in \mathcal{K}} I_\lambda(u) < 0 = I_\lambda(0).$$

Clearly, $\mathcal{E}_k$ and $\mathbb{R}^k$ are isomorphic and $\mathcal{K}$ and $\mathbb{S}^{k-1}$ are homeomorphic. Thus, we conclude that $\gamma(\mathcal{K}) = k$ by Lemma 3.1. Since $f(x, u)$ is odd with respect to $u \in \mathbb{R}$, the functional I_λ is even. Moreover, by (3.1), we know I_λ is bounded from below and satisfies the $(PS)_c$ condition by Lemma 3.2. It follows from Theorem 3.1 that I_λ has at least k pairs of distinct critical points. The arbitrariness of k yields that I_λ has infinitely many pairs distinct critical points in W_λ, that is, problem (1.1) has infinitely many pairs distinct solutions. □

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References

[1] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136, (2012), no. 5, 521–573.
[2] T. Bartsch and Z.-Q. Wang, Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^N$, Commun. Partial Differ. Equ. 20, (1995), no. 9-10, 1725–1741.
[3] H. Brézis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36, (1983), no. 4, 437–477.
[4] R. Servadei and E. Valdinoci, The Brézis–Nirenberg result for the fractional Laplacian, Trans. Amer. Math. Soc. 367, (2015), no. 1, 67–102.
[5] R. Servadei and E. Valdinoci, Fractional Laplacian equations with critical Sobolev exponent, Revista Matemática Complutense 28, (2015), no. 3, 655–676.

[6] X. Ros-Oston and J. Serra, Nonexistence results for nonlocal equations with critical and supercritical nonlinearities, Comm. Partial Differential Equations 40, (2015), no. 1, 115–133.
[7] G. Autuori and P. Pucci, Elliptic problems involving the fractional Laplacian in $\mathbb{R}^N$, J. Differential Equations 255, (2013), no. 8, 2340–2362.
[8] D. Applebaum, Lévy processes–from probability to finance quantum groups, Notices Amer. Math. Soc. 51, (2004), no. 11, 1336–1347.
[9] L. Caffarelli, Non-local diffusions, drifts and games, Nonlinear Partial Differential Equations, Abel Symposia 7 (2012) 37–52.
[10] G. Molica Bisci and D. Repovš, Higher nonlocal problems with bounded potential, J. Math. Anal. Appl. 420, (2014), no. 1, 591–601.
[11] G. Molica Bisci and V. Rădulescu, Ground state solutions of scalar field fractional for Schrödinger equations, Calc. Var. Partial Differential Equations 54, (2015), no. 3, 2985–3008.
[12] G. Molica Bisci and V. Rădulescu, Multiplicity results for elliptic fractional equations with subcritical term, Nonlinear Differential Equations Appl. NoDEA, 22, (2015), no. 4, 721–739.
[13] G. Molica Bisci, V. Rădulescu, R. Servadei, Variational Methods for Nonlocal Fractional Problems, Cambridge University Press, Cambridge, 2016.
[14] N. Laskin, Fractional quantum mechanics and Lévy path integrals, Phys. Lett. A 268, (2000), no. 4-6, 298-305.
[15] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E 66, (2002), 056108.
[16] G. M. Figueiredo, Existence of a positive solution for a Kirchhoff problem type with critical growth via truncation argument, J. Math. Anal. Appl. 401, (2013), no. 2, 706–713.
[17] X. M. He and W. M. Zou, Ground states for nonlinear Kirchhoff equations with critical growth, Ann. Mat. Pura Appl. 193, (2014), no. 2, 473–500.
[18] A. Ourraoui, On a p–Kirchhoff problem involving a critical nonlinearity, C. R. Math. Acad. Sci. Paris Ser. I 352, (2014), no. 4, 295–298.
[19] Y. He, G. B. Li, S. J. Peng, Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, Adv. Nonlinear Stud. 14, (2014), no. 2, 483–510.
[20] J. Liu, J. F. Liao, C. L. Tang, Positive solutions for Kirchhoff-type equations with critical exponent in $\mathbb{R}^N$, J. Math. Anal. Appl. 429, (2015), no. 2, 1153–1172.
[21] S. H. Liang and S. Y. Shi, Soliton solutions to Kirchhoff type problems involving the critical growth in $\mathbb{R}^N$, Nonlinear Anal. 81, (2013), 31–41.
[22] A. Fiscella and E. Valdinoci, A critical Kirchhoff type problem involving a nonlocal operator, Nonliear Anal. 94, (2014), 156–170
[23] P. Pucci, M. Q. Xiang, B. L. Zhang, Existence and multiplicity of entire solutions for fractional p–Kirchhoff equations, Adv. Nonlinear Anal. 5, (2016), no. 1, 27–55.
[24] S. H. Liang, D. Repovš, B. L. Zhang, On the fractional Schrödinger-Kirchhoff equations with electromagnetic fields and critical nonlinearity, Comput. Math. Appl. 75, (2018), no. 5, 1778–1794.
[25] P. Pucci and S. Saldi, Critical stationary Kirchhoff equations in $\mathbb{R}^N$ involving nonlocal operators, Rev. Mat. Iberoam 32, (2016), no. 1, 1–22.
[26] P. Pucci, M. Q. Xiang, B. L. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p–Laplacian in $\mathbb{R}^N$, Calc. Var. Partial Differential Equations 54, (2015), no. 3, 2785–2806.
[27] M. Q. Xiang, B. L. Zhang, M. Ferrara, Existence of solutions for Kirchhoff type problem involving the non–local fractional p–Laplacian, J. Math. Anal. Appl. 424, (2015), no. 2, 1021–1041.
[28] M. Q. Xiang, B. L. Zhang, M. Ferrara, Multiplicity results for the nonhomogeneous fractional p–Kirchhoff equations with concave-convex nonlinearities, Proc. Roy. Soc. A 471, (2015), no. 2177, 14 pp.
[29] X. Mingqi, G. Molica Bisci, G. H. Tian, B. L. Zhang, Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p–Laplacian, Nonlinearity 29, (2016), no. 2, 357–374.
[30] M. Q. Xiang, B. L. Zhang, V. Rădulescu, Multiplicity of solutions for a class of quasilinear Kirchhoff system involving the fractional p–Laplacian, Nonlinearity 29, (2016), no. 10, 3186–3205.
[31] Z. Binlin, A. Fiscella, S. Liang, Infinitely many solutions for critical degenerate Kirchhoff type equations involving the fractional p–Laplacian, Appl. Math. Optim. doi: 10.1007/s00245-017-9458-5.
[32] G. Autuori, A. Fiscella, P. Pucci, Stationary Kirchhoff problems involving a fractional elliptic operator and a critical nonlinearity, Nonlinear Anal. 125, (2015), 699–714.
[33] M. Q Xiang, B. L. Zhang, H. Qiu, Existence of solutions for a critical fractional Kirchhoff type problem in $\mathbb{R}^N$, Sci. China Math. 60, (2017), no. 9, 1647–1660.
[34] M. Q. Xiang, B. L. Zhang, X. Zhang, A nonhomogeneous fractional p–Kirchhoff type problem involving critical exponent in $\mathbb{R}^N$, Adv. Nonlinear Stud. 17, (2017), no. 3, 611–640.
[35] P. L. Lions, The concentration-compactness principle in the calculus of variations, the limit case, Part I. Rev. Mat. Iberoam. 1, (1985), 145–201. [Erratum in Part II, Rev. Mat. Iberoam. 1, (1985), 45–121.]
[36] A. Fiscella and P. Pucci, p–fractional Kirchhoff equations involving critical nonlinearities, Nonlinear Anal. Real World Appl. 35, (2017), 350–378.

[37] J. Byeon and Z.-Q. Wang, Standing waves with a critical frequency for nonlinear Schrődinger equations, Arch. Ration. Mech. Anal. 165, (2002), no. 11-12, 295–316.
[38] D. Cao and E. S. Noussair, Multi-bump standing waves with a critical frequency for nonlinear Schrödinger equations, J. Differ. Equations 203, (2004), no. 2, 292–312.
[39] Y. H. Ding and F. H. Lin, Solutions of perturbed Schrödinger equations with critical nonlinearity, Calc. Var. Partial Differentail Equations 30, (2007), no. 2, 231–249.
[40] Y. H. Ding and J. C. Wei, Semiclassical states for Schrödinger equations with sign-changing potentials, J. Funct. Anal. 251, (2007), no. 2, 546–572.
[41] S. H. Liang and J. H. Zhang, On some p–Laplacian equation with electromagnetic fileds and critical nonlinearity in $\mathbb{R}^N$, J. Math. Phys. 56, (2015), 041504.
[42] Y. Q. Song and S. Y. Shi, Solutions of p–Kirchhoff problems with critical nonlinearity in $\mathbb{R}^N$, J. Nonlonear Sci. Appl. 11, (2018), no. 2, 172–188.
[43] F. Wang and M. Q. Xiang, Multiplicity of solutions for a class of fractional Choquard-Kirchhoff equations involving critical nonlinearity, Anal. Math. Phys. (2017). https://doi.org/10.1007/s13324-017-0174-8.
[44] V. Benci, On critical point theory of indefinite functionals in the presence of symmetries, Trans. Amer. Math. Soc. 274, (1982), no. 2, 533–572.
[45] P. H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence, RI, 1986.
[46] D. C. Clarke, A variant of the Lusternik-Schnirelman theory, Indiana Univ. Math. J. 22, (1972), no. 1, 65–74.