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Anisotropic double-phase problems with indefinite potential: multiplicity of solutions

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Abstract

We consider an anisotropic double-phase problem plus an indefinite potential. The reaction is superlinear. Using variational tools together with truncation, perturbation and comparison techniques and critical groups, we prove a multiplicity theorem producing five nontrivial smooth solutions, all with sign information and ordered. In this process we also prove two results of independent interest, namely a maximum principle for anisotropic double-phase problems and a strong comparison principle for such solutions.

Keywords Anisotropic regularity · Anisotropic maximum principle · Strong comparison · Constant sign and nodal solutions · Critical groups · Double phase

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1 Introduction and origin of double-phase problems

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we deal with the following anisotropic double phase Dirichlet problem

$$\begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) + \xi(z)|u(z)|^{p(z)-2}u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{1}$$

In this problem, we assume that $p, q \in C^1(\overline{\Omega})$ and $1 < q_- \leq q(z) \leq q_+ < p_- \leq p(z) \leq p_+ < p^*(z)$, where $p^*(z) = \frac{Np(z)}{N-p(z)}$ if $p_+ < N$ and $+\infty$ otherwise. The potential function $\xi \in L^\infty(\Omega)$ is sign-changing and so the differential operator (left-hand side) of problem (1) is not coercive. The reaction $f(z, x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$ the mapping $z \mapsto f(z, x)$ is measurable and for a.a. $z \in \Omega$ the function $x \mapsto f(z, x)$ is continuous) which exhibits $(p_+ - 1)$-superlinear growth near $\pm\infty$, but without satisfying the Ambrosetti-Rabinowitz condition (the AR-condition). Using variational tools from the critical point theory, together with truncation, perturbation and comparison techniques and critical groups, we show that the problem has at least five nontrivial smooth solutions, all with sign information and ordered.

The energy functional associated to problem (1) is a *double-phase* variational integral, according to the terminology of Marcellini and Mingione. Problems with unbalanced growth have been studied for the first time by Ball [4,5] in relationship with patterns arising in nonlinear elasticity. More precisely, if Ω is a bounded domain in $\mathbb{R}^N$, $u : \Omega \to \mathbb{R}^N$ is the displacement and if Du is the $N \times N$ matrix of the deformation gradient, then Ball studied the total energy, which can be represented by an integral of the type

$$I(u) = \int_\Omega f(x, Du(x))dx, \tag{2}$$

where the energy function $f = f(x, \xi) : \Omega \times \mathbb{R}^{N\times N} \to \mathbb{R}$ is quasiconvex with respect to ξ. One of the simplest examples considered by Ball is given by functions f of the type

$$f(\xi) = g(\xi) + h(\det \xi),$$

where $\det \xi$ is the determinant of the $N \times N$ matrix ξ, and g, h are nonnegative convex functions, which satisfy the growth conditions

$$g(\xi) \geq c_1 |\xi|^p; \quad \lim_{t\to+\infty} h(t) = +\infty,$$

where c_1 is a positive constant and $1 < p < N$. The condition $p \leq N$ is necessary to study the existence of equilibrium solutions with cavities, that is, minima of the integral (2) that are discontinuous at one point where a cavity forms; in fact, every u with finite energy belongs to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^N)$, and thus it is a continuous function if $p > N$.

The mathematical analysis of double-phase integral functionals has been initiated by Marcellini [20,21]. Marcellini considered continuous functions $f = f(x,u)$ with *unbalanced growth* that satisfy

$$c_1\,|u|^q \le |f(x,u)| \le c_2\,(1+|u|^p) \quad \text{for all } (x,u) \in \Omega \times \mathbb{R},$$

where c_1, c_2 are positive constants and $1 \le q \le p$. These contributions are in relationship with the works of Zhikov [37,38], in order to describe the behavior of phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenisation. These functionals revealed to be important also in the study of duality theory and in the context of the Lavrentiev phenomenon. In particular, Zhikov considered the following three model functional in relation to the Lavrentiev phenomenon:

$$\begin{aligned}
\mathcal{M}(u) &:= \int_\Omega c(x)|\nabla u|^2 dx, \quad 0 < 1/c(\cdot) \in L^t(\Omega),\ t > 1 \\
\mathcal{V}(u) &:= \int_\Omega |\nabla u|^{p(x)} dx, \quad 1 < p(x) < \infty \\
\mathcal{P}_{p,q}(u) &:= \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q) dx, \quad 0 \le a(x) \le L,\ 1 < p < q.
\end{aligned} \tag{3}$$

The functional $\mathcal{M}$ is well-known and there is a loss of ellipticity on the set $\{x \in \Omega;\ c(x) = 0\}$. This functional has been studied at length in the context of equations involving Muckenhoupt weights. The functional $\mathcal{V}$ has also been the object of intensive interest nowadays and a huge literature was developed on it. The energy functional defined by $\mathcal{V}$ was used to build models for strongly anisotropic materials: in a material made of different components, the exponent $p(x)$ dictates the geometry of a composite that changes its hardening exponent according to the point. The functional $\mathcal{P}_{p,q}$ defined in (3) appears as an upgraded version of $\mathcal{V}$. Again, in this case, the modulating coefficient $a(x)$ dictates the geometry of the composite made by two differential materials, with hardening exponents p and q, respectively. The study of non-autonomous functionals characterized by the fact that the energy density changes its ellipticity and growth properties according to the point has been continued in a series of remarkable papers by Mingione *et al.* [6,7,9].

This work continues the recent paper by Papageorgiou, Rădulescu & Repovš [26], where the authors consider parametric equations driven by the $p(z)$-Laplacian plus an indefinite potential term. In the reaction there are the competing effects of a parametric concave term and of a superlinear (convex) perturbation ("concave-convex" problem). The authors focus on positive solutions and they prove a bifurcation-type result describing the changes in the set of positive solutions as the parameter $\lambda > 0$ varies. We also mention the work of Papageorgiou & Vetro [28], who also deal with anisotropic double phase problems with no potential term (that is, $\xi \equiv 0$) and with a superlinear reaction that has a different geometry near zero. They prove a multiplicity theorem producing three nontrivial solutions. However, they do not prove the existence of nodal solutions. Finally, we mention the work of Gasiński & Papageorgiou

[14] on superlinear Neumann problems driven by the $p(z)$-Lapacian. Other anisotropic boundary value problems (including double phase problems) can be found in the book of Rădulescu & Repovš [31] and in the papers of Bahrouni, Rădulescu & Repovš [2,3], Cencelj, Rădulescu & Repovš [8], Papageorgiou, Vetro and Vetro [29], Ragusa and Tachikawa [30], Vetro and Vetro [34], and Zhang & Rădulescu [36].

The features of this paper are the following:

(i) we are concerned with an *anisotropic* model with *double-phase*, namely the problem is driven by *two* differential operators with *variable growth*;

(ii) we develop a refined mathematical analysis (that combines variational and topological methods) in order to study *multiplicity* properties of solutions;

(iii) we establish both a *maximum principle* for anisotropic double-phase problems and a *strong comparison principle* for solutions of anisotropic PDEs with unbalanced growth.

2 Auxiliary results and hypotheses

The study of anisotropic boundary value problems uses variable exponent Lebesgue and Sobolev spaces. A comprehensive presentation of the theory of such spaces can be found in the book of Diening, Harjulehto, Hästo & Ruzička [10].

Let

$$L_1^\infty(\Omega) = \{p \in L^\infty(\Omega) : \ 1 \le \operatorname*{essinf}_\Omega p\}.$$

Given $p \in L_1^\infty(\Omega)$, we define

$$p_- = \operatorname*{essinf}_\Omega p \quad \text{and} \quad p_+ = \operatorname*{esssup}_\Omega p.$$

We also let $M(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable}\}$. We identify two such functions which differ on a Lebesgue null set.

Given $p \in L_1^\infty(\Omega)$, we define the variable exponent Lebesgue space $L^{p(z)}(\Omega)$ by

$$L^{p(z)}(\Omega) = \left\{u \in M(\Omega) : \int_\Omega |u|^{p(z)} dz < \infty\right\}.$$

This space is furnished with the so-called Luxemburg norm defined by

$$\|u\|_{p(z)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|u|}{\lambda}\right)^{p(z)} dz \le 1\right\}.$$

Using these variable exponent Lebesgue spaces, we can define the corresponding variable exponent Sobolev spaces by

$$W^{1,p(z)}(\Omega) = \{u \in L^{p(z)}(\Omega) : \ |Du| \in L^{p(z)}(\Omega)\}.$$

The norm of this space is given by

$$\|u\|_{1,p(z)} = \|u\|_{p(z)} + \|Du\|_{p(z)}.$$

The space $W_0^{1,p(z)}(\Omega)$ is defined to be the $\|\cdot\|_{1,p(z)}$-closure of the compactly supported elements of $W^{1,p(z)}(\Omega)$. If $p \in C^1(\overline{\Omega})$, then

$$W_0^{1,p(z)}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,p(z)}}.$$

When $p_- > 1$, then the spaces $L^{p(z)}(\Omega)$, $W^{1,p(z)}(\Omega)$, $W_0^{1,p(z)}(\Omega)$ are separable and uniformly convex (thus, reflexive too).

The critical Sobolev exponent is defined by

$$p^*(z) = \begin{cases} \frac{Np(z)}{N-p(z)}, & \text{if } p(z) < N \\ +\infty, & \text{if } N \leq p(z). \end{cases}$$

Suppose $p, q \in C(\overline{\Omega})$, $p_+ < N$ and $1 \leq q(z) \leq p^*(z)$ (resp. $1 \leq q(z) < p^*(z)$) for all $z \in \overline{\Omega}$. Then we have

$$\begin{aligned} &W^{1,p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega) \text{ continuously} \\ &(\text{resp., } W^{1,p(z)}(\Omega) \hookrightarrow L^{q(z)(\Omega)} \text{ compactly}). \end{aligned}$$

Let $p, p' \in L_1^\infty(\Omega)$ and assume that $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$ for a.a. $z \in \Omega$. We have $L^{p(z)}(\Omega)^* = L^{p'(z)}(\Omega)$ and the following Hölder-type inequality holds

$$\int_\Omega |uv| dz \leq \left(\frac{1}{p_-} + \frac{1}{p'_-}\right) \|u\|_{p(z)} \|v\|_{p'(z)}$$

for all $u \in W^{1,p(z)}(\Omega)$, $v \in W^{1,p'(z)}(\Omega)$.

When $p \in C^1(\overline{\Omega})$, the Poincaré inequality holds for the space $W_0^{1,p(z)}(\Omega)$, namely there exists $C^* > 0$ such that

$$\|u\|_{p(z)} \leq C^* \|Du\|_{p(z)} \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

The following modular functions are important in the study of these anisotropic spaces:

$$\begin{aligned} \rho_p(u) &= \int_\Omega |u|^{p(z)} dz \text{ for all } u \in L^{p(z)}(\Omega), \\ \rho_p(Du) &= \int_\Omega |Du|^{p(z)} dz \text{ for all } u \in W_0^{1,p(z)}(\Omega). \end{aligned}$$

The next propositions reveal the close relation between these modular functions and the norms of the spaces.

Proposition 1 *If $p \in L_1^\infty(\Omega)$, then the following properties hold:*

(a) for $u \in L^{p(z)}(\Omega)$, $u \neq 0$, we have

$$\|u\|_{p(z)} = \lambda \Leftrightarrow \rho_p\left(\frac{u}{\lambda}\right) = 1;$$

(b) $\|u\|_{p(z)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_p(u) < 1$ *(resp.* $= 1, > 1$*);*
(c) $\|u\|_{p(z)} < 1 \Rightarrow \|u\|_{p(z)}^{p_+} \leq \rho_p(u) \leq \|u\|_{p(z)}^{p_-}$,
$\|u\|_{p(z)} > 1 \Rightarrow \|u\|_{p(z)}^{p_-} \leq \rho_p(u) \leq \|u\|_{p(z)}^{p_+}$*;*
(d) $\|u_n\|_{p(z)} \to 0 \Leftrightarrow \rho_p(u_n) \to 0$*;*
(e) $\|u_n\|_{p(z)} \to +\infty \Leftrightarrow \rho_p(u_n) \to +\infty$.

Proposition 2 *If $p \in C^1(\overline{\Omega})$, then the following properties hold:*

(a) for $u \in W_0^{1,p(z)}(\Omega)$, $u \neq 0$, we have

$$\|u\|_{1,p(z)} = \lambda \Leftrightarrow \rho_p\left(\frac{Du}{\lambda}\right) = 1;$$

(b) $\|u\|_{1,p(z)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho_p(Du) < 1$ *(resp.* $= 1, > 1$*);*
(c) $\|u\|_{1,p(z)} < 1 \Rightarrow \|u\|_{1,p(z)}^{p_+} \leq \rho_p(Du) \leq \|u\|_{1,p(z)}^{p_-}$ *and*
$\|u\|_{1,p(z)} > 1 \Rightarrow \|u\|_{1,p(z)}^{p_-} \leq \rho_p(Du) \leq \|u\|_{1,p(z)}^{p_+}$*;*
(d) $\|u_n\|_{1,p(z)} \to 0 \Leftrightarrow \rho_p(Du_n) \to 0$*;*
(e) $\|u_n\|_{1,p(z)} \to +\infty \Leftrightarrow \rho_p(Du_n) \to +\infty$.

For $p \in C^1(\overline{\Omega})$, we have

$$W_0^{1,p(z)}(\Omega)^* = W^{-1,p'(z)}(\Omega) \quad \left(\frac{1}{p(z)} + \frac{1}{p'(z)} = 1\right).$$

Then we consider the operator $A_p : W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ defined by

$$\langle A_p(u), h\rangle = \int_\Omega |Du|^{p(z)-2}(Du, Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,p(z)}(\Omega).$$

From Gasiński & Papageorgiou [14] (see also Rădulescu & Repovš [31, p. 40]), we have:

Proposition 3 *The map $A_p : W_0^{1,p(z)}(\Omega) \to W^{-1,p'(z)}(\Omega)$ is bounded (that is, it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too) and of type $(S)_+$, that is,*

"$u_n \xrightarrow{w} u$ *in* $W_0^{1,p(z)}(\Omega)$, $\limsup_{n\to\infty}\langle A_p(u_n), u_n - u\rangle \leq 0 \Rightarrow u_n \to u$ *in* $W_0^{1,p(z)}(\Omega)$.

Our hypotheses on the exponents p, q and the potential function ξ are:

H_0: $p, q \in C^1(\overline{\Omega})$, $1 < q_- \leq q(z) \leq q_+ < p_- \leq p(z) \leq p_+ < p^*(z)$ for all $z \in \overline{\Omega}$, $\xi \in L^\infty(\Omega)$.

For every $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$ and then given $u \in W_0^{1,p(z)}(\Omega)$ we define $u^\pm(z) = u(z)^\pm$ for all $z \in \Omega$. We know that

$$u^\pm \in W_0^{1,p(z)}(\Omega),\ u = u^+ - u^-,\ |u| = u^+ + u^-.$$

Given $u, v \in W_0^{1,p(z)}(\Omega)$ with $u \leq v$, we define:

$$[u, v] = \{h \in W_0^{1,p(z)}(\Omega) :\ u(z) \leq h(z) \leq v(z) \text{ for a.a. } z \in \Omega\},$$
$$\mathrm{int}_{C_0^1(\overline{\Omega})}[u, v] = \text{the interior in } C_0^1(\overline{\Omega}) \text{ of } [u, v],$$
$$[u) = \{h \in W_0^{1,p(z)}(\Omega) :\ u(z) \leq h(z) \text{ for a.a. } z \in \Omega\}.$$

A set $S \subseteq W_0^{1,p(z)}(\Omega)$ is said to be "downward directed" (resp., "upward directed"), if for all $u_1, u_2 \in S$, we can find $u \in S$ such that $u \leq u_1$, $u \leq u_2$ (resp., for all $v_1, v_2 \in S$, we can find $v \in S$ such that $v_1 \leq v$, $v_2 \leq v$).

Let X be a Banach space and $\varphi \in C^1(X, \mathbb{R})$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if the following property holds:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that
$\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded,
$(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$,
admits a strongly convergent subsequence" .

For $\varphi(\cdot)$ we define

$$K_\varphi = \{u \in X :\ \varphi'(u) = 0\} \text{ (critical set of } \varphi),$$

and for $c \in \mathbb{R}$, we denote $\varphi^c = \{u \in X :\ \varphi(u) \leq c\}$.

If (Y_1, Y_2) is a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$, for every $k \in \mathbb{N}_0$, we denote by $H_k(Y_1, Y_2)$ the kth relative singular homology group with integer coefficients. Then for $u \in K_\varphi$ isolated and $c = \varphi(u)$, we define the "kth critical group" of $\varphi(\cdot)$ at u, by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}), k \in \mathbb{N}_0,$$

with U a neighborhood of u, such that $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that this definition is independent of the isolating neighborhood U.

The regularity theory for anisotropic problems will lead us to the Banach space

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) :\ u|_{\partial\Omega} = 0\}.$$

This is an ordered Banach space with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega} < 0\right\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$.

In what follows, we denote by $\|\cdot\|$ the norm of the Sobolev space $W_0^{1,p(z)}(\Omega)$. On account of the Poincaré inequality, we have

$$\|u\| = \|Du\|_{p(z)} \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

Next, we will prove two auxiliary results which are actually of independent interest. The first is a strong maximum principle for anisotropic double phase problems. Our result complements the analogous result by Zhang [35]. His conditions on the differential operator do not cover double phase problems (see conditions (3)-(7) in [35]).

So, let $f \in L^\infty(\Omega)$ and consider the following double phase Dirichlet problem

$$-\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = f(z) \text{ in } \Omega,\ u|_{\partial\Omega} = 0. \tag{4}$$

By an "upper solution" (resp., "lower solution") of problem (4), we mean a function $u \in W^{1,p(z)}(\Omega)$ such that $u|_{\partial\Omega} \geq 0$ (resp., $u|_{\partial\Omega} \leq 0$) and

$$\langle A_{p(z)}(u), h\rangle + \langle A_{q(z)}(u), h\rangle \geq \int_\Omega f(z)u(z)dz \text{ (resp., } \leq)$$
$$\text{for all h} \in W_0^{1,p(z)}(\Omega),\ h \geq 0.$$

Proposition 4 *If hypotheses H_0 hold, $u \in C^1(\overline{\Omega})$, $u \neq 0$ is an upper solution for (4) and $u(z) \geq 0$ for all $z \in \overline{\Omega}$, then $u \in \operatorname{int} C_+$.*

Proof First we show that $u(z) > 0$ for all $z \in \Omega$.

Arguing by contradiction, suppose we can find $z_1, z_2 \in \Omega$ and an open ball $B_{2\rho}(z_2)$ such that $z_1 \in \partial B_{2\rho}(z_2)$, $u(z_1) = 0$ and $u|_{B_{2\rho}(z_2)} > 0$.

Let $m = \inf\{u(z) : z \in B_\rho(z_2)\} > 0$. We have

$$u(z_1) = 0,\ Du(z_1) = 0 \text{ and } \frac{m}{\rho} \to 0^+ \text{ as } \rho \to 0^+$$
$$\text{(by l'Hospital's rule)}. \tag{5}$$

We introduce the following items

$$\begin{aligned}
&\Omega_1=\{z\in\Omega:\rho<|z-z_2|<2\rho\},\ q_1=q(z_1),\ a=\sup\{|\nabla p(z)|:\ z\in\Omega_1\}\\
&\eta=8a+2,\ k=-\eta\ln\frac{m}{\rho}+\frac{2(N-1)}{\rho},\\
&v(t)=m\left[\frac{e^{\frac{kt}{q_1-1}}-1}{e^{\frac{k\rho}{q_1-1}}-1}\right]\ \text{for all } t\in[0,\rho].
\end{aligned}$$

We can easily check that

$$\left(\frac{m}{\rho}\right)^3\le v'(t)\le 1 \text{ for all } t\in[0,\rho]. \tag{6}$$

Choose $\rho>0$ small so that

$$\frac{m}{\rho}<1 \text{ (see (5)) and } \frac{q(z)-1}{q_1-1}\ge\frac{1}{2} \text{ for all } z\in\Omega_1.$$

To simplify things, we may take without any loss of generality $z_2=0$. We set $r=|z-z_2|$, $t=2\rho-r$. For $t\in[0,\rho]$ and $r\in[\rho,2\rho]$, we set

$$\begin{aligned}
y(r)&=v(2\rho-r)=v(t),\\
&\Rightarrow y'(r)=-v'(t),\ y''(r)=v''(t).
\end{aligned}$$

From (4) we have

$$\begin{aligned}
&\operatorname{div}\left[|Dy|^{p(z)-2}Dy+|Dy|^{q(z)-2}Dy\right]+f(z)\\
&=(p(z)-1)(v'(t))^{p(z)-1}v''(t)-\frac{N-1}{r}(v'(t))^{p(z)-1}\\
&\quad-(v'(t))^{p(z)-1}\ln v'(t)\sum_{k=1}^{N}\frac{\partial p}{\partial z_k}\frac{z_k}{r}\\
&\quad+(q(z)-1)(v'(t))^{q(z)-1}v''(t)-\frac{N-1}{r}(v'(t))^{q(z)-1}\\
&\quad-(v'(t))^{q(z)-1}\ln v'(t)\sum_{k=1}^{N}\frac{\partial q}{\partial z_k}\frac{z_k}{r}\\
&\quad+f(z)\\
&\ge 2v(t)^{p(z)}\left[\frac{1}{2}k+M\ln v'(t)-\frac{N-1}{r}\right]\\
&\quad+f(z)\ \text{(see (6) and recall that } q(\cdot)<p(\cdot))
\end{aligned}$$

$$\geq -\ln \frac{m}{\rho} v'(t)^{p(z)-1} + f(z) \geq 0 \text{ for } \rho > 0 \text{ small,}$$
$$\Rightarrow y(\cdot) \text{ is a lower solution of (4) on } \Omega_1.$$

Note that $y \leq u$ on $\partial\Omega_1$. So, by Lemma 2.3 of Zhang [35] we have that

$$y(|z|) \leq u(z) \text{ for all } z \in \Omega_1.$$

Hence we have

$$\begin{aligned} &\lim_{\mu\to 0^+} \frac{u(z_1+\mu(z_2-z_1))-u(z_1)}{\mu} \\ &\quad \geq \lim_{\mu\to 0^+} \frac{y(|z_1+\mu(z_2-z_1)|)-y(|z_1|)}{\mu} = v'(0) > 0 \end{aligned} \tag{7}$$

which contradicts (5). Therefore we infer that

$$u(z) > 0 \text{ for all } z \in \Omega.$$

Next, let $z_1 \in \partial\Omega$ and let $\rho > 0$ be small. We set $z_2 = z_1 - 2\rho n(z_1)$ and have

$$B_{2\rho}(z_2) \subseteq \Omega \text{ and } z_1 \in \partial B_{2\rho}(z_2).$$

Let $\Omega_1' = \{z \in \Omega : \rho < |z - z_2| < 2\rho\}$ and choose $0 < \beta < \inf\{u(z) : z \in \partial B_\rho(z_2)\}$ small. From the first part of the proof, we know that there exists a lower solution $y \in C^1(\overline{\Omega}_1') \cap C^2(\Omega_1')$ of (4) such that $y \leq u$ in Ω_1, $y(z_1) = 0$ and $\frac{\partial u}{\partial n}(z_1) \leq \frac{\partial y}{\partial n}(z_1) < 0$ (see (7)). We conclude that $u \in \operatorname{int} C_+$. □

The second auxiliary result is a strong comparison principle which complements Proposition 2.4 of Papageorgiou, Rădulescu & Repovš [26] and extends to anisotropic problems Proposition 2.10 of Papageorgiou, Rădulescu & Repovš [24].

In what follows, we denote by D_+ the following open cone in $C^1(\overline{\Omega})$:

$$D_+ = \left\{ u \in C^1(\overline{\Omega}) : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n}|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Proposition 5 *If hypotheses H_0 hold, $\xi, h, g \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$*

$$0 < \eta \leq g(z) - h(z) \text{ for a.a. } z \in \Omega,$$

and $u, v \in C^1(\overline{\Omega})$ satisfy $u \leq v$ on $\overline{\Omega}$ and

$$\begin{aligned} -\Delta_{p(z)}u - \Delta_{q(z)}u + \xi(z)|u|^{p(z)-2}u &= h(z) \text{ in } \Omega \\ -\Delta_{p(z)}v - \Delta_{q(z)}v + \xi(z)|v|^{p(z)-2}v &= g(z) \text{ in } \Omega, \end{aligned}$$

then $v - u \in D_+$.

Proof Let $y = v - u$. Then $y \in C^1(\overline{\Omega})$, $y \geq 0$. Also let $A(z) = (a_{ij}(z))_{i,j=1}^N$ be the $N \times N$ matrix with entries defined by

$$\begin{aligned} a_{ij}(z) = \int_0^1 [(1-t)Du(z) + tDv(z)] \Big[\delta_{ij} + (p(z) - 2) \\ \times \frac{D_i((1-t)u + tv)D_j((1-t)u + tv)}{|(1-t)Du + tDv|^2} \\ +(q(z) - 2) \frac{D_i((1-t)u + tv)D_j((1-t)u + tv)}{|(1-t)Du + tDv|^2} \Big] dz \end{aligned}$$

with δ_{ij} being the Kronecker symbol, that is, $\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$

Then $a_{ij} \in W^{1,\infty}(\Omega)$ and by the mean value theorem we have

$$-\operatorname{div}(A(z)Dy) = g(z) - h(z) - \xi(z)\left[|v|^{p(z)-2}v - |u|^{p(z)-2}u \right] \text{ in } \Omega \tag{8}$$

(see also Guedda & Véron [16]).

Suppose that there exists $z_0 \in \Omega$ such that $u(z_0) = v(z_0)$. Hence $y(z_0) = 0$. From our hypotheses and since the function $(z, x) \mapsto |x|^{p(z)-2}x$ is uniformly continuous on $\overline{\Omega} \times \mathbb{R}$, we see that we can find $\delta > 0$ small such that

$$g(z) - h(z) - \xi(z) \left| |v(z)|^{p(z)-2}v(z) - |u(z)|^{p(z)-2}u(z) \right| \geq \frac{\eta}{2} > 0$$

for a.a. $z \in B_\delta(z_0) = \{z \in \Omega : \ |z - z_0| < \delta\}$. Then from (8) we have

$$\begin{aligned} -\operatorname{div}(A(z)Dy(z)) \geq \frac{\eta}{2} > 0 \text{ for a.a. } z \in B_\delta(z_0), \\ \Rightarrow y(z) > 0 \text{ for all } z \in B_\delta(z_0) \\ \text{(see Theorem 4 of Vazquez [33]),} \end{aligned}$$

a contradiction since $y(z_0) = 0$. So, we have

$$y(z) = v(z) - u(z) > 0 \text{ for all } z \in \Omega.$$

Let $K_0 = \{z \in \partial\Omega : \ y(z) = 0\}$. If $K_0 \neq \emptyset$, then by Proposition 4, we have

$$\begin{aligned} \frac{\partial y}{\partial n}(z_0) < 0 \text{ for all } z_0 \in K_0, \\ \Rightarrow y = v - u \in D_+. \end{aligned}$$

The proof is now complete. □

Now we introduce the hypotheses on the reaction $f(z, x)$.

H : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leq a(z)[1+|x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p_+ < r < p^*(z)$;

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then $\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^{p_+}} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) if $\beta(z,x) = f(z,x)x - p_+F(z,x)$, then there exists $\eta \in L^1(\Omega)$ such that

$$\beta(z,x) \leq \beta(z,y) + \eta(z) \text{ for a.a. } z \in \Omega,\ \text{all } 0 \leq x \leq y \text{ or } y \leq x \leq 0;$$

(iv) $\lim\limits_{x\to 0} \dfrac{f(z,x)}{|x|^{q_- -2}x} = +\infty$ uniformly for a.a. $z \in \Omega$ and there exists $1 < \tau < q_-$ such that

$$0 \leq \liminf_{x\to 0} \frac{\tau F(z,x) - f(z,x)x}{|x|^{p_+}} \text{ uniformly for a.a. } z \in \Omega;$$

(v) we can find $C_0, \hat{C} > 0$ such that

$$f(z,C_0) - \xi(z)C_0^{p(z)-1} \leq -\vartheta_+ < 0 < \vartheta_- \leq f(z,-\hat{C}) + \xi(z)\hat{C}^{p(z)-1}$$

for a.a. $z \in \Omega$.

Remark 1 Hypotheses $H(ii)$, (iii) imply that $f(z,\cdot)$ is $(p_+ - 1)$ superlinear. We point out that we do not use the AR-condition, which is common in the literature when dealing with superlinear equations. Instead we use the quasimonotonicity hypothesis $H(iii)$ on $\beta(z,\cdot)$. This assumption is a slight generalization of the condition used by Li & Yang [19]. Similar conditions were used by Mugnai & Papageorgiou [22] (isotropic problems) and by Papageorgiou, Rădulescu & Repovš [26,27], Papageorgiou & Vetro [28] (anisotropic problems). With this condition we incorporate in our framework also superlinear functions with "slower" growth near $\pm\infty$ which fail to satisfy the AR-condition. For example, consider the function with the exponents $p, q \in C^1(\overline{\Omega})$

$$f(z,x) = \begin{cases} |x|^{p_+-2}x\ln|x| + C - 1, & \text{if } x < -1 \\ |x|^{\tau-2}x - C|x|^{s-2}x, & \text{if } -1 \leq x \leq 1 \\ |x|^{p_+-2}x\ln|x| + 1 - C, & \text{if } 1 < x \end{cases}$$

with $1 < \tau < q_- < p_+ \leq s$ and $1 - \operatorname*{essinf}_\Omega \xi < C$. This function satisfies hypotheses H but fails to satisfy the AR-condition (see [1]). Hypotheses $H(iv)$, (v) imply that $f(z,\cdot)$ near zero has a kind of oscillatory behavior. Finally, we mention that another superlinearity condition for anisotropic equations was used by Gasiński & Papageorgiou [14].

3 Constant sign solutions

In this section we produce constant sign solutions.

We first produce two constant solutions. One solution is positive in the order interval $[0, C_0]$ and the other solution is negative in the order interval $[-\hat{C}, 0]$. To produce these two solutions, we do not need the hypotheses concerning the asymptotic behavior of $f(z,\cdot)$ (hypotheses $H(ii)$, (iii)).

Proposition 6 *If hypotheses H_0 and $H(i)$, (iv), (v) hold, then problem (1) admits two constant sign solutions*

$$u_0 \in [0, C_0] \cap \operatorname{int} C_+ \text{ and } v_0 \in [-\hat{C}, 0] \cap (-\operatorname{int} C_+).$$

Proof First we produce the positive solution.

Let $\vartheta > \|\xi\|_\infty$ and consider the following truncation perturbation of $f(z,\cdot)$:

$$\hat{f}_+(z,x) = \begin{cases} f(z,x^+) + \vartheta(x^+)^{p(z)-1}, & \text{if } x \le -C_0 \\ f(z,C_0) + \vartheta C_0^{p(z)-1}, & \text{if } -C_0 < x. \end{cases} \tag{9}$$

This is a Carathéodory function. We set $\hat{F}_+(z,x) = \int_0^x \hat{f}_+(z,s)ds$ and consider the C^1-functional $\hat{\varphi}_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}\hat{\varphi}_+(u) = &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz \\ &+ \int_\Omega \frac{[\xi(z)+\vartheta]}{p(z)}|u|^{p(z)}dz - \int_\Omega \hat{F}_+(z,u)dz\end{aligned}$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

From (9), the Poincaré inequality and since $\vartheta > \|\xi\|_\infty$, we infer that $\hat{\varphi}_+(\cdot)$ is coercive. Also, from the anisotropic Sobolev embedding theorem (see Section 2), we see that $\hat{\varphi}_+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\varphi}_+(u_0) = \inf\{\hat{\varphi}_+(u) : \ u \in W_0^{1,p(z)}(\Omega)\}. \tag{10}$$

On account of hypothesis $H(iv)$, given any $\eta > 0$, we can find $\delta = \delta(\eta) \in (0, C_0)$ such that

$$F(z,x) \ge \frac{\eta}{q_-}|x|^{q_-} \text{ for a.a. } z \in \Omega, \text{ all } |x| \le \delta. \tag{11}$$

Let $u \in \operatorname{int} C_+$ and choose $t \in (0,1)$ small such that $tu(z) \in [0,\delta]$ for all $z \in \overline{\Omega}$, $t\|u\| \le 1$ and $t\|u\|_{1,q(z)} \le 1$. We have

$$\hat{\varphi}_+(tu) \le \frac{C_1 t^{p_-}}{p_-}\|u\|^{p_-} + \frac{t^{q_-}}{q_-}\|Du\|_{q(z)}^{q_-} - \frac{\eta}{q_-}t^{q_-}\|u\|_{q(z)}^{q_+}$$
for some $C_1 > 0$ (see (9),(11) and hypotheses H_0).

Since $\eta > 0$ is arbitrary, choosing $\eta > 0$ big, we have

$$\begin{aligned} & \hat{\varphi}_+(tu) < 0 \\ \Rightarrow\ & \hat{\varphi}_+(u_0) < 0 = \hat{\varphi}_+(0) \text{ (see (10))}, \\ \Rightarrow\ & u_0 \neq 0. \end{aligned}$$

From (10) we have

$$\begin{aligned} & \hat{\varphi}'_+(u_0) = 0, \\ & \Rightarrow \langle A_{p(z)}(u_0), h\rangle + \langle A_{q(z)}(u_0), h\rangle + \int_\Omega [\xi(z) + \vartheta]|u_0|^{p(z)-2}u_0 h dz \\ & \quad = \int_\Omega \hat{f}_+(z, u_0) h dz \end{aligned} \tag{12}$$

for all $h \in W_0^{1,p(z)}(\Omega)$.

In (12) we choose $h = -u_0^- \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} & \int_\Omega |Du_0^-|^{p(z)} dz + \int_\Omega |Du_0^-|^{q(z)} dz + \int_\Omega [\xi(z) + \vartheta](u_0^-)^{p(z)} dz = 0 \text{ (see (9))}, \\ & \Rightarrow u_0 \geq 0,\ u_0 \neq 0 \text{ (recall that } \vartheta > \|\xi\|_\infty). \end{aligned}$$

Also, in (12) we choose $(u_0 - C_0)^+ \in W_0^{1,p(z)}(\Omega)$. Then

$$\begin{aligned} & \int_\Omega |D(u_0 - C_0)^+|^{p(z)} dz + \int_\Omega |D(u_0 - C_0)^+|^{q(z)} dz + \int_\Omega [\xi(z) \\ & \quad + \vartheta] u_0^{p(z)-1}(u_0 - C_0)^+ dz \\ & \quad = \int_\Omega [f(z, C_0) + \vartheta C_0^{p(z)-1}](u_0 - C_0)^+ dz \text{ (see (9))} \\ & \leq \int_\Omega [\xi(z) + \vartheta] C_0^{p(z)-1}(u_0 - C_0)^+ dz \text{ (see hypothesis } H(v)), \\ & \quad \Rightarrow \int_\Omega [\xi(z) + \vartheta][u_0^{p(z)-1} - C_0^{p(z)-1}](u_0 - C_0)^+ dz \leq 0, \\ & \Rightarrow u_0 \leq C_0 \text{ (since } \vartheta > \|\xi\|_\infty). \end{aligned}$$

Therefore we have proved that

$$u_0 \in [0, C_0],\ \ u_0 \neq 0. \tag{13}$$

From (13), (9) and (12), we have

$$-\Delta_{p(z)} u_0(z) - \Delta_{q(z)} u_0(z) + \xi(z) u_0(z)^{p(z)-1} = f(z, u_0(z)) \text{ in } \Omega. \tag{14}$$

From Fan & Zhao [12] (see also Gasiński & Papageorgiou [14]), we have $u_0 \in L^\infty(\Omega)$. Then Theorem 1.3 of Fan [11] implies that $u_0 \in C_+ \setminus \{0\}$ and so applying Proposition 4, we conclude that $u_0 \in \operatorname{int} C_+$.

For the negative solution, we consider the following truncation perturbation of $f(z, \cdot)$:

$$\hat{f}_-(z, x) = \begin{cases} f(z, -\hat{C}) - \vartheta \hat{C}^{p(z)-1}, & \text{if } x < -\hat{C} \\ f(z, -x^-) - \vartheta (x^-)^{p(z)-1}, & \text{if } -\hat{C} \leq x \end{cases} \quad (\vartheta > \|\xi\|_\infty).$$

This is a Carathéodory function. We set $\hat{F}_-(z, x) = \int_0^x \hat{f}_-(z, s) ds$ and consider the C^1-functional $\hat{\varphi}_- : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \hat{\varphi}_-(u) = & \int_\Omega \frac{1}{p(z)} |Du|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |Du|^{q(z)} dz \\ & + \int_\Omega \frac{[\xi(z) + \vartheta]}{p(z)} |u|^{p(z)} dz - \int_\Omega \hat{F}_-(z, u) dz \end{aligned}$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

Working as above, using this time $\hat{\varphi}_-(\cdot)$, we produce a negative solution

$$v_0 \in [-\hat{C}, 0] \cap (-\operatorname{int} C_+).$$

The proof is now complete. $\square$

By introducing an extra mild condition on $f(z, \cdot)$, we can improve the conclusion of the previous proposition. With this stronger conclusion, we will be able to produce in the sequel additional constant sign solutions.

The new conditions on the reaction $f(z, x)$ are the following:

H': $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$, hypotheses $H'(i) \to (v)$ are the same as the corresponding hypotheses $H(i) \to (v)$ and

(vi) for every $\rho > 0$, there exists $\hat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$, the function

$$x \mapsto f(z, x) + \hat{\xi}_\rho |x|^{p(z)-2} x$$

is nondecreasing on $[-\rho, \rho]$.

Using this perturbed monotonicity condition on $f(z, \cdot)$, we obtain the following improved version of Proposition 6.

Proposition 7 *If hypotheses H_0, H' hold, then problem (1) admits two constant sign solutions*

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [0, C_0] \text{ and } v_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [-\hat{C}, 0].$$

Proof From Proposition 6, we already have two solutions

$$u_0 \in [0, C_0] \cap \operatorname{int} C_+ \text{ and } v_0 \in [-\hat{C}, 0] \cap (-\operatorname{int} C_+). \tag{15}$$

Let $\rho = C_0$ and let $\hat{\xi}_\rho > 0$ be as postulated by hypothesis $H'(vi)$. Clearly we can always have $\hat{\xi}_\rho > \|\xi\|_\infty$. Then

$$\begin{aligned}
&-\Delta_{p(z)} u_0 - \Delta_{q(z)} u_0 + [\xi(z) + \hat{\xi}_\rho] u_0^{p(z)-1} \\
&= f(z, u_0) + \hat{\xi}_\rho u_0^{p(z)-1} \\
&\leq f(z, C_0) + \hat{\xi}_\rho C_0^{p-1} \text{ (see (15) and hypothesis } H'(vi)) \\
&\leq [\xi(z) + \hat{\xi}_\rho] C_0^{p-1} - \vartheta_+ \\
&\leq -\Delta_{p(z)} C_0 - \Delta_{q(z)} C_0 + [\xi(z) + \hat{\xi}_\rho] C_0^{p-1} \text{ in } \Omega, \\
\Rightarrow\ & u_0(z) < C_0 \text{ for all } z \in \overline{\Omega} \text{ (see Proposition 5)}, \\
\Rightarrow\ & u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [0, C_0].
\end{aligned}$$

Similarly we show that $v_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [-\hat{C}, 0]$. □

We will use these two solutions $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$, in order to produce two more constant sign smooth solutions localized with respect to u_0 and v_0 respectively.

Proposition 8 *If hypotheses H_0, H' hold, then problem (1) admits two more constant sign solutions*

$$\begin{aligned}
&\hat{u} \in \operatorname{int} C_+, u_0 \leq \hat{u}, u_0 \neq \hat{u}, \\
&\hat{v} \in -\operatorname{int} C_+, \hat{v} \leq v_0, v_0 \neq \hat{v}.
\end{aligned}$$

Proof Let $u_0 \in \operatorname{int} C_+$ and $v_0 \in -\operatorname{int} C_+$ be the two constant sign solutions from Proposition 6. From Proposition 7 we have

$$u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [0, C_0] \text{ and } v_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})} [-\hat{C}, 0]. \tag{16}$$

First we will produce the second positive solution. To this end, we introduce the following truncation perturbation of $f(z, \cdot)$:

$$g_+(z, x) = \begin{cases} f(z, u_0(z)) + \vartheta u_0(z)^{p(z)-1}, & \text{if } x \leq u_0(z) \\ f(z, x) + \vartheta x^{p-1}, & \text{if } u_0(z) < x \end{cases} \quad (\vartheta > \|\xi\|_\infty). \tag{17}$$

This is a Carathéodory function. We set $G_+(z, x) = \displaystyle\int_0^x g_+(z, s) ds$ and consider the C^1-functional $\Psi_+ : W_0^{1, p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\Psi_+(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{[\xi(z)+\vartheta]}{p(z)}|u|^{p(z)}dz - \int_\Omega G_+(z,u)dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

Claim 1: Ψ_+ satisfies the C-condition.

Let $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(z)}(\Omega)$ be a sequence such that

$$|\Psi_+(u_n)| \leq C_2 \text{ for some } C_2 > 0, \text{ all } n \in \mathbb{N} \tag{18}$$

$$(1+\|u_n\|)\Psi_+'(u_n) \to 0 \text{ in } W^{-1,p'(z)}(\Omega) = (W_0^{1,p(z)}(\Omega))^* \text{ as } n \to \infty. \tag{19}$$

From (19) we have

$$\left|\langle A_{p(z)}(u_n)h\rangle + \langle A_{q(z)}(u_n),h\rangle + \int_\Omega [\xi(z)+\vartheta]|u_n|^{p(z)-2}u_n h dz - \int_\Omega g_+(z,u_n)hdz\right| \leq \frac{\varepsilon_n\|h\|}{1+\|u_n\|}$$
$$\text{for all } h \in W_0^{1,p(z)}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{20}$$

In (20) we choose $h = -u_n^- \in W_0^{1,p(z)}(\Omega)$. Then

$$\int_\Omega |Du_n^-|^{p(z)}dz + \int_\Omega |Du_n^-|^{q(z)}dz + \int_\Omega [\xi(z)+\vartheta](u_n^-)^{p(z)}dz \leq C_3$$
$$\text{for some } C_3 > 0, \text{ all } n \in \mathbb{N} \text{ (see (15))},$$
$$\Rightarrow \{u_n^-\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded}$$
$$\text{(recall that } \vartheta > \|\xi\|_\infty \text{ and see Proposition 2).} \tag{21}$$

In (20) we choose $h = u_n^+ \in W_0^{1,p(z)}(\Omega)$ and using (17), we have

$$-\int_\Omega |Du_n^+|^{p(z)}dz - \int_\Omega |Du_n^+|^{q(z)}dz - \int_\Omega [\xi(z)+\vartheta](u_n^+)^{p(z)}dz + \int_\Omega f(z,u_n^+)u_n^+dz \leq C_4$$
$$\text{for some } C_4 > 0, \text{ all } n \in \mathbb{N}. \tag{22}$$

From (18) and (21) we have

$$\frac{1}{p_+}\Big[\int_\Omega |Du_n^+|^{p(z)}dz + \int_\Omega |Du_n^+|^{q(z)}dz + \int_\Omega [\xi(z)+\vartheta](u_n^+)^{p(z)}dz\Big] - \int_\Omega F(z,u_n^+)dz \leq C_5$$
$$\text{for some } C_5 > 0, \text{ all } n \in \mathbb{N} \text{ (see (17))},$$

$$\Rightarrow \int_\Omega |Du_n^+|^{p(z)}dz + \int_\Omega |Du_n^+|^{q(z)}dz + \int_\Omega [\xi(z)+\vartheta](u_n^+)^{p(z)}dz$$
$$- \int_\Omega p_+ F(z,u_n^+)dz \le p_+ C_5$$
$$\text{for all } n \in \mathbb{N}. \tag{23}$$

We add (22) and (23) and obtain

$$\int_\Omega [f(z,u_n^+)u_n^+ - p_+ F(z,u_n^+)]dz \le C_6 \tag{24}$$

for some $C_6 < 0$, all $n \in \mathbb{N}$.

Using (24) we will show that $\{u_n^+\}_{n\ge 1} \subseteq W_0^{1,p(z)}(\Omega)$ is bounded. Arguing by contradiction, we assume that at least for a subsequence we have

$$\|u_n^+\| \to \infty \text{ as } n \to \infty. \tag{25}$$

Let $y_n = \frac{u_n^+}{\|u_n^+\|}$ for $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $\mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega) \text{ as } n \to \infty,\ \ y \ge 0. \tag{26}$$

Let $\Omega_+ = \{z \in \Omega : \ y(z) > 0\}$. First we assume that $|\Omega_+|_N > 0$ (by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$). We have

$$u_n^+(z) \to +\infty \text{ for a.a. } z \in \Omega_+,$$
$$\Rightarrow \frac{F(z,u_n^+(z))}{u_n^+(z)^{p_+}} \to +\infty \text{ for a.a. } z \in \Omega_+$$
$$\text{(see hypothesis } H'(ii)),$$
$$\Rightarrow \int_{\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz \to +\infty \text{ (by Fatou's lemma)}. \tag{27}$$

On account of hypotheses $H'(i)$, (ii), we have

$$F(z,x) \ge -C_7 \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{some } C_7 > 0. \tag{28}$$

We have

$$\int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz = \int_{\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz + \int_{\Omega\setminus\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz$$
$$\ge \int_{\Omega_+} \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz - \frac{C_7|\Omega|_N}{\|u_n^+\|^{p_+}} \text{ (see (28))},$$
$$\Rightarrow \lim_{n\to\infty} \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz = +\infty \text{ (see (27) and (25))}. \tag{29}$$

From (18), (17) and (21), we have

$$
\begin{aligned}
& -\frac{1}{p_+}\Big[\int_\Omega \frac{1}{\|u_n^+\|^{p_+-p(z)}}|Dy_n|^{p(z)} + \int_\Omega \frac{1}{\|u_n\|^{p_+-q(z)}}|Dy_n|^{q(z)}dz \\
& \quad + \int_\Omega \frac{1}{\|u_n^+\|^{p_+-p(z)}}[\xi(z)+\vartheta]y_n^{p(z)}dz\Big] + \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz \le C_8 \\
& \quad \text{for some } C_8>0, \ \text{all } n\in\mathbb{N} \\
\Rightarrow \quad & \int_\Omega \frac{F(z,u_n^+)}{\|u_n^+\|^{p_+}}dz \le C_9 \text{ for some } C_9>0, \ \text{all } n\in\mathbb{N} \\
& \quad (\text{since } q_+ < p(z) \le p_+ \text{ for all } z\in\overline{\Omega}).
\end{aligned} \tag{30}
$$

Comparing (29) and (28), we have a contradiction.
Next we assume that $|\Omega_+|_N = 0$. Then $y \equiv 0$.
Let $t_n \in [0,1]$ be such that

$$
\Psi_+(t_n u_n^+) = \max\{\Psi_+(tu_n^+) : \ 0 \le t < 1\}, \ n\in\mathbb{N}. \tag{31}
$$

For $k > 1$, we set $v_n = k^{1/p_-} y_n$ for all $n \in \mathbb{N}$. We have

$$
\begin{aligned}
& v_n \xrightarrow{w} 0 \text{ in } W_0^{1,p(z)}(\Omega) \text{ as } n\to\infty \\
& (\text{see (26) and recall that } y=0).
\end{aligned} \tag{32}
$$

From (26) it follows that

$$
\int_\Omega \frac{1}{p(z)}[\xi(z)+\vartheta]v_n^{p(z)}dz \to 0 \text{ and } \int_\Omega G_+(z,v_n)dz \to 0 \text{ as } n\to\infty. \tag{33}
$$

On account of (25), we see that we can find $n_0 \in \mathbb{N}$ such that

$$
\frac{k^{1/p_-}}{\|u_n^+\|} \in (0,1] \text{ for all } n \ge n_0. \tag{34}
$$

From (31) and (34), we have

$$
\begin{aligned}
\Psi_+(t_n u_n^+) &\ge \Psi_+(v_n) \\
&= \int_\Omega \frac{1}{p(z)}|Dv_n|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Dv_n|^{q(z)}dz + \int_\Omega \frac{[\xi(z)+\vartheta]}{p(z)}|v_n|^{p(z)}dz \\
&\quad - \int_\Omega G_+(z,v_n)dz \text{ for all } n \ge n_0, \\
&\ge \frac{1}{p_+}k + \int_\Omega \frac{[\xi(z)+\vartheta]}{p(z)}v_n^{p(z)}dz - \int_\Omega G_+(z,v_n)dz \\
&\quad (\text{since } k>1, \ \|y_n\|=1 \text{ and by Proposition 2}) \\
&\ge \frac{1}{2p_+}k \text{ for all } n \ge n_1 \ge n_0 \text{ (see (33))}.
\end{aligned}
$$

But $k > 1$ is arbitrary. So, we infer that

$$\Psi_+(t_n u_n^+) \to +\infty \text{ as } n \to \infty. \tag{35}$$

We have

$$\begin{aligned}&\Psi_+(0) = 0 \text{ and } \Psi_+(u_n^+) \leq C_{10} \text{ for some } C_{10} > 0, \text{ all } n \in \mathbb{N}\\ &\text{(see (18) and (21)).}\end{aligned} \tag{36}$$

From (35) and (36) it follows that

$$t_n \in (0, 1) \text{ for all } n \geq n_2. \tag{37}$$

From (31) and (37), we have

$$\begin{aligned}&t_n \frac{d}{dt}\Psi_+(tu_n^+)|_{t=t_n} = 0,\\ &\Rightarrow \langle \Psi_+'(t_n u_n^+), t_n u_n^+\rangle = 0 \text{ for all } n \geq n_2 \text{ (use the chain rule).}\end{aligned} \tag{38}$$

Then for all $n \geq n_2$, we have

$$\begin{aligned}
&\Psi_+(t_n u_n^+)\\
&= \Psi_+(t_n u_n^+) - \frac{1}{p_+}\langle \Psi_+'(t_n u_n^+), t_n u_n^+\rangle \text{ (see (38))}\\
&\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right] |D(t_n u_n^+)|^{p(z)} dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right] |D(t_n u_n^+)|^{q(z)} dz\\
&+ \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right] [\xi(z) + \vartheta](t_n u_n^+)^{p(z)} dz + \frac{1}{p_+}\int_\Omega \beta(z, t_n u_n^+) dz + C_{11}\\
&\quad \text{for some } C_{11} > 0 \text{ (see (17))}\\
&\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right] |Du_n^+|^{p(z)} dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p_+}\right] |Du_n^+|^{q(z)} dz\\
&+ \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p_+}\right] [\xi(z) + \vartheta](u_n^+)^{p(z)} dz + \frac{1}{p_+}\int_\Omega \beta(z, u_n^+) dz + C_{12}\\
&\quad \text{for some } C_{12} > 0 \text{ (see hypothesis } H'(iii) \text{ and recall that } t_n \leq 1)\\
&\leq \Psi_+(u_n^+) + C_{13} \text{ for some } C_{13} > 0 \text{ (see (24))},\\
\Rightarrow \quad &\Psi_+(u_n^+) \to +\infty \text{ (see (35))}
\end{aligned}$$

which contradicts (36). This means that

$$\begin{aligned}&\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded}\\ \Rightarrow\ &\{u_n\}_{n\geq 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded (see (21)).}\end{aligned}$$

We may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{39}$$

In (20) we choose $h = u_n - u \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (39). Then

$$\begin{aligned}
&\lim_{n\to\infty} \left[\langle A_{p(z)}(u_n), u_n - u\rangle + \langle A_{q(z)}(u_n), u_n - u\rangle\right] = 0,\\
\Rightarrow\ &\limsup_{n\to\infty} \left[\langle A_{p(z)}(u_n), u_n - u\rangle + \langle A_{q(z)}(u), u_n - u\rangle\right] \le 0\\
&\text{(since } A_{q(z)}(\cdot) \text{ is monotone)},\\
\Rightarrow\ &\limsup_{n\to\infty} \langle A_{p(z)}(u_n), u_n - u\rangle \le 0 \text{ (see (39))},\\
\Rightarrow\ &u_n \to u \text{ in } W_0^{1,p(z)}(\Omega) \text{ (see Proposition 3)},\\
\Rightarrow\ &\Psi_+(\cdot) \text{ satisfies the } C\text{-condition.}
\end{aligned}$$

This proves Claim 1.

On account of hypothesis $H'(ii)$, for every $u \in \operatorname{int} C_+$, we have that

$$\Psi_+(tu) \to -\infty \text{ as } t \to +\infty. \tag{40}$$

Claim 2: $K_{\Psi_+} \subseteq [u_0) \cap \operatorname{int} C_+$.

Let $u \in K_{\Psi_+}$. Then

$$\langle A_{p(z)}(u), h\rangle + \langle A_{q(z)}(u), h\rangle + \int_\Omega [\xi(z) + \vartheta]|u|^{p(z)-2}uh dz = \int_\Omega g_+(z, u) dz \tag{41}$$

for all $h \in W_0^{1,p(z)}(\Omega)$.

In (41) we choose $h = (u_0 - u)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned}
&\langle A_{p(z)}(u), (u_0 - u)^+\rangle + \langle A_{q(z)}(u), (u_0 - u)^+\rangle\\
&\quad + \int_\Omega [\xi(z) + \vartheta]|u|^{p-2}u(u_0 - u)^+ dz\\
&= \int_\Omega [f(z, u_0) + \vartheta u_0^{p-1}](u_0 - u)^+ dz \text{ (see (17))}\\
&= \langle A_{p(z)}(u_0), (u_0 - u)^+\rangle + \langle A_{q(z)}(u_0), (u_0 - u)^+\rangle\\
&\quad + \int_\Omega [\xi(z) + \vartheta]u_0^{p(z)-1}(u_0 - u)^+ dz,\\
\Rightarrow\ &u_0 \le u \text{ (since } \vartheta > \|\xi\|_\infty).
\end{aligned}$$

Using the anisotropic regularity theory (see Fan [11]) we deduce that $u \in \operatorname{int} C_+$. This proves Claim 2.

Recall that $u_0(z) < C_0$ for all $z \in \overline{\Omega}$. On account of Claim 2, we may assume that

$$K_{\Psi_+} \cap [u_0, C_0] = \{u_0\} \tag{42}$$

or otherwise we already have a second positive solution bigger than u_0 (see (17)) and so we are done.

We consider the following truncation of $g_+(z, \cdot)$

$$\hat{g}_+(z, x) = \begin{cases} g_+(z, x), & \text{if } x \leq C_0 \\ g_+(z, C_0), & \text{if } C_0 < x. \end{cases} \tag{43}$$

This is a Carathéodory function. We set $\hat{G}_+(z, x) = \int_0^x \hat{g}_+(z, s)ds$ and consider the C^1-functional $\hat{\Psi}_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \hat{\Psi}_+(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz \\ + \int_\Omega \frac{[\xi(z) + \vartheta]}{p(z)}|u|^{p(z)}dz - \int_\Omega \hat{G}_+(z, u)dz \end{aligned}$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

From (43) and since $\vartheta > \|\xi\|_\infty$, we see that $\hat{\Psi}_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\Psi}_+(\tilde{u}_0) = \inf\{\hat{\Psi}_+(u) : u \in W_0^{1,p(z)}(\Omega)\}. \tag{44}$$

Claim 3: $K_{\hat{\Psi}_+} \subseteq [u_0, C_0] \cap \operatorname{int} C_+$.

Let $u \in K_{\hat{\Psi}_+}$. As in the proof of Claim 2, we show that

$$u_0 \leq u.$$

Next, in (41) we choose $h = (u - C_0)^+ \in W_0^{1,p(z)}(\Omega)$. We have

$$\begin{aligned} &\langle A_{p(z)}(u), (u - C_0)^+\rangle + \langle A_{q(z)}(u), (u - C_0)^+\rangle \\ &\quad + \int_\Omega [\xi(z) + \vartheta]u^{p(z)-1}(u - C_0)^+dz \\ &= \int_\Omega g_+(z, C_0)(u - C_0)^+dz \text{ (see (43))} \\ &= \int_\Omega [f(x, C_0) + \vartheta C_0^{p(z)-1}](u - C_0)^+dz \text{ (see (17))} \\ &\leq \int_\Omega [\xi(z) + \vartheta]C_0^{p(z)-1}(u - C_0)^+dz \text{ (see hypothesis } H'(v)), \\ &\Rightarrow u \leq C_0. \end{aligned}$$

So, we have proved that $u \in [u_0, C_0]$. From this and the anisotropic regularity theory (see Fan [11]), we conclude that $K_{\hat{\Psi}_+} \subseteq [u_0, C_0] \cap \operatorname{int} C_+$. This proves Claim 3.

Note that

$$\Psi_+|_{[0,C_0]} = \hat{\Psi}_+|_{[0,C_0]} \text{ and } \Psi'_+|_{[0,C_0]} = \hat{\Psi}'_+|_{[0,C_0]} \text{ (see (17), (43)).} \tag{45}$$

Then from (42) and (45) it follows that $K_{\hat{\Psi}_+} = \{u_0\}$. Hence from (44) we have that $\tilde{u}_0 = u_0$ and since $u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[0, C_0]$ (see Proposition 7), from (45) we infer that

$$\begin{aligned} & u_0 \text{ is local } C_0^1(\overline{\Omega}) \text{ -minimizer of } \Psi_+(\cdot), \\ \Rightarrow\ & u_0 \text{ is local } W_0^{1,p(z)}(\Omega) \text{ -minimizer of } \Psi_+(\cdot) \\ & \text{(see Gasiński \& Papageorgiou [14, Proposition 3.3]).} \end{aligned} \tag{46}$$

On account of Claim 2, we may assume that

$$K_{\Psi_+} \text{ is finite.} \tag{47}$$

Otherwise we already have an infinity of positive smooth solutions bigger than u_0 and so we are done.

From (46), (47) and Theorem 5.7.6 of Papageorgiou, Rădulescu & Repovš [25, p. 449], we see that we can find $\rho \in (0, 1)$ small such that

$$\Psi_+(u_0) < \inf\{\Psi_+(u) : \|u - u_0\| = \rho\} = m_+. \tag{48}$$

Claim 1, (40) and (48) permit the use of the mountain pass theorem. So, we can find $\hat{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{u} \in K_{\Psi_+} \subseteq [u_0) \cap \operatorname{int} C_+ \text{ (see Claim 2) and } m_+ \le \Psi_+(\hat{u}) \text{ (see (48)).} \tag{49}$$

From (48) and (49) we see that

$$\begin{aligned} & \hat{u} \in \operatorname{int} C_+ \text{ is the second positive solution of (1),} \\ & u_0 \le \hat{u}, u_0 \ne \hat{u}. \end{aligned}$$

To produce the second negative solution, we argue similarly starting from the Carathéodory function

$$g_-(z, x) = \begin{cases} f(z, x) + \vartheta |x|^{p(z)-2} x, & \text{if } x \le -\hat{C} \\ f(z, -\hat{C}) - \vartheta \hat{C}^{p(z)-1}, & \text{if } -\hat{C} < x. \end{cases}$$

The proof is now complete. ◻

We introduce the following sets

$$
\begin{aligned}
S_+ = & \text{ set of positive solutions of problem (1)}, \\
S_- = & \text{ set of negative solutions of problem (1)}.
\end{aligned}
$$

We already know that

$$\emptyset \neq S_+ \subseteq \operatorname{int} C_+ \text{ and } \emptyset \neq S_- \subseteq -\operatorname{int} C_+.$$

Moreover, S_+ is downward directed and S_- is upward directed (see Papageorgiou, Rădulescu & Repovš [23]). We will show that there exist extremal constant sign solutions, that is, a smallest positive solution and a biggest negative solution. In the next section, we will use these extremal constant sign solutions in order to produce a nodal (sign-changing) solution.

Proposition 9 *If hypotheses H_0, H hold, then there exist $u^* \in S_+$ and $v^* \in S_-$ such that*

$$u^* \leq u \text{ for all } u \in S_+ \text{ and } v \leq v^* \text{ for all } v \in S_-.$$

Proof Invoking Lemma 3.10 of Hu & Papageorgiou [17, p. 178], we can find a decreasing sequence $\{u_n\}_{n \geq 1} \subseteq S_+$ such that

$$\inf_{n \geq 1} u_n = \inf S_+.$$

We have

$$
\begin{aligned}
& \langle A_{p(z)}(u_n), h \rangle + \langle A_{q(z)}(u_n), h \rangle = \int_\Omega f(z, u_n) h dz \qquad (50) \\
& \text{for all } h \in W_0^{1,p(z)}(\Omega), \text{ all } n \in \mathbb{N}, \\
& 0 \leq u_n \leq u_1 \text{ for all } n \in \mathbb{N}. \qquad (51)
\end{aligned}
$$

If in (50) we choose $h = u_n \in W_0^{1,p(z)}(\Omega)$ and use (51) and hypothesis $H(i)$, we see that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p(z)}(\Omega) \text{ is bounded.} \qquad (52)$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } u_n \to u^* \text{ in } L^r(\Omega). \qquad (53)$$

In (50) we choose $h = u_n - u^* \in W_0^{1,p(z)}(\Omega)$, pass to the limit as $n \to \infty$ and use (53). Then, as before (see the proof of Proposition 8, Claim 1), we obtain

$$\limsup_{n \to \infty} \langle A_{p(z)}(u_n), u_n - u^* \rangle \leq 0,$$

$$\begin{aligned}&\Rightarrow u_n \to u^* \text{ in } W_0^{1,p(z)}(\Omega),\\&\Rightarrow u^* \in S_+ \cup \{0\}.\end{aligned} \tag{54}$$

We need to show that $u^* \neq 0$.

On account of hypotheses $H(i)$, (iv), given any $\eta > 0$, we can find $C_{14} = C_{14}(\eta) > 0$ such that

$$f(z,x)x \geq \eta|x|^{q_-} - C_{14}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{55}$$

We consider the following auxiliary anisotropic Dirichlet problem:

$$\begin{cases}-\Delta_{p(z)}u - \Delta_{q(z)}u + |\xi(z)||u(z)|^{p-2}u = \eta|u|^{q_--2}u - C_{14}|u|^{r-2}u \text{ in } \Omega,\\ u|_{\partial\Omega} = 0.\end{cases} \tag{56}$$

Claim 1: Problem (56) admits a unique positive solution $\overline{u} \in \operatorname{int} C_+$ and since the problem is odd, then $\overline{v} = -\overline{u} \in -\operatorname{int} C_+$ is the unique negative solution of (56).

First we show the existence of a positive solution. So, we consider the C^1-functional $\tau_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}\tau_+(u) = &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{|\xi(z)|}{p(z)}|u|^{p(z)}dz\\ &+ \frac{C_{14}}{r}\|u^+\|_r^r - \frac{\eta}{q_-}\|u^+\|_{q_-}^{q_-} \text{ for all } u \in W_0^{1,p(z)}(\Omega).\end{aligned}$$

Since $q_- \leq q(z) < p(z) < r$ for all $z \in \overline{\Omega}$, we see that $\tau_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\overline{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\tau_+(\overline{u}) = \inf\{\tau_+(u) : u \in W_0^{1,p(z)}(\Omega)\}. \tag{57}$$

Fix $u \in \operatorname{int} C_+$. For $t \in (0,1)$, we have

$$\tau_+(tu) \leq \frac{t^{p_-}}{p_-}\rho_p(Du) + \frac{t^{q_-}}{q_-}\left[\rho_q(Du) - \eta\rho_{q_-}(u)\right] + \frac{t^r}{r}\|u\|_r^r.$$

Recall that $\eta > 0$ is arbitrary. So, choosing $\eta > \frac{\rho_q(Du)}{\rho_{q_-}(u)}$ and $t \in (0,1)$ even smaller if necessary, we have that

$$\begin{aligned}&\tau_+(tu) < 0,\\&\Rightarrow \tau_+(\overline{u}) < 0 = \tau_+(0) \text{ (see (57))},\\&\Rightarrow \overline{u} \neq 0.\end{aligned}$$

From (57) we have

$$\begin{aligned}
&\tau_+'(\overline{u}) = 0,\\
\Rightarrow\ &\langle A_{p(z)}(\overline{u}), h\rangle + \langle A_{q(z)}(\overline{u}), h\rangle + \int_\Omega |\xi(z)||\overline{u}|^{p(z)-2}\overline{u}h dz\\
=\ &\eta \int_\Omega (\overline{u}^+)^{p_- -1} h dz - C_{14}\int_\Omega (\overline{u}^+)^{r-1} h dz \text{ for all } h \in W_0^{1,p(z)}(\Omega).
\end{aligned}$$

Choose $h = -\overline{u}^- \in W_0^{1,p(z)}(\Omega)$. We obtain

$$\begin{aligned}
&\rho_p(D\overline{u}^-) + \rho_q(D\overline{u}^-) + \int_\Omega |\xi(z)|(u^-)^{p(z)} dz = 0,\\
\Rightarrow\ &\overline{u} \geq 0,\ \overline{u} \neq 0.
\end{aligned}$$

So, $\overline{u}$ is a positive solution of (56) and from the anisotropic regularity theory and Proposition 4, we have

$$\overline{u} \in \operatorname{int} C_+. \tag{58}$$

Suppose that $\tilde{u} \in W_0^{1,p(z)}(\Omega)$ is another positive solution of (56). Again we have

$$\tilde{u} \in \operatorname{int} C_+. \tag{59}$$

We consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \int_\Omega \frac{1}{p(z)}|Du^{\frac{1}{q_-}}|^{p(z)} + \int_\Omega \frac{1}{q(z)}|Du^{\frac{1}{q_-}}|^{q(z)} dz + \int_\Omega \frac{|\xi(z)|}{p(z)} u^{\frac{p(z)}{q_-}} dz, \\ \qquad \text{if } u \geq 0,\ u^{1/q_-} \in W_0^{1,p(z)}(\Omega) \\ +\infty,\ \text{otherwise.} \end{cases}$$

On account of Theorem 2.2 of Takač & Giacomoni [32], we have that $j(\cdot)$ is convex. Let $\operatorname{dom} j = \{u \in L^1(\Omega) :\ j(u) < \infty\}$ (the effective domain of $j(\cdot)$).

From (58), (59) and Proposition 4.1.22 of Papageorgiou, Rădulescu & Repovš [25, p. 274], we have

$$\frac{\overline{u}}{\tilde{u}},\ \frac{\tilde{u}}{\overline{u}} \in L^\infty(\Omega).$$

Let $h = \overline{u}^{q_-} - \tilde{u}^{q_-}$. Then for $|t| \leq 1$ small we have

$$\overline{u}^{q_-} + th \in \operatorname{dom} j \text{ and } \tilde{u}^{q_-} + th \in \operatorname{dom} j.$$

Hence the functional $j(\cdot)$ is Gâteaux differentiable at $\overline{u}^q$ and at $\tilde{u}^q$ in the direction h. Moreover, on account of the convexity of $j(\cdot)$ we obtain that $j'(\cdot)$ is monotone.

We have

$$j'(\overline{u}^{q_-})(h) = \int_\Omega \left[|D\overline{u}|^{p(z)-2} + |D\overline{u}|^{q(z)-2}\right]\left(D\overline{u}, D\left(\overline{u} - \frac{\tilde{u}^{q_-}}{\overline{u}^{q_-}}\right)\right)_{\mathbb{R}^N} dz$$
$$+ \int_\Omega |\xi(z)|\overline{u}\left(\overline{u}^{q_-} - \tilde{u}^{q_-}\right) dz \tag{60}$$
$$j'(\tilde{u}^{q_-})(h) = \int_\Omega \left[|D\tilde{u}|^{p(z)-2} + |D\tilde{u}|^{q(z)-2}\right]\left(D\tilde{u}, D\left(\tilde{u} - \frac{\overline{u}^{q_-}}{\tilde{u}^{q_-}}\right)\right)_{\mathbb{R}^N} dz$$
$$+ \int_\Omega |\xi(z)|\tilde{u}(\tilde{u}^{q_-} - \overline{u}^{q_-}) dz \tag{61}$$

From (60), (61), the monotonicity of $j'(\cdot)$ and using the distributional interpolation of the inequality (see also Takač & Giacomoni [32, Remark 2.6]), we have

$$0 \le C_{14} \int_\Omega \left[\tilde{u}^{r-q_-} - \overline{u}^{r-q_-}\right]\left(\overline{u}^{q_-} - \tilde{u}^{q_-}\right) dz \le 0,$$
$$\Rightarrow \overline{u} = \tilde{u}.$$

This proves Claim 1.

Claim 2: $\overline{u} \le u$ for all $u \in S_+$ and $v \le \overline{v}$ for all $v \in S_-$.

Let $u \in S_+$ and consider the Carathéodory function $\gamma_+(z, x)$ defined by

$$\gamma_+(z,x) = \begin{cases} \eta(x^+)^{q_--1} - C_{14}(x^+)^{r-1}, & \text{if } x \le u(z) \\ \eta u(z)^{q_--1} - C_{14}u(z)^{r-1}, & \text{if } u(z) < x. \end{cases} \tag{62}$$

We set $\Gamma_+(z,x) = \int_0^x \gamma_+(z,s)ds$ and consider the C^1-functional $\hat{\tau}_+ : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\hat{\tau}_+(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{|\xi(z)|}{p(z)}|u|^{p(z)}dz$$
$$- \int_\Omega \Gamma_+(z,u)dz \text{ for all } u \in W_0^{1,p(z)}(\Omega).$$

From (62) it is clear that $\hat{\tau}_+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. Therefore we can find $\tilde{u} \in W_0^{1,p(z)}(\Omega)$ such that

$$\hat{\tau}_+(\tilde{u}) = \inf\{\hat{\tau}_+(u)\ u \in W_0^{1,p(z)}(\Omega)\} < 0 = \hat{\tau}_+(0)$$
(see proof of Claim 1).

We have

$$\hat{\tau}_+'(\tilde{u}) = 0,\ \tilde{u} \ne 0,$$
$$\Rightarrow \tilde{u} \in [0, u] \text{ (as before using (62) and (55))},$$

$$\Rightarrow \tilde{u} = \overline{u} \in \operatorname{int} C_+ \text{ (see (62) and Claim 1)},$$
$$\Rightarrow \overline{u} \leq u \text{ for all } u \in S_+.$$

Similarly we show that $v \leq \overline{v}$ for all $v \in S_-$.
This proves Claim 2.
From (54) and Claim 2, we have

$$\overline{u} \leq u^*, \text{ hence } u^* \neq 0,$$
$$\Rightarrow u^* \in S_+ \text{ and } u^* = \inf S_+.$$

For the biggest negative solution the reasoning is similar. In this case, since S_- is upward directed, we can find $\{v_n\}_{n\geq 1} \subseteq S_-$ increasing such that

$$\sup_{n\geq 1} v_n = \sup S_-.$$

Then working as above, we obtain $v^* \in W_0^{1,p(z)}(\Omega)$ such that

$$v^* \in S_- \subseteq -\operatorname{int} C_+ \text{ and } v \leq v^* \text{ for all } v \in S_-.$$

The proof is now complete. □

4 Nodal solutions

In this section, using the extremal constant sign solutions from Proposition 9, we will obtain a nodal (sign changing) solution.

In what follows $\varphi : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ is the energy functional for problem (1) defined by

$$\varphi(u) = \int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{\xi(z)}{p(z)}|u|^{p(z)}dz - \int_\Omega F(z,u)dz$$

for all $u \in W_0^{1,p(z)}(\Omega)$.

We have that $\varphi \in C^1(W_0^{1,p(z)}(\Omega))$.

Proposition 10 *If hypotheses H_0, H' hold, then problem* (1) *admits a nodal solution*

$$y_0 \in [v^*, u^*] \cap C_0^1(\overline{\Omega})$$

with u^* and v^* being the two extremal constant sign solutions from Proposition 9.

Proof As before let $\vartheta > \|\xi\|_\infty$ and introduce the Carathéodory function $\hat{\tau}(z,x)$ defined by

$$\hat{\tau}(z,x) = \begin{cases} f(z,v^*(z)) + \vartheta|v^*(z)|^{p(z)-2}v^*(z), & \text{if } x < v^*(z) \\ f(z,x) + \vartheta|x|^{p(z)-2}x, & \text{if } v^* \le x \le u^*(z) \\ f(z,u^*(z)) + \vartheta u^*(z)^{p(z)-1}, & \text{if } u^*(z) < x. \end{cases} \tag{63}$$

Also, we consider the positive and negative truncations of $\hat{\tau}(z,\cdot)$, namely the Carathéodory functions

$$\hat{\tau}_\pm(z,x) = \hat{\tau}(z, \pm x^\pm). \tag{64}$$

We set $\hat{T}(z,x) = \displaystyle\int_0^x \hat{\tau}(z,x)ds$, $\hat{T}_\pm(z,x) = \int_0^x \hat{\tau}_\pm(z,s)ds$ and consider the C^1-functionals $\tilde{\varphi}, \tilde{\varphi}_\pm : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \tilde{\varphi}(u) = &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{[\xi(z)+\vartheta]}{p(z)}|u|^{p(z)}dz \\ &- \int_\Omega \hat{T}(z,u)dz \text{ for all } u \in W_0^{1,p(z)}(\Omega) \\ \tilde{\varphi}_\pm(u) = &\int_\Omega \frac{1}{p(z)}|Du|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|Du|^{q(z)}dz + \int_\Omega \frac{[\xi(z)+\vartheta]}{p(z)}|u|^{p(z)}dz \\ &- \int_\Omega \hat{T}_\pm(z,u)dz \text{ for all } u \in W_0^{1,p(z)}(\Omega). \end{aligned}$$

Using (63) and (64), as before (see the proof of Proposition 8, Claim 3) we can check that

$$K_{\tilde{\varphi}} \subseteq [v^*, u^*] \cap C_0^1(\overline{\Omega}),\ K_{\tilde{\varphi}_+} \subseteq [0, u^*] \cap C_+,\ K_{\tilde{\varphi}_-} \subseteq [v^*, 0] \cap (-C_+).$$

The extremality of u^* and v^* implies that

$$K_{\tilde{\varphi}} \subseteq [v^*, u^*] \cap C_0^1(\overline{\Omega}),\ K_{\tilde{\varphi}_+} = \{0, u^*\},\ K_{\tilde{\varphi}_-} = \{0, v^*\}. \tag{65}$$

Claim 1: $u^* \in \operatorname{int} C_+$ and $v^* \in -\operatorname{int} C_+$ are local minimizers of $\tilde{\varphi}(\cdot)$.

From (63) and (64) and since $\vartheta > \|\xi\|_\infty$ it is clear that $\tilde{\varphi}_+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}^* \in W_0^{1,p(z)}(\Omega)$ such that

$$\tilde{\varphi}_+(\tilde{u}^*) = \int \inf\left[\tilde{\varphi}_+(u) : u \in W_0^{1,p(z)}(\Omega)\right]. \tag{66}$$

On account of hypothesis $H'(iv)$, we have

$$\begin{aligned} &\tilde{\varphi}_+(\tilde{u}^*) < 0 = \tilde{\varphi}_+(0), \\ \Rightarrow\ &\tilde{u}^* \neq 0, \\ \Rightarrow\ &\tilde{u}^* = u^* \in \operatorname{int} C_+ \text{ (see (66) and (65)).} \end{aligned}$$

Clearly $\tilde{\varphi}|_{C_+} = \tilde{\varphi}_+|_{C_+}$. Hence it follows that

$$\begin{aligned} & u^* \text{ is a local } C_0^1(\overline{\Omega}) \text{ -minimizer of } \tilde{\varphi}(\cdot), \\ \Rightarrow\ & u^* \text{ is a local } W_0^{1,p(z)}(\Omega) \text{ -minimizer of } \tilde{\varphi}(\cdot) \text{ (see [14]).} \end{aligned}$$

Similarly for $v^* \in -\mathrm{int}\, C_+$ using this time the functional $\tilde{\varphi}_-(\cdot)$.
This proves Claim 1.
On account of Claim 1 we have

$$C_k(\tilde{\varphi}, u^*) = C_k(\tilde{\varphi}, v^*) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0 \tag{67}$$

with $\delta_{k,0}$ being the Kronecker symbol defined by $\delta_{k,0} = \begin{cases} 1, & \text{if } k = 0 \\ 0, & \text{if } k \neq 0 \end{cases}$ for all $k \in \mathbb{N}_0$ (see Proposition 6.2.5 of Papageorgiou, Rădulescu & Repovš [25, p. 479]).

Claim 2: $C_k(\tilde{\varphi}, 0) = 0$ for all $k \in \mathbb{N}_0$.

On account of hypotheses $H'(i)(iv)$, given $\eta > 0$, we can find $C_{15} > 0$ such that

$$F(z, x) \geq \eta |x|^{q_-} - C_{15}|x|^r \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then for $u \in W_0^{1,p(z)}(\Omega)$ and $0 < t < 1$ we have

$$\begin{aligned} \varphi(tu) \leq\ & t^{p_+}\left[\rho_p(Du) + \|\xi\|_\infty \rho_p(u) + C_{15}\|u\|_r^r\right] + t^{q_-}\left[\rho_p(Du) - \eta\|u\|_{q_-}^{q_-}\right] \\ & \text{(recall that } q_- < p_- \leq p_+ < r). \end{aligned}$$

Since $\eta > 0$ is arbitrary, we choose $\eta > 0$ big so that

$$\varphi(tu) < 0 \text{ for all } 0 < t < t^* < 1. \tag{68}$$

Let $u \in W_0^{1,p(z)}(\Omega)$, $0 < \|u\| < 1$, $\varphi(u) = 0$. We have

$$\begin{aligned} & \frac{d}{dt}\varphi(tu)|_{t=1} \\ & = \langle \varphi'(u), u\rangle \text{ (by the chain rule)} \\ & = \langle A_{p(z)}(u), u\rangle + \langle A_{q(z)}(u), u\rangle + \int_\Omega \xi(z)|u|^{p(z)}dz - \int_\Omega f(z, u)u dz \\ & \geq \left[1 - \frac{q_-}{p_+}\right]\rho_p(Du) + \left[1 - \frac{q_-}{p_+}\right]\int_\Omega \xi(z)|u|^{p(z)}dz \\ & \quad +(q_- - \tau)\int_\Omega F(z, u)dz + \int_\Omega [\tau F(z, u) - f(z, u)u]dz \text{ (since } \varphi(u) = 0). \end{aligned} \tag{69}$$

On account of hypothesis $H'(iv)$, given $\eta > 0$, we can find $\delta = \delta(\eta) \in (0, 1)$ small such that

$$F(z,x) \geq \frac{\eta}{q_-}|x|^{q_-} \geq \frac{\eta}{q_-\delta^{p_+-q_-}}|x|^{p_+}$$
$$\text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

If we combine this with hypothesis $H'(i)$, we obtain

$$F(z,x) \geq \frac{\eta}{q_-\delta^{p_+-q_-}}|x|^{p_+} - C_{16}|x|^r$$
$$\text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } C_{16} > 0. \tag{70}$$

Also, from the second part of hypothesis $H'(iv)$ and from $H'(i)$, we see that given $\varepsilon > 0$, we can find $C_{17} > 0$ such that

$$\tau F(z,x) - f(z,x)x \geq -\varepsilon|x|^{p_+} - C_{17}|x|^r$$
$$\text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{71}$$

We return to (69) and use (70) and (71). Then

$$\frac{d}{dt}\varphi(tu)|_{t=1}$$
$$\geq \left[1 - \frac{q_-}{p_+} - \varepsilon C_{18}\right]\|u\|^{p_+} + \left[\frac{\eta}{q_-\delta^{p_+-q_-}} - \left(1 - \frac{q_-}{p_+}\right)\|\xi\|_\infty\right]\|u\|_{p(z)}^{p_+} - C_{19}\|u\|^r$$
$$\text{for some } C_{18}, C_{19} > 0.$$

Recall that $\eta, \varepsilon > 0$ are arbitrary. So, we choose $\varepsilon > 0$ small and $\eta > 0$ big (recall that $\eta \to \delta(\eta)$ is decreasing) such that

$$\frac{d}{dt}\varphi(tu)|_{k=1} \geq C_{20}\|u\|^{p_+} - C_{19}\|u\|^r \text{ for some } C_{20} > 0.$$

Since $p_+ < r$, we can find $\rho \in (0,1)$ small such that

$$\frac{d}{dt}\varphi(tu)|_{t=1} > 0$$
$$\text{for all } u \in W_0^{1,p(z)}(\Omega), \text{ with } 0 < \|u\| \leq \rho, \varphi(u) = 0. \tag{72}$$

Let $u \in W_0^{1,p(z)}(\Omega)$ with $0 < \|u\| \leq \rho$, $\varphi(u) = 0$. We will show that

$$\varphi(tu) \leq 0 \text{ for all } t \in [0,1]. \tag{73}$$

Arguing by contradiction, suppose we can find $t_0 \in (0,1)$ such that

$$\varphi(t_0u) > 0.$$

Recall that $\varphi(u) = 0$ and $\varphi(\cdot)$ is continuous. So, we can find $t_1 \in (t_0, 1]$ such that $\varphi(t_1 u) = 0$. We consider the first time instant after t_0 for which this is true. So, we define

$$\begin{aligned} & t_* = \min\{t \in [t_0, 1] : \varphi(tu) = 0\} > t_0 > 0, \\ \Rightarrow & \varphi(tu) > 0 \text{ for all } t \in [t_0, t_*). \end{aligned} \tag{74}$$

Let $y = t_* u$. We have $0 < \|y\| \leq \|u\| \leq \rho$ and $\varphi(y) = 0$. So, from (72) it follows that

$$\frac{d}{dt}\varphi(ty)|_{t=1} > 0. \tag{75}$$

From (74) we have

$$\begin{aligned} & \varphi(y) = \varphi(t_* u) = 0 < \varphi(tu) \text{ for all } t_0 \leq t < t_*, \\ \Rightarrow & \frac{d}{dt}\varphi(ty)|_{t=1} = t_* \frac{d}{dt}\varphi(tu)|_{t=t_*} = t_* \lim_{t \to t_*} \frac{\varphi(tu)}{t - t_*} \leq 0. \end{aligned} \tag{76}$$

Comparing (75) and (76), we obtain a contradiction. Therefore relation (73) is true.

From (65) we see that we may assume that $K_{\tilde{\varphi}}$ is finite. Otherwise we already have an infinity of nodal solutions (due to the extremality of u^* and v^*). So, $0 \in K_\varphi$ is isolated (recall that $K_\varphi|_{[v^*, u^*]} = K_{\tilde{\varphi}}|_{[v^*, u^*]}$) and so we can have $\rho \in (0, 1)$ small such that $K_\varphi \cap \overline{B}_\rho = \{0\}$ where $\overline{B}_\rho = \{u \in W_0^{1,p(z)}(\Omega) : \|u\| \leq \rho\}$. Let $h : [0, 1] \times (\varphi^0 \cap \overline{B}_\rho) \to \varphi^0 \cap \overline{B}_\rho$ be the deformation defined by

$$h(t, u) = (1 - t)u \text{ for all } (t, u) \in [0, 1] \times (\varphi^0 \cap \overline{B}_\rho).$$

On account of (73), this deformation is well defined and shows that

$$\varphi^0 \cap \overline{B}_\rho \text{ is contractible.} \tag{77}$$

Fix $u \in \overline{B}_\rho$ with $\varphi(u) > 0$. We show that there exists unique $t(u) \in (0, 1)$ such that

$$\varphi(t(u)u) = 0. \tag{78}$$

Note that

$$\varphi(u) > 0 \text{ and } t \mapsto \varphi(tu) \text{ is continuous.}$$

So, from (68) and Bolzano's theorem, we see that such a $t(u) \in (0, 1)$ exist. We show the uniqueness of this $t(u)$. Suppose we could find $0 < t_1 < t_2 < 1$ such that

$$\begin{aligned} & \varphi(t_1 u) = \varphi(t_2 u) = 0, \\ \Rightarrow & \varphi(t t_2 u) \leq 0 \text{ for all } t \in [0, 1] \text{ (see (73)).} \end{aligned}$$

Then for $\mu(t) = \varphi(tt_2u)$, $t \in [0, 1]$, $\frac{t_1}{t_2} \in (0, 1)$ is a maximizer of $\mu(\cdot)$ and so

$$\frac{t_1}{t_2}\frac{d}{dt}\mu(t)|_{t=\frac{t_1}{t_2}} = \frac{t_1}{t_2}\frac{d}{dt}\varphi(tt_2u)|_{t=\frac{t_1}{t_2}} = \frac{d}{dt}\varphi(tt_1u)|_{t=1} = 0,$$

which contradicts (72). Therefore the time instant $t(u) \in (0, 1)$ is unique.

We have

$$\begin{cases} \varphi(tu) < 0, & \text{for } t \in (0, t(u)) \\ \varphi(tu) > 0, & \text{for } t \in (t(u), 1]. \end{cases} \tag{79}$$

Let $k : \overline{B}_\rho \setminus \{0\} \to [0, 1]$ be defined by

$$k(u) = \begin{cases} 1, & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \varphi(u) \le 0 \\ t(u), & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \varphi(u) > 0. \end{cases} \tag{80}$$

We can easily check that $k(\cdot)$ is continuous. Then we introduce the map $\hat{k} : \overline{B}_\rho \setminus \{0\} \to (\overline{B}_\rho \cap \varphi^0) \setminus \{0\}$ defined by

$$\hat{k}(u) = \begin{cases} u, & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \varphi(u) \le 0 \\ k(u)u, & \text{if } u \in \overline{B}_\rho \setminus \{0\},\ \varphi(u) > 0. \end{cases}$$

This map is continuous and

$$\begin{aligned} &\hat{k}|_{(\overline{B}_\rho\cap\varphi^0)\setminus\{0\}} = \mathrm{id}|_{(\overline{B}_\rho\cap\varphi^0)\setminus\{0\}} \\ &\Rightarrow (\overline{B}_\rho \cap \varphi^0) \setminus \{0\} \text{ is a retract of } \overline{B}_\rho \setminus \{0\}. \end{aligned}$$

But since the space is infinite dimensional, $\overline{B}_\rho \setminus \{0\}$ is contractible (see Gasiński & Papageorgiou [15, pp. 677-678]). A retract of a contractible space is itself contractible. So

$$(\overline{B}_\rho \cap \varphi^0) \setminus \{0\} \text{ is contractible.} \tag{81}$$

From (77) and (81) it follows that

$$\begin{aligned} &H_k(\overline{B}_\rho \cap \varphi^0, (\overline{B}_\rho \cap \varphi^0) \setminus \{0\}) = 0 \text{ for all } k \in \mathbb{N}_0 \\ &\text{(see Papageorgiou, R\u{a}dulescu \&Repov\v{s} [25, p. 469])} \\ \Rightarrow\ &C_k(\varphi, 0) = 0 \text{ for all } k \in \mathbb{N}_0. \end{aligned} \tag{82}$$

We consider the homotopy

$$h(t, u) = (1 - t)\varphi(u) + t\tilde{\varphi}(u), \text{ for all } t \in [0, 1], \text{ all } u \in W_0^{1,p(z)}(\Omega).$$

Suppose we could find $\{t_n\}_{n\ge1} \subseteq [0, 1]$ and $\{u_n\}_{n\ge1} \subseteq W_0^{1,p(z)}(\Omega)$ such that

$$t_n \to t,\ u_n \to 0 \text{ in } W_0^{1,p(z)}(\Omega) \text{ and } h'_u(t_n, u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{83}$$

From the equation in (83) and Theorem 4.1 of Fan & Zhao [12], we know that

$$u_n \in L^\infty(\Omega) \text{ and } \|u_n\|_\infty \le C_{21} \text{ for some } C_{21} > 0, \text{ all } n \in \mathbb{N}.$$

Then from Fan [11, Theorem 1.3] (see also Fukagai & Narukawa [13, Lemma 3.3] and Lieberman [18]), we can find $\alpha \in (0, 1)$ and $C_{22} > 0$ such that

$$u_n \in C_0^{1,\alpha}(\overline{\Omega}), \ \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \le C_{22} \text{ for all } n \in \mathbb{N}.$$

The compact embedding of $C_0^{1,\alpha}(\overline{\Omega})$ into $C_0^1(\overline{\Omega})$ and (83) imply that

$$u_n \to 0 \text{ in } C_0^1(\overline{\Omega}) \text{ as } n \to \infty,$$
$$\Rightarrow u_n \in [v^*, u^*] \text{ for all } n \ge n_0.$$

But recall that we have assumed that $K_{\tilde{\varphi}}$ is finite (see (65)). So, (83) can not happen and the homotopy invariance property of critical groups (see Papageorgiou, Rădulescu & Repovš [25, p. 505]) implies that

$$C_k(\tilde{\varphi}, 0) = C_k(\varphi, 0) \text{ for all } k \in \mathbb{N}_0,$$
$$\Rightarrow C_k(\tilde{\varphi}, 0) = 0 \text{ for all } k \in \mathbb{N}_0 \text{ (see (82))}.$$

This proves Claim 2.

We may assume that

$$\tilde{\varphi}(v^*) \le \tilde{\varphi}(u^*).$$

The reasoning is similar if the opposite inequality holds.

Recall that $K_{\tilde{\varphi}}$ is finite. Then Claim 1 implies that we can find $\rho \in (0, 1)$ small such that

$$\tilde{\varphi}(v^*) \le \tilde{\varphi}(u^*) < \inf\left[\tilde{\varphi}(u) : \ \|u - u^*\| = \rho\right] = \tilde{m}, \ \|v^* - u^*\| > \rho$$
$$\text{(see Proposition 5.7.6 of Papageorgiou, R\breve{a}dulescu \&Repov\check{s} [25, p. 449]).} \tag{84}$$

Evidently $\tilde{\varphi}$ is coercive (see (63)) and so it satisfies the C-condition (see Proposition 5.1.15 of Papageorgiou, Rădulescu & Repovš [25, p. 369]). This fact and (84) permit the use of the mountain pass theorem. Therefore we can find $y_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$y_0 \in K_{\tilde{\varphi}} \subseteq [v^*, u^*] \cap C_0^1(\overline{\Omega}) \text{ (see (65))}, \tilde{m} \le \tilde{\varphi}(y_0) \text{ (see (84))}. \tag{85}$$

Also, from Theorem 6.5.8 of Papageorgiou, Rădulescu & Repovš [25, p. 527], we have

$$
\begin{aligned}
& C_1(\tilde{\varphi}, y_0) \neq \emptyset \\
\Rightarrow\ & y_0 \notin \{0, u^*, v^*\} \text{ (see Claim 2 and (84))}, \\
\Rightarrow\ & y_0 \in C_0^1(\overline{\Omega}) \text{ is a nodal solution of (1).}
\end{aligned}
$$

The proof is now complete. □

So, summarizing we can state the following multiplicity theorem for problem (1).

Theorem 11 *If hypotheses H_0, H' hold, then problem* (1) *has at least five nontrivial smooth solutions*

$$
\begin{aligned}
& u_0, \hat{u} \in \operatorname{int} C_+, u_0 \leq \hat{u}, u_0 \neq \hat{u}, u_0(z) < C_0 \textit{ for all } z \in \overline{\Omega}, \\
& v_0, \hat{v} \in -\operatorname{int} C_+, \hat{v} \leq v_0, v_0 \neq \hat{v}, -\hat{C} < v_0(z) \textit{ for all } z \in \overline{\Omega}, \\
& y_0 \in [v_0, u_0] \cap C_0^1(\overline{\Omega}) \textit{ nodal.}
\end{aligned}
$$

Remark 2 We emphasize that in the above multiplicity theorem we provide sign information for the solutions and moreover, the solutions are linearly ordered that is, we have $\hat{v} \leq v_0 \leq y_0 \leq u_0 \leq \hat{u}$.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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