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D. Rădulescu & Dušan D. Repovš**

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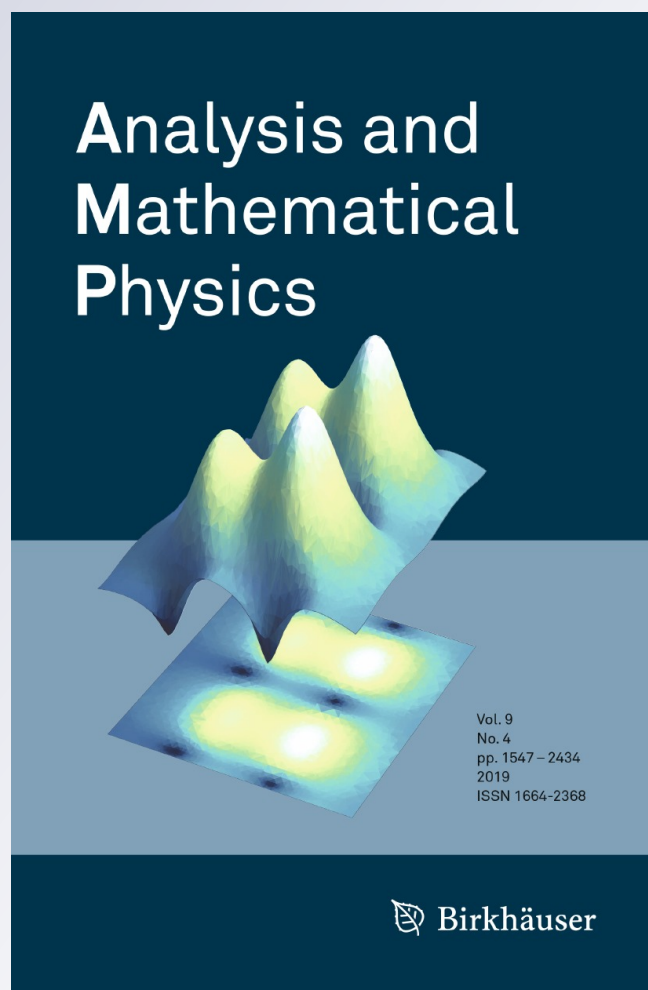
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# Nonlinear singular problems with indefinite potential term

Nikolaos S. Papageorgiou<sup>1,2</sup> · Vicențiu D. Rădulescu<sup>2,3,4</sup> · Dušan D. Repovš<sup>2,5</sup>

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## Abstract

We consider a nonlinear Dirichlet problem driven by a nonhomogeneous differential operator plus an indefinite potential. In the reaction we have the competing effects of a singular term and of concave and convex nonlinearities. In this paper the concave term will be parametric. We prove a bifurcation-type theorem describing the changes in the set of positive solutions as the positive parameter  $\lambda$  varies.

**Keywords** Nonhomogeneous differential operator · Indefinite potential · Singular term · Concave and convex nonlinearities · Truncation · Comparison principles · Nonlinear regularity · Nonlinear maximum principle

**Mathematics Subject Classification** 35J75 · 35J92 · 35P30

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✉ Vicențiu D. Rădulescu  
vicentiu.radulescu@imar.ro

Nikolaos S. Papageorgiou  
npapg@math.ntua.gr

Dušan D. Repovš  
dusan.repovs@guest.arnes.si

- <sup>1</sup> Department of Mathematics, Zografou Campus, National Technical University, 15780 Athens, Greece
- <sup>2</sup> Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
- <sup>3</sup> Faculty of Applied Mathematics, AGH University of Science and Technology, 30-059 Kraków, Poland
- <sup>4</sup> Department of Mathematics, University of Craiova, 200585 Craiova, Romania
- <sup>5</sup> Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia

## 1 Introduction

Let  $\Omega \subseteq \mathbb{R}^N$  be a bounded domain with a  $C^2$ -boundary  $\partial\Omega$ . In this paper we study the following nonlinear nonhomogeneous parametric singular problem:

$$\left\{ \begin{array}{l} -\operatorname{div} a(Du(z)) + \xi(z)u(z)^{p-1} = \vartheta(u(z)) + \lambda u(z)^{q-1} + f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \quad u > 0, \quad \lambda > 0, \quad 1 < q < p < \infty. \end{array} \right\} \quad (P_\lambda)$$

The map  $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$  involved in the differential operator of  $(P_\lambda)$  is strictly monotone, continuous (hence maximal monotone, too) and satisfies certain other regularity and growth conditions which are listed in hypotheses  $H(a)$  below (see Sect. 2). These conditions are general enough to incorporate in our framework many differential operators of interest such as the  $p$ -Laplacian and the  $(p, q)$ -Laplacian (that is, the sum of a  $p$ -Laplacian and a  $q$ -Laplacian). The operator  $u \mapsto \operatorname{div} a(Du)$  is not homogeneous and this is a source of difficulties in the analysis of problem  $(P_\lambda)$ . The potential function  $\xi \in L^\infty(\Omega)$  is indefinite (that is, sign changing). So the operator  $u \mapsto -\operatorname{div} a(Du) + \xi(z)|u|^{p-2}u$  is not coercive and this is one more difficulty in the analysis of problem  $(P_\lambda)$ . In the reaction (the right-hand side of  $(P_\lambda)$ ), the term  $\vartheta(\cdot)$  is singular at  $x = 0$ , while the perturbation contains the combined effects of a parametric concave term  $x \mapsto \lambda x^{q-1}$  ( $x \geq 0$ ) (recall that  $q < p$ ), with  $\lambda > 0$  being the parameter and of a Carathéodory function  $f(z, x)$  (that is, for all  $x \in \mathbb{R}$  the mapping  $z \mapsto f(z, x)$  is measurable and for almost all  $z \in \Omega$  the mapping  $x \mapsto f(z, x)$  is continuous), which is assumed to exhibit  $(p - 1)$ -superlinear growth near  $+\infty$ , but without satisfying the usual for superlinear problems Ambrosetti-Rabinowitz condition (the AR-condition for short). So in problem  $(P_\lambda)$  we have the competing effects of singular, concave and convex terms.

Using variational methods related to the critical point theory, combined with suitable truncation, perturbation and comparison techniques, we produce a critical parameter value  $\lambda^* > 0$  such that

- (i) for all  $\lambda \in (0, \lambda^*)$  problem  $(P_\lambda)$  has at least two positive solutions;
- (ii) for  $\lambda = \lambda^*$  problem  $(P_\lambda)$  has at least one positive solution;
- (iii) for all  $\lambda > \lambda^*$  problem  $(P_\lambda)$  has no positive solutions.

This work continues the recent paper by Papageorgiou et al. [16], where  $\xi \equiv 0$  and in the reaction the parametric term is the singular one. It is also related to the works of Papageorgiou and Smyrlis [17] and Papageorgiou and Winkert [19], where the differential operator is the  $p$ -Laplacian,  $\xi \equiv 0$  and no concave terms are allowed. Singular  $p$ -Laplacian equations with no potential term and reactions of special form were considered by Chu et al. [2], Giacomoni et al. [5], Li and Gao [10], Mohammed [12], Perera and Zhang [20], and Papageorgiou et al. [14].

## 2 Mathematical background and hypotheses

In this section we present the main mathematical tools which we will use in the analysis of problem  $(P_\lambda)$ . We also fix our notation and state the hypotheses on the data of the problem.

So, let  $X$  be a Banach space,  $X^*$  its topological dual, and let  $\varphi \in C^1(X)$ . We say that  $\varphi(\cdot)$  satisfies the ‘‘C-condition’’, if the following property holds:

‘‘Every sequence  $\{u_n\}_{n \geq 1} \subseteq X$  such that  $\{\varphi(u_n)\}_{n \geq 1} \subseteq \mathbb{R}$  is bounded and  $(1 + \|u_n\|_X)\varphi'(u_n) \rightarrow 0$  in  $X^*$  as  $n \rightarrow \infty$ , admits a strongly convergent subsequence’’.

This is a compactness-type condition on the functional  $\varphi(\cdot)$ , which leads to the minimax theory of the critical values of  $\varphi(\cdot)$  (see, for example, Papageorgiou et al. [15]). We denote by  $K_\varphi$  the critical set of  $\varphi$ , that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

The main spaces in the analysis of problem  $(P_\lambda)$  are the Sobolev space  $W_0^{1,p}(\Omega)$  ( $1 < p < \infty$ ) and the Banach space  $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$ . We denote by  $\|\cdot\|$  the norm of  $W_0^{1,p}$ . By the Poincaré inequality we have

$$\|u\| = \|Du\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The Banach space  $C_0^1(\Omega)$  is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n} \Big|_{\partial\Omega} < 0 \right\},$$

with  $n(\cdot)$  being the outward unit normal on  $\partial\Omega$ .

We will also use two additional ordered Banach spaces. The first one is

$$C_0(\overline{\Omega}) = \{u \in C(\overline{\Omega}) : u|_{\partial\Omega} = 0\}.$$

This cone is ordered with positive (order) cone

$$K_+ = \{u \in C_0(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } K_+ = \{u \in K_+ : c_u \hat{d} \leq u \text{ for some } c_u > 0\},$$

where  $\hat{d}(z) = d(z, \partial\Omega)$  for all  $z \in \bar{\Omega}$ . On account of Lemma 14.16 of Gilbarg and Trudinger [6, p. 355], we have

$$“c_u \hat{d} \leq u \text{ for some } c_u > 0 \text{ if and only if } \hat{c}_u \hat{u}_1 \leq u \text{ for some } \hat{c}_u > 0”, \tag{1}$$

with  $\hat{u}_1$  being the positive,  $L^p$ -normalized (that is,  $\|\hat{u}_1\|_p = 1$ ) eigenfunction corresponding to the principal eigenvalue  $\hat{\lambda}_1 > 0$  of the Dirichlet  $p$ -Laplacian. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski and Papageorgiou [4, pp. 737–738]), imply that  $\hat{u}_1 \in \text{int } C_+$ .

The second ordered space is  $C^1(\bar{\Omega})$  with positive (order) cone

$$\hat{C}_+ = \left\{ u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega}, \frac{\partial u}{\partial n} \Big|_{\partial\Omega \cap u^{-1}(0)} < 0 \right\}.$$

Clearly, this cone has a nonempty interior.

Concerning ordered Banach spaces with an order cone which has a nonempty interior (solid order cone), we have the following result which will be useful in our analysis (see Papageorgiou et al. [15, Proposition 4.1.22]).

**Proposition 1** *If  $X$  is an ordered Banach space with positive (order) cone  $K$ ,  $\text{int } K \neq \emptyset$ , and  $e \in \text{int } K$ , then for every  $u \in X$  we can find  $\lambda_u > 0$  such that  $\lambda_u e - u \in K$ .*

Let  $l \in C^1(0, \infty)$  with  $l(t) > 0$  for all  $t > 0$ . We assume that

$$0 < \hat{c} \leq \frac{l'(t)t}{l(t)} \leq c_0, c_1 t^{p-1} \leq l(t) \leq c_2 [t^{s-1} + t^{p-1}] \tag{2}$$

for all  $t > 0$ , and some  $c_1, c_2 > 0, 1 \leq s < p$ .

Then the conditions on the map  $a(\cdot)$  are the following:

$H(a) : a(y) = a_0(|y|)y$  for all  $y \in \mathbb{R}^N$ , with  $a_0(t) > 0$  for all  $t > 0$  and

- (i)  $a_0 \in C^1(0, +\infty)$ ,  $t \mapsto a_0(t)$  is strictly increasing on  $(0, +\infty)$ ,  $a_0(t)t \rightarrow 0^+$  as  $t \rightarrow 0^+$  and

$$\lim_{t \rightarrow 0^+} \frac{a'_0(t)t}{a_0(t)} > -1;$$

- (ii) there exists  $c_3 > 0$  such that

$$|\nabla a(y)| \leq c_3 \frac{l(|y|)}{|y|} \text{ for all } y \in \mathbb{R}^N \setminus \{0\};$$

- (iii)  $(\nabla a(y)\xi, \xi)_{\mathbb{R}^N} \geq \frac{l(|y|)}{|y|} |\xi|^2$  for all  $y \in \mathbb{R}^N \setminus \{0\}, \xi \in \mathbb{R}^N$ ;

- (iv) if  $G_0(t) = \int_0^t a_0(s)s ds$ , then there exists  $\tau \in (q, p]$  such that

$$\limsup_{t \rightarrow 0^+} \frac{\tau G_0(t)}{t^\tau} \leq c^*$$

and  $0 \leq pG_0(t) - a_0(t)t^2$  for all  $t > 0$ .

**Remark 1** Hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  are dictated by the nonlinear regularity theory of Lieberman [10] and the nonlinear maximum principle of Pucci and Serrin [21]. Hypothesis  $H(a)(iv)$  serves the needs of our problem, but in fact, it is a mild condition and it is satisfied in all cases of interest (see the examples below). These conditions were used by Papageorgiou and Rădulescu [13] and by Papageorgiou et al. [16].

Hypotheses  $H(a)$  imply that the primitive  $G_0(\cdot)$  is strictly increasing and strictly convex. We set  $G(y) = G_0(|y|)$  for all  $y \in \mathbb{R}^N$ . Evidently,  $G(\cdot)$  is convex,  $G(0) = 0$  and

$$\nabla G(y) = G'_0(|y|) \frac{y}{|y|} = a_0(|y|)y = a(y) \text{ for all } y \in \mathbb{R}^N \setminus \{0\}, \nabla G(0) = 0,$$

that is,  $G(\cdot)$  is the primitive of  $a(\cdot)$ . From the convexity of  $G(\cdot)$  we have

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N. \tag{3}$$

Using hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  and (2), we can easily obtain the following lemma, which summarizes the main properties of the map  $a(\cdot)$ .

**Lemma 2** *If hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  hold, then*

- (a) *the map  $y \mapsto a(y)$  is continuous, strictly monotone (hence maximal monotone, too);*
- (b)  *$|a(y)| \leq c_4(|y|^{s-1} + |y|^{p-1})$  for some  $c_4 > 0$ , and all  $y \in \mathbb{R}^N$ ;*
- (c)  *$(a(y), y)_{\mathbb{R}^N} \geq \frac{c_1}{p-1}|y|^p$  for all  $y \in \mathbb{R}^N$ .*

Using this lemma and (3), we obtain the following growth estimates for the primitive  $G(\cdot)$ .

**Corollary 3** *If hypotheses  $H(a)(i)$ ,  $(ii)$ ,  $(iii)$  hold, then  $\frac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5(1 + |y|^p)$  for some  $c_5 > 0$ , and all  $y \in \mathbb{R}^N$ .*

The examples that follow confirm that the framework provided by hypotheses  $H(a)$  is broad and includes many differential operators of interest (see [13]).

**Example 1** (a)  $a(y) = |y|^{p-2}y$  with  $1 < p < \infty$ .

This map corresponds to the  $p$ -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|Du|^{p-2}Du) \text{ for all } u \in W_0^{1,p}(\Omega).$$

(b)  $a(y) = |y|^{p-2}y + \mu|y|^{q-2}y$  with  $1 < q < p < \infty$ ,  $\mu \geq 0$ .

This map corresponds to the  $(p, q)$ -Laplace differential operator defined by

$$\Delta_p u + \Delta_q u \text{ for all } u \in W_0^{1,p}(\Omega).$$

Such operators arise in models of physical processes. We mention the works of Cherfils and Ilyasov [1] (reaction-diffusion systems) and Zhikov [22] (homogenization of composites consisting of two materials with distinct hardening exponent in elasticity theory).

(c)  $a(y) = (1 + |y|^2)^{\frac{p-2}{2}} y$  with  $1 < p < \infty$ .

This map corresponds to the modified capillary operator.

(d)  $a(y) = |y|^{p-2} y \left(1 + \frac{1}{1+|y|^p}\right)$  with  $1 < p < \infty$ .

The hypotheses on the potential term  $\xi(\cdot)$  and on the singular part  $\vartheta(\cdot)$  of the reaction are the following:

$$H(\xi) : \xi \in L^\infty(\Omega).$$

$H(\vartheta) : \vartheta : (0, +\infty) \rightarrow (0, +\infty)$  is a locally Lipschitz function such that

(i) for some  $\gamma \in (0, 1)$  we have

$$0 < c_6 \leq \liminf_{x \rightarrow 0^+} \vartheta(x)x^\gamma \leq \limsup_{x \rightarrow 0^+} \vartheta(x)x^\gamma \leq c_7;$$

(ii)  $\vartheta(\cdot)$  is nonincreasing.

**Remark 2** In the literature we almost always encounter the following particular singular term

$$\vartheta(x) = x^{-\gamma} \text{ for all } x > 0, \text{ with } 0 < \gamma < 1.$$

Of course, hypotheses  $H(\vartheta)$  provide a much more general framework and can accommodate also singularities like the ones that follow:

$$\begin{aligned} \vartheta_1(x) &= x^{-\gamma} [1 + \ln(1 + x)], \quad x > 0, \\ \vartheta_2(x) &= x^{-\gamma} e^{-x}, \quad x > 0 \\ \vartheta_3(x) &= \begin{cases} x^{-\gamma} (1 - \eta \sin x) & 0 < x \leq \frac{\pi}{2} \\ x^{-\gamma} (1 - \eta) & \text{if } \frac{\pi}{2} < x \end{cases} \text{ with } 0 < \gamma < 1. \end{aligned}$$

The following strong comparison principle can be found in Papageorgiou et al. [16, Proposition 6] (see also Papageorgiou and Smyrlis [17, Proposition 4]).

**Proposition 4** *If hypotheses  $H(a)$ ,  $H(\vartheta)$  hold,  $\hat{\xi} \in L^\infty(\Omega)$ ,  $\hat{\xi}(z) \geq 0$  for almost all  $z \in \Omega$ ,  $h_1, h_2 \in L^\infty(\Omega)$  satisfy*

$$0 < c_8 \leq h_2(z) - h_1(z) \text{ for almost all } z \in \Omega$$

and  $u, v \in C^{1,\alpha}(\overline{\Omega})$  satisfy  $0 < u(z) \leq v(z)$  for all  $z \in \Omega$  and for almost all  $z \in \Omega$  we have

- $-\operatorname{div} a(Du(z)) - \vartheta(u(z)) + \hat{\xi}(z)u(z)^{p-1} = h_1(z)$
- $-\operatorname{div} a(Dv(z)) - \vartheta(v(z)) + \hat{\xi}(z)v(z)^{p-1} = h_2(z),$

then  $v - u \in \operatorname{int} \hat{C}_+$ .

In what follows,  $p^*$  is the critical Sobolev exponent corresponding to  $p$ , that is,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N. \\ +\infty & \text{if } N \leq p. \end{cases}$$



Now we introduce our hypotheses on the nonlinearity  $f(z, x)$ .

$H(f) : f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}_+$  is a Carathéodory function such that  $f(z, 0) = 0$  for almost all  $z \in \Omega$  and

- (i)  $f(z, x) \leq a(z)(1 + x^{r-1})$  for almost all  $z \in \Omega$ , and all  $x \geq 0$ , with  $a \in L^\infty(\Omega)$ ,  $p < r < p^*$ ;
- (ii) if  $F(z, x) = \int_0^x f(z, s)ds$ , then  $\lim_{x \rightarrow +\infty} \frac{F(z,x)}{x^p} = +\infty$  uniformly for almost all  $z \in \Omega$ ;
- (iii) there exists  $\sigma \in ((r - p) \max\{\frac{N}{p}, 1\}, p^*)$ ,  $\sigma > q$  such that

$$0 < \hat{\beta}_0 \leq \liminf_{x \rightarrow +\infty} \frac{f(z, x)x - pF(z, x)}{x^\sigma} \text{ uniformly for almost all } z \in \Omega;$$

- (iv)  $\limsup_{x \rightarrow 0^+} \frac{f(z,x)}{x^{r-1}} \leq \eta_0$  uniformly for almost all  $z \in \Omega$ ;
- (v) for every  $\rho > 0$ , there exists  $\hat{\xi}_\rho > 0$  such that for almost all  $z \in \Omega$  the function

$$x \mapsto f(z, x) + \hat{\xi}_\rho x^{\rho-1}$$

is nondecreasing on  $[0, \rho]$ .

**Remark 3** Since our aim is to find positive solutions and the above hypotheses concern the positive semiaxis  $\mathbb{R}_+ = [0, +\infty)$ , we may assume that

$$f(z, x) = 0, \text{ for almost all } z \in \Omega, \text{ and all } x \leq 0. \tag{4}$$

Hypotheses  $H(f)(ii)$ ,  $(iii)$  imply that

$$\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \text{ uniformly for almost all } z \in \Omega. \tag{5}$$

So, the nonlinearity  $f(z, \cdot)$  is  $(p - 1)$ -superlinear near  $+\infty$ . However, this superlinearity of  $f(z, \cdot)$  is not formulated using the AR-condition. We recall that the AR-condition (unilateral version due to (4)), says that there exist  $\gamma > p$  and  $M > 0$  such that

$$0 < \gamma F(z, x) \leq f(z, x)x \text{ for almost all } z \in \Omega, \text{ and all } x \geq M, \tag{6a}$$

$$0 < \text{ess inf}_\Omega F(\cdot, M). \tag{6b}$$

If we integrate (6a) and use (6b), we obtain the weaker condition

$$\begin{aligned} c_9 x^\gamma &\leq F(z, x) \text{ for almost all } z \in \Omega, \text{ all } x \geq M, \text{ and some } c_9 > 0, \\ &\Rightarrow c_9 x^{\gamma-1} \leq f(z, x) \text{ for almost all } z \in \Omega, \text{ and all } x \geq M. \end{aligned} \tag{7}$$

Therefore the AR-condition implies that  $f(z, \cdot)$  exhibits at least  $(\gamma - 1)$ -polynomial growth. Evidently, (7) implies the much weaker condition (5). In this work instead of the standard AR-condition, we employ the less restrictive hypothesis  $H(f)(iii)$ .

In this way we incorporate in our framework also  $(p - 1)$ -superlinear terms with “slower” growth near  $+\infty$ , which fail to satisfy the AR-condition. The following function satisfies hypotheses  $H(f)$  but fails to satisfy the AR-condition (for the sake of simplicity we drop the  $z$ -dependence)

$$f(x) = x^{p-1} \ln(1 + x) \text{ for all } x \geq 0.$$

Finally, let us fix the notation which we will use throughout this work. For  $x \in \mathbb{R}$  we set  $x^\pm = \max\{\pm x, 0\}$ . Then for  $u \in W_0^{1,p}(\Omega)$  we define  $u^\pm(z) = u(z)^\pm$  for almost all  $z \in \Omega$ . It follows that

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If  $u, v \in W_0^{1,p}(\Omega)$  and  $u \leq v$ , then we define.

$$[u, v] = \{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \text{ for almost all } z \in \Omega\},$$

$$[u] = \{y \in W_0^{1,p}(\Omega) : u(z) \leq y(z) \text{ for almost all } z \in \Omega\}.$$

Also, by  $\text{int}_{C_0^1(\overline{\Omega})}[u, v]$  we denote the interior in the  $C_0^1(\overline{\Omega})$ -norm topology of the set  $[u, v] \cap C_0^1(\overline{\Omega})$ .

By  $A : W_0^{1,p}(\Omega) \rightarrow W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \left(\frac{1}{p} + \frac{1}{p'} = 1\right)$  we denote the nonlinear operator defined by

$$\langle A(u), h \rangle = \int_{\Omega} (a(Du), Dh)_{\mathbb{R}^N} dz \text{ for all } u, h \in W_0^{1,p}(\Omega).$$

We know (see Gasinski and Papageorgiou [4]), that  $A(\cdot)$  is continuous, strictly monotone (hence maximal monotone, too) and of type  $(S)_+$ , that is,

$$\text{“if } u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ and } \limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0, \text{ then } u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega).”$$

We introduce the following two sets related to problem  $(P_\lambda)$ :

$$\mathcal{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ admits a positive solution}\},$$

$$S_\lambda = \text{the set of positive solutions for problem } (P_\lambda).$$

We let  $\lambda^* = \sup \mathcal{L}$ .

### 3 Positive solutions

We start by considering the following purely singular problem:

$$-\text{div } a(Du(z)) + \xi^+(z)u(z)^{p-1} = \vartheta(u(z)) \text{ in } \Omega, \quad u|_{\partial\Omega} = 0, \quad u > 0. \quad (8)$$

From Papageorgiou et al. [16, Proposition 10], we have the following property.

**Proposition 5** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$  hold, then problem (8) admits a unique positive solution  $v \in \text{int } C_+$ .*

Let  $\beta > \|\xi\|_\infty$ . Then hypotheses  $H(f)(i)$ ,  $(iv)$  and since  $1 < q < p < r$ , imply that we can find  $c_{10}, c_{11} > 0$  such that

$$\lambda x^{q-1} + f(z, x) \leq \lambda c_{10} x^{q-1} + c_{11} x^{r-1} - \beta x^{p-1} \text{ for almost all } z \in \Omega, \text{ and all } x \geq 0. \tag{9}$$

Let  $k_\lambda(x) = \lambda c_{10} x^{q-1} + c_{11} x^{r-1} - \beta x^{p-1}$  for all  $x \geq 0$ . With  $v \in \text{int } C_+$  from Proposition 5, we consider the following auxiliary Dirichlet problem:

$$\left\{ \begin{array}{l} -\text{div } a(Du(z)) + \xi(z)u(z)^{p-1} = \vartheta(v(z)) + k_\lambda(u(z)) \text{ in } \Omega \\ u|_{\partial\Omega} = 0, u > 0. \end{array} \right\} \tag{10}_\lambda$$

For this problem we prove the following result.

**Proposition 6** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$  hold, then for all small enough  $\lambda > 0$  problem  $(10)_\lambda$  has a smallest positive solution*

$$\bar{u}_\lambda \in \text{int } C_+.$$

**Proof** Recall that  $v \in \text{int } C_+$  (see Proposition 5). Hence  $v \in \text{int } K_+$  (see (1)). For  $s > N$  we consider the function  $\hat{u}_1^{1/s} \in K_+$ . According to Proposition 1, we can find  $\mu > 0$  such that

$$\begin{aligned} \hat{u}_1^{1/s} &\leq \mu v, \\ \Rightarrow v^{-\gamma} &\leq \mu^\gamma \hat{u}_1^{-\gamma/s}. \end{aligned} \tag{11}$$

From the Lemma in Lazer and McKenna [9, p. 726], we have

$$\begin{aligned} \hat{u}_1^{-\gamma/s} &\in L^s(\Omega), \\ \Rightarrow v^{-\gamma} &\in L^s(\Omega) \text{ (see (11)).} \end{aligned} \tag{12}$$

Hypotheses  $H(\vartheta)$  imply that we can find  $c_{12} > 0$  and  $\delta > 0$  such that

$$0 \leq \vartheta(x) \leq c_{12} x^{-\gamma} \text{ for all } 0 \leq x \leq \delta \text{ and } 0 \leq \vartheta(x) \leq \vartheta(\delta) \text{ for all } x > \delta. \tag{13}$$

It follows from (12), (13) that

$$\vartheta(v(\cdot)) \in L^s(\Omega) \text{ (} s > N \text{)}.$$

Let  $\hat{k}_\lambda(x) = \lambda c_{10} x^{q-1} + c_{11} x^{r-1}$  for all  $x \geq 0$  and set  $\hat{K}_\lambda(x) = \int_0^x \hat{k}_\lambda(s) ds$ . We consider the  $C^1$ -functional  $\psi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\psi_\lambda(u) = \int_\Omega G(Du) dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta] |u|^p dz - \int_\Omega \hat{K}_\lambda(u^+) dz - \int_\Omega \vartheta(v) u^+ dz$$

for all  $u \in W_0^{1,p}(\Omega)$

$$\begin{aligned} &\geq \frac{c_1}{p(p-1)} \|Du\|_p^p + \frac{1}{p} \int_{\Omega} [\xi(z) + \beta] |u|^p dz - \frac{\lambda c_{10}}{q} \|u\|_q^q \\ &\quad - \frac{c_{11}}{r} \|u\|_r^r - \int_{\Omega} \vartheta(v) |u| dz \\ &\text{(see Corollary 3)} \\ &\geq c_{12} \|u\|^p - c_{13} [\lambda \|u\| + \|u\|^r] \\ &\text{for some } c_{12}, c_{13} > 0 \text{ and all } 0 < \lambda \leq 1 \text{ (recall that } \beta > \|\xi\|_{\infty} \text{ and } 1 < q < r) \\ &= [c_{12} - c_{13}(\lambda \|u\|^{1-p} + \|u\|^{r-p})] \|u\|^p. \end{aligned} \tag{14}$$

We introduce the function  $\mathfrak{S}_{\lambda}(t) = \lambda t^{1-p} + t^{r-p}, t > 0$ . Evidently,  $\mathfrak{S}_{\lambda} \in C^1(0, +\infty)$  and since  $1 < p < r$ , we see that

$$\mathfrak{S}_{\lambda}(t) \rightarrow +\infty \text{ as } t \rightarrow 0^+ \text{ and as } t \rightarrow +\infty.$$

So, we can find  $t_0 > 0$  such that

$$\begin{aligned} \mathfrak{S}_{\lambda}(t_0) &= \inf\{\mathfrak{S}_{\lambda}(t) : t > 0\}, \\ &\Rightarrow \mathfrak{S}'_{\lambda}(t_0) = 0, \\ &\Rightarrow \lambda(p-1)t_0^{-p} = (r-p)t_0^{r-p-1} \\ &\Rightarrow t_0 = \left[ \frac{\lambda(p-1)}{r-p} \right]^{\frac{1}{r-1}}. \end{aligned}$$

Since  $\frac{p-1}{r-1} < 1$ , it follows that

$$\mathfrak{S}_{\lambda}(t_0) \rightarrow 0 \text{ as } \lambda \rightarrow 0^+.$$

So, we can find  $\lambda_0 \in (0, 1]$  such that

$$\mathfrak{S}_{\lambda}(t_0) \leq \frac{c_{12}}{c_{13}} \text{ for all } \lambda \in (0, \lambda_0].$$

For  $\rho = t_0$ , we see from (14) that

$$\psi_{\lambda}|_{\partial \bar{B}_{\rho}} > 0, \tag{15}$$

where  $\bar{B}_{\rho} = \{u \in W_0^{1,p}(\Omega) : \|u\| \leq \rho\}$  and  $\partial \bar{B}_{\rho} = \{u \in W_0^{1,p}(\Omega) : \|u\| = \rho\}$ .

We fix  $\lambda \in (0, \lambda_0]$ . Hypothesis  $H(a)(iv)$  implies that we can find  $c_0^* > c^*$  and  $\delta > 0$  such that

$$G(y) \leq \frac{c_0^*}{\tau} |y|^{\tau} \text{ for all } |y| \leq \delta.$$

Let  $u \in \text{int } C_+$  and choose small enough  $t \in (0, 1)$  such that

$$t|Du(z)| \leq \delta \text{ for all } z \in \bar{\Omega}.$$

Then we have

$$\psi_\lambda(tu) \leq \frac{t^\tau c_0^*}{\tau} \|Du\|_\tau^\tau + \frac{t^p}{p} \int_\Omega [\xi(z) + \beta] |u|^p dz - \frac{\lambda t^q}{q} \|u\|_q^q.$$

Since  $q < \tau \leq p$ , choosing  $t \in (0, 1)$  even smaller if necessary, we have

$$\begin{aligned} \psi_\lambda(tu) &< 0, \\ \Rightarrow \inf_{\overline{B}_\rho} \psi_\lambda &< 0. \end{aligned} \tag{16}$$

The functional  $\psi_\lambda(\cdot)$  is sequentially weakly lower semicontinuous and by the Eberlein-Smulian theorem and the reflexivity of  $W_0^{1,p}(\Omega)$ , the set  $\overline{B}_\rho$  is sequentially weakly compact. So, by the Weierstrass-Tonelli theorem, we can find  $\bar{u} \in W_0^{1,p}(\Omega)$  such that

$$\psi_\lambda(\bar{u}) = \inf\{\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)\} \quad (\lambda \in (0, \lambda_0]). \tag{17}$$

From (15), (16) and (17) it follows that

$$\begin{aligned} 0 &< \|\bar{u}\| < \rho, \\ \Rightarrow \psi'_\lambda(\bar{u}) &= 0 \text{ (see (17)),} \\ \Rightarrow \langle A(\bar{u}), h \rangle + \int_\Omega [\xi(z) + \beta] |\bar{u}|^{p-2} \bar{u} h dz &= \int_\Omega [\vartheta(v) + \hat{k}_\lambda(\bar{u}^+)] h dz \\ \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{18}$$

In (18) we choose  $h = -\bar{u}^- \in W_0^{1,p}(\Omega)$ . Using Lemma 2(c) and since  $\beta > \|\xi\|_\infty$  we obtain

$$\begin{aligned} c_{14} \|\bar{u}^-\|^p &\leq 0 \text{ for some } c_{14} > 0, \\ \Rightarrow \bar{u} &\geq 0, \bar{u} \neq 0. \end{aligned}$$

Then from (18) we have

$$\begin{aligned} -\operatorname{div} a(D\bar{u}(z)) + \xi(z)\bar{u}(z)^{p-1} &= \vartheta(v(z)) + k_\lambda(\bar{u}(z)) \text{ for almost all } z \in \Omega, \\ \Rightarrow \bar{u} \in W_0^{1,p}(\Omega) &\text{ is a positive of problem } (10)_\lambda \text{ for } \lambda \in (0, \lambda_0]. \end{aligned} \tag{19}$$

From (19) and Theorem 7.1 of Ladyzhenskaya and Uraltseva [8, p. 286], we have  $\bar{u} \in L^\infty(\Omega)$ . Hence  $k_\lambda(\bar{u}(\cdot)) \in L^\infty(\Omega)$ . Recall that  $\vartheta(v(\cdot)) \in L^s(\Omega)$  with  $s > N$ . From Theorem 9.15 of Gilbarg and Trudinger [6, p. 241], we know that there exists a unique solution  $y_0 \in W^{2,s}(\Omega)$  to the following linear Dirichlet problem

$$-\Delta y(z) = \vartheta(v(z)) \text{ in } \Omega, \quad y|_{\partial\Omega} = 0.$$

By the Sobolev embedding theorem, we have

$$W^{2,s}(\Omega) \hookrightarrow C^{1,\alpha}(\overline{\Omega}) \text{ with } \alpha = 1 - \frac{N}{s} > 0.$$

Let  $\eta_0(z) = Dy_0(z)$ . Then  $\eta_0 \in C^\alpha(\overline{\Omega}, \mathbb{R}^N)$  and we have

$$-\operatorname{div}(a(D\bar{u}(z)) - \eta_0(z)) + \xi(z)\bar{u}(z)^{p-1} = k_\lambda(\bar{u}(z)) \text{ for almost all } z \in \Omega.$$

The regularity theory of Lieberman [11] implies that  $\bar{u} \in C_+ \setminus \{0\}$ . Moreover, from (19) we have

$$\begin{aligned} \operatorname{div} a(D\bar{u}(z)) &\leq \|\xi\|_\infty \bar{u}(z)^{p-1} \text{ for almost all } z \in \Omega, \\ \Rightarrow \bar{u} &\in \operatorname{int} C_+ \end{aligned}$$

(from the nonlinear maximum principle, see Pucci and Serrin [21, pp. 111,120]).

Let  $\hat{S}_\lambda$  denote the set of positive solutions of problem  $(10)_\lambda$ . We have just seen that  $\emptyset \neq \hat{S}_\lambda \subseteq \operatorname{int} C_+$  for  $\lambda \in (0, \lambda_0]$ . Moreover, from Papageorgiou et al. [16, Proposition 18], we know that  $\hat{S}_\lambda$  is downward directed (that is, if  $u_1, u_2 \in \hat{S}_\lambda$ , then we can find  $u \in \hat{S}_\lambda$  such that  $u \leq u_1, u \leq u_2$ ). So, by Lemma 3.10 of Hu and Papageorgiou [7, p. 178], we can find a decreasing sequence  $\{\bar{u}_n\}_{n \geq 1} \subseteq \hat{S}_\lambda$  such that

$$\inf \hat{S}_\lambda = \inf_{n \geq 1} \bar{u}_n.$$

For every  $n \in \mathbb{N}$  we have

$$\langle A(\bar{u}_n), h \rangle + \int_\Omega \xi(z)\bar{u}_n^{p-1} h dz = \int_\Omega [\vartheta(v) + k_\lambda(\bar{u}_n)] h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{20}$$

Choosing  $h = \bar{u}_n \in W_0^{1,p}$  and since  $0 \leq \bar{u}_n \leq \bar{u}_1$  for all  $n \in \mathbb{N}$ , using Lemma 2(c), we see that  $\{\bar{u}_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$  is bounded. So, we have

$$\bar{u}_n \overset{w}{\rightharpoonup} \bar{u}_\lambda \text{ in } W_0^{1,p}(\Omega). \tag{21}$$

Next, in (20) we choose  $h = \bar{u}_n - \bar{u} \in W_0^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$  and use (21). Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \langle A(\bar{u}_n), \bar{u}_n - \bar{u}_\lambda \rangle &= 0, \\ \Rightarrow \bar{u}_n &\rightarrow \bar{u}_\lambda \text{ in } W_0^{1,p}(\Omega), \bar{u}_\lambda \geq 0 \\ \text{(recall that } A(\cdot) &\text{ is of type } (S)_+, \text{ see Sect. 2).} \end{aligned} \tag{22}$$

We pass to the limit as  $n \rightarrow \infty$  in (20) and use (22). Then

$$\begin{aligned} \langle A(\bar{u}_\lambda), h \rangle + \int_\Omega \xi(z)\bar{u}_\lambda^{p-1} h dz &= \int_\Omega [\vartheta(v) + k_\lambda(\bar{u}_\lambda)] h dz \\ \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow \bar{u}_\lambda &\text{ is a nonnegative solution of } (10)_\lambda \end{aligned}$$

Note that for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned}
 -\operatorname{div} a(D\bar{u}_n(z)) + \xi^+(z)\bar{u}_n(z)^{p-1} &\geq \vartheta(v(z)) + k_\lambda(\bar{u}_n(z)) \\
 &\geq \vartheta(v(z)) = -\operatorname{div} a(Dv(z)) + \xi^+(z)v(z)^{p-1} \\
 &\text{for almost all } z \in \Omega, \\
 \Rightarrow v &\leq \bar{u}_n \text{ for all } n \in \mathbb{N}
 \end{aligned}$$

(by the weak comparison principle, see Damascelli [3, Theorem 1.2])

$$\Rightarrow v \leq \bar{u}_\lambda \text{ (see (22)), hence } \bar{u}_\lambda \neq 0. \tag{23}$$

Therefore  $\bar{u}_\lambda \in \hat{S}_\lambda \subseteq \operatorname{int} C_+$  and  $\bar{u}_\lambda = \inf \hat{S}_\lambda$ . □

We will use  $\bar{u}_\lambda \in \operatorname{int} C_+$  from Proposition 6 to show the nonemptiness of  $\mathcal{L}$ .

**Proposition 7** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$ ,  $H(f)$  hold, then  $\mathcal{L} \neq \emptyset$  and  $S_\lambda \subseteq \operatorname{int} C_+$ .*

**Proof** From (9) we have

$$\lambda x^{q-1} + f(z, x) \leq k_\lambda(x) \text{ for almost all } z \in \Omega, \text{ and all } x \geq 0, \lambda > 0. \tag{24}$$

For  $\lambda \in (0, \lambda_0]$  we have

$$\begin{aligned}
 -\operatorname{div} a(D\bar{u}_\lambda(z)) + \xi(z)\bar{u}_\lambda(z)^{p-1} &= \vartheta(v(z)) + k_\lambda(\bar{u}_\lambda(z)) \\
 &\text{(see Proposition 6)} \\
 &\geq \vartheta(\bar{u}_\lambda(z)) + k_\lambda(\bar{u}_\lambda(z)) \\
 &\text{(see (23) and hypothesis } H(\vartheta)(ii)) \\
 &\geq \vartheta(\bar{u}_\lambda(z)) + \lambda\bar{u}_\lambda(z)^{q-1} + f(z, \bar{u}_\lambda(z)) \\
 &\text{for almost all } z \in \Omega \text{ (see (24)).} \tag{25}
 \end{aligned}$$

With  $\beta > \|\xi\|_\infty$  and  $\lambda \in (0, \lambda_0]$ , we consider the following truncation-perturbation of the reaction in problem  $(P_\lambda)$ :

$$\gamma_\lambda(z, x) = \begin{cases} \vartheta(v(z)) + \lambda v(z)^{q-1} + f(z, v(z)) + \beta v(z)^{p-1} & \text{if } x < v(z) \\ \vartheta(x) + \lambda x^{q-1} + f(z, x) + \beta x^{p-1} & \text{if } v(z) \leq x \leq \bar{u}_\lambda(z) \\ \vartheta(\bar{u}_\lambda(z)) + \lambda\bar{u}_\lambda(z)^{q-1} + f(z, \bar{u}_\lambda(z)) + \beta\bar{u}_\lambda(z)^{p-1} & \text{if } \bar{u}_\lambda(z) < x. \end{cases} \tag{26}$$

This is a Carathéodory function. We set  $\Gamma_\lambda(z, x) = \int_0^x \gamma_\lambda(z, s)ds$  and consider the functional  $\hat{\sigma}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned}
 \hat{\sigma}_\lambda(u) &= \int_\Omega G(Du)dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta]|u|^p dz - \int_\Omega \Gamma_\lambda(z, u)dz \\
 &\text{for all } u \in W_0^{1,p}(\Omega).
 \end{aligned}$$

Using Proposition 3 of Papageorgiou and Smyrlis [17], we see that  $\hat{\sigma}_\lambda \in C^1(W_0^{1,p}(\Omega))$ . Also, from (26), Corollary 3 and since  $\beta > \|\xi\|_\infty$ , we see that  $\hat{\sigma}_\lambda(\cdot)$  is coercive. In addition, it is sequentially weakly lower semicontinuous. So, we can find  $u_\lambda \in W_0^{1,p}(\Omega)$  such that

$$\begin{aligned} \hat{\sigma}_\lambda(u_\lambda) &= \inf\{\hat{\sigma}(u) : u \in W_0^{1,p}(\Omega)\}, \\ &\Rightarrow \hat{\sigma}'_\lambda(u_\lambda) = 0, \\ &\Rightarrow \langle A(u_\lambda, h) \rangle + \int_\Omega [\xi(z) + \beta]|u_\lambda|^{p-2}u_\lambda h dz \\ &= \int_\Omega \gamma_\lambda(z, u_\lambda) h dz \\ &\text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{27}$$

In (27) first we choose  $h = (u_\lambda - \bar{u}_\lambda)^+ \in W_0^{1,p}(\Omega)$ . Then we have

$$\begin{aligned} &\langle A(u_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \beta]u_\lambda^{p-1}(u_\lambda - \bar{u}_\lambda)^+ dz \\ &= \int_\Omega [\vartheta(\bar{u}_\lambda) + \lambda \bar{u}_\lambda^{q-1} + f(z, \bar{u}_\lambda) + \beta \bar{u}_\lambda^{p-1}](u_\lambda - \bar{u}_\lambda)^+ dz \text{ (see (26))} \\ &\leq \langle A(\bar{u}_\lambda), (u_\lambda - \bar{u}_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \beta]\bar{u}_\lambda^{p-1}(u_\lambda - \bar{u}_\lambda)^+ dz \text{ (see (25))}, \\ &\Rightarrow u_\lambda \leq \bar{u}_\lambda \text{ (since } \beta > \|\xi\|_\infty \text{)}. \end{aligned}$$

Next, in (27) we choose  $h = (v - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ . Then we have

$$\begin{aligned} &\langle A(u_\lambda), (v - u_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \beta]|u_\lambda|^{p-2}u_\lambda(v - u_\lambda)^+ dz \\ &= \int_\Omega [\vartheta(v) + \lambda v^{q-1} + f(z, v) + \beta v^{p-1}](v - u_\lambda)^+ dz \text{ (see (26))} \\ &\geq \int_\Omega [\vartheta(v) + \beta v^{p-1}](v - u_\lambda)^+ dz \text{ (since } f \geq 0 \text{)} \\ &= \langle A(v), (v - u_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \beta]v^{p-1}(v - u_\lambda)^+ dz \text{ (see Proposition 5)}, \\ &\Rightarrow v \leq u_\lambda. \end{aligned}$$

So, we have proved that

$$u_\lambda \in [v, \bar{u}_\lambda] \quad (\lambda \in (0, \lambda_0)). \tag{28}$$

It follows from (26), (27) and (28) that

$$\begin{aligned} -\operatorname{div} a(Du_\lambda(z)) + \xi(z)u_\lambda(z)^{p-1} &= \vartheta(u_\lambda(z)) + \lambda u_\lambda(z)^{q-1} + f(z, u_\lambda(z)) \\ &\text{for almost all } z \in \Omega. \end{aligned}$$



Note that  $\vartheta_\lambda(u_\lambda) \leq \vartheta(v)$  (see (28) and hypothesis  $H(\vartheta)(ii)$ ) and  $\vartheta(v) \in L^s(\Omega)$ . So, as before (see the proof of Proposition 6), we infer that

$$u_\lambda \in \text{int } C_+.$$

Therefore we have seen that

$$(0, \lambda_0] \subseteq \mathcal{L}, \text{ hence } \mathcal{L} \neq \emptyset$$

$$\text{and } S_\lambda \subseteq \text{int } C_+.$$

The proof is now complete. □

For  $\eta > 0$ , let  $\tilde{u}_\eta \in \text{int } C_+$  be the unique solution of the following Dirichlet problem

$$-\text{div } a(Du(z)) + \xi^+(z)u(z)^{p-1} = \eta \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

By Proposition 9 of Papageorgiou et al. [16], we see that given  $u \in S_\lambda \subseteq \text{int } C_+$  (that is,  $\lambda \in \mathcal{L}$ ), we can find small  $\eta > 0$  such that

$$\tilde{u}_\eta \leq u \text{ and } \eta \leq \vartheta(\tilde{u}_\eta). \tag{29}$$

We will use this to obtain a lower bound for the elements of  $S_\lambda$ .

**Proposition 8** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$ ,  $H(f)$  hold and  $\lambda \in \mathcal{L}$ , then  $v \leq u$  for all  $u \in S_\lambda$ .*

**Proof** Let  $u \in S_\lambda \subseteq \text{int } C_+$ . Then on account of (29) we can define the following Carathéodory function

$$e(z, x) = \begin{cases} \vartheta(\tilde{u}_\eta(z)) & \text{if } x < \tilde{u}_\eta(z) \\ \vartheta(x) & \text{if } \tilde{u}_\eta(z) \leq x \leq u(z) \\ \vartheta(u(z)) & \text{if } u(z) < x. \end{cases} \tag{30}$$

We set  $E(z, x) = \int_0^x e(z, s)ds$  and consider the functional  $\mu : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\mu(u) = \int_\Omega G(Du)dz + \frac{1}{p} \int_\Omega \xi^+(z)|u|^p dz - \int_\Omega E(z, u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

As before, Proposition 3 of Papageorgiou and Smyrlis [17] implies that  $\mu \in C^1(W_0^{1,p}(\Omega))$ . The coercivity of  $\mu(\cdot)$  (see (30)) and the sequential weak lower semi-continuity guarantee the existence of  $\tilde{v} \in W_0^{1,p}(\Omega)$  such that

$$\begin{aligned}
\mu(\tilde{v}) &= \inf\{\mu(u) : u \in W_0^{1,p}(\Omega)\}, \\
&\Rightarrow \mu'(\tilde{v}) = 0, \\
&\Rightarrow \langle A(\tilde{v}), h \rangle + \int_{\Omega} \xi^+(z) |\tilde{v}|^{p-2} \tilde{v} h dz = \int_{\Omega} e(z, \tilde{v}) h dz \text{ for all } h \in W_0^{1,p}(\Omega).
\end{aligned}
\tag{31}$$

In (31) we choose  $h = (\tilde{v} - u)^+ \in W_0^{1,p}(\Omega)$ . Then we have

$$\begin{aligned}
&\langle A(\tilde{v}), (\tilde{v} - u)^+ \rangle + \int_{\Omega} \xi^+(z) \tilde{v}^{p-1} (\tilde{v} - u)^+ dz \\
&= \int_{\Omega} \vartheta(u) (\tilde{v} - u)^+ dz \text{ (see (30))} \\
&\leq \int_{\Omega} [\vartheta(u) + \lambda u^{q-1} + f(z, u)] (\tilde{v} - u)^+ dz \text{ (since } u \in \text{int } C_+, f \geq 0) \\
&\leq \langle A(u), (\tilde{v} - u)^+ \rangle + \int_{\Omega} \xi^+(z) u^{p-1} (\tilde{v} - u)^+ dz \text{ (since } u \in S_{\lambda}), \\
&\Rightarrow \tilde{v} \leq u.
\end{aligned}$$

Similarly, if in (31) we choose  $h = (\tilde{u}_{\eta} - \tilde{v})^+ \in W_0^{1,p}(\Omega)$ , then we have

$$\begin{aligned}
&\langle A(\tilde{v}), (\tilde{u}_{\eta} - \tilde{v})^+ \rangle + \int_{\Omega} \xi^+(z) |\tilde{v}|^{p-2} \tilde{v} (\tilde{u}_{\eta} - \tilde{v})^+ dz \\
&= \int_{\Omega} \vartheta(\tilde{u}_{\eta}) (\tilde{u}_{\eta} - \tilde{v})^+ dz \text{ (see (30))} \\
&\geq \int_{\Omega} \eta (\tilde{u}_{\eta} - \tilde{v})^+ dz \text{ (see (29))} \\
&= \langle A(\tilde{u}_{\eta}), (\tilde{u}_{\eta} - \tilde{v})^+ \rangle + \int_{\Omega} \xi^+(z) \tilde{u}_{\eta}^{p-1} (\tilde{u}_{\eta} - \tilde{v})^+ dz, \\
&\Rightarrow \tilde{u}_{\eta} \leq \tilde{v}.
\end{aligned}$$

So, we have proved that

$$\tilde{v} \in [\tilde{u}_{\eta}, u]. \tag{32}$$

It follows from (30), (31), (32) that  $\tilde{v}$  is a positive solution of (18). Then on account of Proposition 5, we have

$$\begin{aligned}
\tilde{v} &= v \in \text{int } C_+, \\
&\Rightarrow v \leq u \text{ for all } u \in S_{\lambda} \text{ (see (32)).}
\end{aligned}$$

The proof is now complete.  $\square$

Next, we show a structural property of the set  $\mathcal{L}$ , namely that  $\mathcal{L}$  is an interval. Moreover, we establish a kind of strong monotonicity property for the solution set  $S_{\lambda}$ .

**Proposition 9** *If hypotheses  $H(a), H(\xi), H(\vartheta), H(f)$  hold,  $\lambda \in \mathcal{L}, 0 < \mu < \lambda$  and  $u_\lambda \in S_\lambda \subseteq \text{int } C_+$ , then  $\mu \in \mathcal{L}$  and there exists  $u_\mu \in S_\mu \subseteq \text{int } C_+$  such that  $u_\lambda - u_\mu \in \text{int } C_+$ .*

**Proof** From Proposition 8 we know that  $v \leq u_\lambda$ . Then with  $\beta > \|\xi\|_\infty$  we can define the following truncation-perturbation of the reaction in problem  $(P_\mu)$ :

$$e_\mu(z, x) = \begin{cases} \vartheta(v(z)) + \mu v(z)^{q-1} + f(z, v(z)) + \beta v(z)^{p-1} & \text{if } x < v(z) \\ \vartheta(x) + \mu x^{q-1} + f(z, x) + \beta x^{p-1} & \text{if } v(z) \leq x \leq u_\lambda(z) \\ \vartheta(u_\lambda(z)) + \mu u_\lambda(z)^{q-1} + f(z, u_\lambda(z)) + \beta u_\lambda(z)^{p-1} & \text{if } u_\lambda(z) < x. \end{cases} \tag{33}$$

Evidently,  $e_\mu(z, x)$  is a Carathéodory function. We set  $E_\mu(z, x) = \int_0^x e_\mu(z, s) ds$  and consider the  $C^1$ -functional  $\hat{\psi}_\mu : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\begin{aligned} \hat{\psi}_\mu(u) &= \int_\Omega G(Du) dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta] |u|^p dz \\ &\quad - \int_\Omega E_\mu(z, u) dz \text{ for all } u \in W_0^{1,p}(\Omega). \end{aligned}$$

Clearly,  $\hat{\psi}_\mu(\cdot)$  is coercive (see (33) and recall that  $\beta > \|\xi\|_\infty$ ). It is also sequentially weakly lower semicontinuous. So, we can find  $u_\mu \in W_0^{1,p}(\Omega)$  such that

$$\begin{aligned} \hat{\psi}_\mu(u_\mu) &= \inf \{ \hat{\psi}_\mu(u) : u \in W_0^{1,p}(\Omega) \}, \\ \Rightarrow \hat{\psi}'_\mu(u_\mu) &= 0, \\ \Rightarrow \langle A(u_\mu), h \rangle + \int_\Omega [\xi(z) + \beta] |u_\mu|^{p-2} u_\mu h dz \\ &= \int_\Omega e_\mu(z, u_\mu) h dz \text{ for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{34}$$

In (34) we first use  $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$ . Then

$$\begin{aligned} &\langle A(u_\mu), (u_\mu - u_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \beta] u_\mu^{p-1} (u_\mu - u_\lambda)^+ dz \\ &= \int_\Omega [\vartheta(u_\lambda) + \mu u_\lambda^{q-1} + f(z, u_\lambda) + \beta u_\lambda^{p-1}] (u_\mu - u_\lambda)^+ dz \text{ (see (33))} \\ &\leq \int_\Omega [\vartheta(u_\lambda) + \lambda u_\lambda^{q-1} + f(z, u_\lambda) + \beta u_\lambda^{p-1}] (u_\mu - u_\lambda)^+ dz \text{ (since } \lambda > \mu) \\ &= \langle A(u_\lambda), (u_\mu - u_\lambda)^+ \rangle + \int_\Omega [\xi(z) + \beta] u_\lambda^{p-1} (u_\mu - u_\lambda)^+ dz \text{ (since } u_\lambda \in S_\lambda), \\ &\Rightarrow u_\mu \leq u_\lambda \text{ (recall that } \beta > \|\xi\|_\infty). \end{aligned}$$

Next, in (34) we use  $h = (v - u_\mu)^+ \in W_0^{1,p}(\Omega)$ . Then from Proposition 5 and since  $f \geq 0$ , we obtain

$$v \leq u_\mu.$$

We have proved that

$$u_\mu \in [v, u_\lambda]. \tag{35}$$

It follows from (33), (34), (35) that  $u_\mu \in S_\mu \subseteq \text{int } C_+$  and so  $\mu \in \mathcal{L}$ .

Let  $\rho = \|u_\lambda\|_\infty$  and let  $\hat{\xi}_\rho > 0$  as postulated by hypothesis  $H(f)(v)$ . We have

$$\begin{aligned} & -\text{div } a(Du_\mu) + [\hat{\xi}(z) + \hat{\xi}_\rho]u_\mu^{p-1} - \vartheta(u_\mu) \\ & = \mu u_\mu^{q-1} + f(z, u_\mu) + \hat{\xi}_\rho u_\mu^{p-1} \\ & \leq \lambda u_\lambda^{q-1} + f(z, u_\lambda) + \hat{\xi}_\rho u_\lambda^{p-1} \text{ (see hypothesis } H(f)(vi), \\ & \text{(35) and recall that } \mu < \lambda) \\ & = -\text{div } a(Du_\lambda) + [\hat{\xi}(z) + \hat{\xi}_\rho]u_\lambda^{p-1} - \vartheta(u_\lambda). \end{aligned} \tag{36}$$

From (36) and Proposition 4 of Papageorgiou and Smyrlis [17], we obtain

$$u_\lambda - u_\mu \in \text{int } C_+.$$

The proof is now complete. □

**Proposition 10** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$ ,  $H(f)$  hold, then  $\lambda^* < +\infty$ .*

**Proof** Recall that by hypotheses  $H(f)(ii)$ ,  $(iii)$ , we have

$$\lim_{x \rightarrow +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \text{ uniformly for almost all } z \in \Omega.$$

So, we can find  $M > 0$  such that

$$f(z, x) \geq x^{p-1} \text{ for almost all } z \in \Omega, \text{ and all } x \geq M. \tag{37}$$

Hypotheses  $H(\vartheta)$  imply that we can find small  $\delta \in (0, 1]$  such that

$$\vartheta(x) \geq \vartheta(\delta) \geq 1 \geq \delta^{p-1} \geq x^{p-1} \text{ for all } x \in (0, \delta]. \tag{38}$$

Finally, hypotheses  $H(f)(i)$ ,  $(v)$  imply that we can find big  $\lambda_0 > 0$  such that

$$\lambda_0 x^{q-1} + f(z, x) \geq x^{p-1} \text{ for almost all } z \in \Omega \text{ and all } \delta \leq x \leq M. \tag{39}$$

Combining (37), (38), (39) we have

$$\vartheta(x) + \lambda_0 x^{q-1} + f(z, x) \geq x^{p-1} \text{ for almost all } z \in \Omega \text{ and all } x \geq 0. \tag{40}$$

Let  $\lambda > \lambda_0$  and assume that  $\lambda \in \mathcal{L}$ . Then according to Proposition 7 we can find  $u_\lambda \in S_\lambda \subseteq \text{int } C_+$ . Let  $\Omega_0 \subseteq \Omega$  be an open set with  $\overline{\Omega}_0 \subseteq \Omega$  and  $C^2$ -boundary  $\partial\Omega_0$ . We have

$$0 < m_0 = \min_{\overline{\Omega}_0} u_\lambda.$$

For  $\epsilon > 0$ , let  $m_0^\epsilon = m_0 + \epsilon$  and with  $\rho = \|u_\lambda\|_\infty$ , let  $\hat{\xi}_\rho > 0$  be as postulated by hypothesis  $H(f)(v)$ . We can always take  $\hat{\xi}_\rho > \|\xi\|_\infty$ . We have

$$\begin{aligned} & -\text{div } a(Dm_0^\epsilon) + [\xi(z) + \hat{\xi}_\rho](m_0^\epsilon)^{p-1} - \vartheta(m_0^\epsilon) \\ & \leq [\xi(z) + \hat{\xi}_\rho]m_0^{p-1} + \chi(\epsilon) - \vartheta(m_0) \\ & \quad \text{with } \chi(\epsilon) \rightarrow 0^+ \text{ as } \epsilon \rightarrow 0^+ \text{ (see hypotheses } H(\vartheta)) \\ & < [\xi(z) + \hat{\xi}_\rho]u_\lambda^{p-1} + u_\lambda^{p-1} - \vartheta(u_\lambda) + \chi(\epsilon) \\ & < [\xi(z) + \hat{\xi}_\rho]u_\lambda^{p-1} + \lambda_0 u_\lambda^{p-1} + f(z, u_\lambda) + \chi(\epsilon) \text{ (see (40))} \\ & = [\xi(z) + \hat{\xi}_\rho]u_\lambda^{p-1} + \lambda u_\lambda^{q-1} + f(z, u_\lambda) - (\lambda - \lambda_0)u_\lambda^{q-1} + \chi(\epsilon) \\ & < [\xi(z) + \hat{\xi}_\rho]u_\lambda^{p-1} + \lambda u_\lambda^{q-1} + f(z, u_\lambda) \text{ for } \epsilon > 0 \text{ small enough} \\ & = -\text{div } a(Du_\lambda) + [\xi(z) + \hat{\xi}_\rho]u_\lambda^{p-1} - \vartheta(u_\lambda) \\ & \quad \text{for almost all } z \in \Omega_0 \text{ (recall that } u_\lambda \in S_\lambda). \end{aligned} \tag{41}$$

Then from (40) and Proposition 4, we see that for small enough  $\epsilon > 0$  we have

$$u_\lambda - m_0^\epsilon \in \text{int } \hat{C}_+(\overline{\Omega}_0),$$

which contradicts the definition of  $m_0$ . Hence  $\lambda \notin \mathcal{L}$  and so  $\lambda^* \leq \lambda_0 < +\infty$ . □

By Propositions 9 and 10 it follows that

$$(0, \lambda^*) \subseteq \mathcal{L} \subseteq (0, \lambda^*]. \tag{42}$$

**Proposition 11** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$ ,  $H(f)$  hold and  $\lambda \in (0, \lambda^*)$ , then problem  $(P_\lambda)$  admits at least two positive solutions*

$$u_0, \hat{u} \in \text{int } C_+, \quad u_0 \neq \hat{u}.$$

**Proof** Let  $0 < \mu < \lambda < \eta < \lambda^*$ . We have  $\mu, \eta \in \mathcal{L}$  (see (42)). On account of Proposition 9 we can find  $u_\mu \in S_\mu \subseteq \text{int } C_+$ ,  $u_0 \in S_\lambda \subseteq \text{int } C_+$ ,  $u_\eta \in S_\eta \subseteq \text{int } C_+$  such that

$$\begin{aligned} & u_0 - u_\eta \in \text{int } C_+ \text{ and } u_\eta - u_0 \in \text{int } C_+, \\ & \Rightarrow u_0 \in \text{int } C_0^1(\overline{\Omega})[u_\mu, u_\eta]. \end{aligned} \tag{43}$$

With  $\beta > \|\xi\|_\infty$ , we introduce the Carathéodory function  $d_\lambda(z, x)$  defined by

$$d_\lambda(z, x) = \begin{cases} \vartheta(u_\mu(z)) + \lambda u_\mu(z)^{q-1} + f(z, u_\mu(z)) + \beta u_\mu(z)^{p-1} & \text{if } x \leq u_\mu(z) \\ \vartheta(x) + \lambda x^{q-1} + f(z, x) + \beta x^{p-1} & \text{if } u_\mu(z) < x. \end{cases} \tag{44}$$

We set  $D_\lambda(z, x) = \int_0^x d_\lambda(z, s)ds$  and consider the functional  $\varphi_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\varphi_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta]|u|^p dz - \int_\Omega D_\lambda(z, u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We know that  $\varphi_\lambda \in C^1(W_0^{1,p}(\Omega))$  (see Papageorgiou and Smyrlis [17, Proposition 3]). Also, let

$$\hat{d}_\lambda(z, x) = \begin{cases} d_\lambda(z, x) & \text{if } x \leq u_\eta(z) \\ d_\lambda(z, u_\eta(z)) & \text{if } u_\eta(z) < x. \end{cases} \tag{45}$$

This is a Carathéodory function. We set  $\hat{D}_\lambda(z, x) = \int_0^x \hat{d}_\lambda(z, s)ds$  and consider the  $C^1$ -functional  $\hat{\varphi}_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$  defined by

$$\hat{\varphi}_\lambda(u) = \int_\Omega G(Du)dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta]|u|^p dz - \int_\Omega \hat{D}_\lambda(z, u)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Using (44) and (45) and the nonlinear regularity theory (see the proof of Proposition 7), we show that

$$K_{\varphi_\lambda} \subseteq [u_\mu] \cap \text{int } C_+, \tag{46}$$

$$K_{\hat{\varphi}_\lambda} \subseteq [u_\mu, u_\eta] \cap \text{int } C_+. \tag{47}$$

From (47) we see that we can assume that

$$K_{\hat{\varphi}_\lambda} = \{u_0\} \tag{48}$$

or otherwise we already have a second positive solution for  $(P_\lambda)$  (see (45)) and so we are done.

Clearly,  $\hat{\varphi}_\lambda(\cdot)$  is coercive (see (45)) and sequentially weakly lower semicontinuous. So, we can find  $\hat{u}_0 \in W_0^{1,p}(\Omega)$  such that

$$\begin{aligned} \hat{\varphi}_\lambda(\hat{u}_0) &= \inf\{\hat{\varphi}_\lambda(u) : u \in W_0^{1,p}(\Omega)\}, \\ &\Rightarrow \hat{u}_0 \in K_{\hat{\varphi}_\lambda}, \\ &\Rightarrow \hat{u}_0 = u_0 \text{ (see (48)).} \end{aligned} \tag{49}$$

But from (44) and (45) we see that

$$\hat{\varphi}_\lambda|_{[u_\mu, u_\eta]} = \varphi_\lambda|_{[u_\mu, u_\eta]}. \tag{50}$$

It follows from (43), (49), (50) that

$$\begin{aligned}
 &u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \varphi_\lambda, \\
 &\Rightarrow u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \varphi_\lambda \text{ (see [5]).}
 \end{aligned}
 \tag{51}$$

On account of (44) and (46), we may assume that

$$K_{\varphi_\lambda} \text{ is finite.} \tag{52}$$

Otherwise we already have an infinity of positive smooth solutions. From (51), (52) and Theorem 5.7.6 of Papageorgiou et al. [15], we see that we can find small  $\rho \in (0, 1)$  such that

$$\varphi_\lambda(u_0) < \inf\{\varphi_\lambda(u) : \|u - u_0\| = \rho\} = m_\rho. \tag{53}$$

Hypothesis  $H(f)(ii)$  and Corollary 3 imply that if  $u \in \text{int } C_+$ , then

$$\varphi_\lambda(tu) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \tag{54}$$

**Claim 1**  $\varphi_\lambda$  satisfies the  $C$ -condition.

Consider a sequence  $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$  such that

$$|\varphi_\lambda(u_n)| \leq c_{15} \text{ for some } c_{15} > 0, \text{ and all } n \in \mathbb{N}, \tag{55}$$

$$(1 + \|u_n\|)\varphi'_\lambda(u_n) \rightarrow \text{in } W^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^* \text{ as } n \rightarrow \infty. \tag{56}$$

From (56) we have

$$\begin{aligned}
 &\left| \langle A(u_n), h \rangle + \int_\Omega [\xi(z) + \beta]|u_n|^{p-2}u_n h dz - \int_\Omega d_\lambda(z, u_n)h dz \right| \leq \frac{\epsilon_n \|h\|}{1 + \|u_n\|} \\
 &\text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \epsilon_n \rightarrow 0^+.
 \end{aligned}
 \tag{57}$$

In (57) we choose  $h = -u_n^- \in W_0^{1,p}(\Omega)$ . From (44) and Lemma 2, we have

$$\begin{aligned}
 &\frac{c_1}{p-1} \|Du_n^-\|_p^p + \int_\Omega [\xi(z) + \beta](u_n^-)^p dz \leq \epsilon_n + c_{16} \|u_n^-\| \\
 &\text{for some } c_{16} > 0, \text{ and all } n \in \mathbb{N}, \\
 &\Rightarrow \{u_n^-\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (recall that } \beta > \|\xi\|_\infty).
 \end{aligned}
 \tag{58}$$

Next, in (57) we choose  $h = u_n^+ \in W_0^{1,p}(\Omega)$ . Then

$$\begin{aligned}
 & - \int_{\Omega} (a(Du_n^+), Du_n^+)_{\mathbb{R}^N} dz - \int_{\Omega} [\xi(z) + \beta](u_n^+)^p dz \\
 & + \int_{\Omega} [\lambda(u_n^+)^q + f(z, u_n^+)u_n^+] dz \leq c_{17}
 \end{aligned}$$

for some  $c_{17} > 0$  and all  $n \in \mathbb{N}$  (see (44) and hypothesis  $H(\vartheta)(ii)$ ). (59)

From (55) and (58) we obtain

$$\int_{\Omega} pG(Du_n^+) dz + \int_{\Omega} [\xi(z) + \beta](u_n^+)^p dz - \int_{\Omega} \left[ \frac{\lambda p}{q} (u_n^+)^q + pF(z, u_n^+) \right] dz \leq c_{18}$$

for some  $c_{18} > 0$  and all  $n \in \mathbb{N}$ . (60)

Adding (59) and (60) and using hypothesis  $H(a)(iv)$ , we obtain

$$\int_{\Omega} [f(z, u_n^+)u_n^+ - pF(z, u_n^+)] dz \leq c_{19} + \lambda \left[ \frac{p}{q} - 1 \right] \|u_n^+\|_q^q$$

for some  $c_{19} > 0$ , all  $n \in \mathbb{N}$ . (61)

From hypotheses  $H(f)(i)$ , (iii) we see that we can find  $\hat{\beta}_1 \in (0, \hat{\beta}_0)$  and  $c_{20} > 0$  such that

$$\hat{\beta}_1 x^\sigma - c_{20} \leq f(z, x)x - pF(z, x) \text{ for almost all } z \in \Omega \text{ and all } x \geq 0. \tag{62}$$

Using (62) in (61) and recalling that  $q < \sigma$  (see hypothesis  $H(f)(iii)$ ) we obtain that

$$\{u_n^+\}_{n \geq 1} \subseteq L^\sigma(\Omega) \text{ is bounded.} \tag{63}$$

First, suppose that  $N \neq p$ . It is clear from hypothesis  $H(f)(iii)$  that we may assume that  $\sigma < r < p^*$  (recall that  $p^* = +\infty$  if  $N \leq p$ ). Let  $t \in (0, 1)$  be such that

$$\frac{1}{r} = \frac{1-t}{\sigma} + \frac{t}{p^*}.$$

From the interpolation inequality (see, for example, Papageorgiou and Winkert [18, Proposition 2.3.17, p.116]), we have

$$\begin{aligned}
 \|u_n^+\|_r & \leq \|u_n^+\|_\sigma^{1-t} \|u_n^+\|_{p^*}^t, \\
 & \Rightarrow \|u_n^+\|_r^r \leq c_{21} \|u_n^+\|_{p^*}^t
 \end{aligned}$$

for some  $c_{21} > 0$  and all  $n \in \mathbb{N}$  (see (63)). (64)

From hypothesis  $H(f)(i)$ , we have

$$f(z, x)x \leq c_{22}[1 + x^r] \text{ for almost all } z \in \Omega, \text{ all } x \geq 0 \text{ and some } c_{22} > 0. \tag{65}$$



In (57) we choose  $h = u_n^+ \in W_0^{1,p}(\Omega)$  and use Lemma 2. Then

$$\begin{aligned} & \frac{c_1}{p-1} \|Du_n^+\|_p^p + \int_{\Omega} [\xi(z) + \beta](u_n^+)^p dz \leq \epsilon_n + \int_{\Omega} d_{\lambda}(z, u_n) u_n^+ dz, \\ & \Rightarrow \frac{c_1}{p-1} \|Du_n^+\|_p^p \leq c_{23} + \int_{\Omega} [\lambda(u_n^+)^q + f(z, u_n^+) u_n^+] dz \\ & \quad \text{for some } c_{23} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (44))} \\ & \leq c_{24} [1 + \lambda \|u_n^+\|_q^q + \|u_n^+\|^{tr}] \\ & \quad \text{for some } c_{24} > 0 \text{ and all } n \in \mathbb{N} \text{ (see (64) and (65)).} \end{aligned} \tag{66}$$

The hypothesis on  $\sigma$  (see  $H(f)(iii)$ ) implies that  $tr < p$ . Also we have  $q < p$ . Therefore it follows from (66) that

$$\{u_n^+\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.} \tag{67}$$

If  $p = N$ , then  $p^* = +\infty$  and by the Sobolev embedding theorem, we have that  $W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega)$  for all  $1 \leq s < \infty$ . So, we need to replace in the previous argument  $p^*$  by  $s > r > \sigma$  big enough. More precisely, as before, let  $t \in (0, 1)$  be such that

$$\begin{aligned} \frac{1}{r} &= \frac{1-t}{\sigma} + \frac{t}{s}, \\ \Rightarrow tr &= \frac{s(r-\sigma)}{s-\sigma} \rightarrow r-\sigma \text{ as } s \rightarrow +\infty. \end{aligned}$$

Recall that  $r - \sigma < p$  (see hypothesis  $H(f)(iii)$ ). Hence for large enough  $s > r$

$$tr = \frac{s(r-\sigma)}{s-\sigma} < p.$$

Then for such large  $s > r$ , the previous argument is valid and we again obtain (67).

From (58) and (67) we have that  $\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega)$  is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega). \tag{68}$$

In (57) we choose  $h = u_n - u \in W_0^{1,p}(\Omega)$ , pass to the limit as  $n \rightarrow \infty$ , and use (68). Then

$$\begin{aligned} & \lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0, \\ & \Rightarrow u_n \rightarrow u \text{ in } W_0^{1,p}(\Omega) \\ & \quad \text{(using the } (S)_+ \text{ property of } A(\cdot), \text{ see Sect. 2),} \\ & \Rightarrow \varphi_{\lambda}(\cdot) \text{ satisfies the } C - \text{condition.} \end{aligned}$$

This proves Claim 1.

From (53), (54) and Claim 1, we see that we can apply the mountain pass theorem. So, we can find  $\hat{u} \in W_0^{1,p}(\Omega)$  such that

$$\hat{u} \in K_{\varphi_\lambda} \subseteq [u_\mu] \cap \text{int } C_+ \text{ (see (46)) and } m_\rho \leq \varphi_\lambda(\hat{u}) \text{ (see (53)).} \tag{69}$$

It follows from (44) and (69) that

$$\hat{u} \in S_\lambda \subseteq \text{int } C_+ \text{ and } u_0 \neq \hat{u}.$$

The proof is now complete. □

**Proposition 12** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$ ,  $H(f)$  hold, then  $\lambda^* \in \mathcal{L}$ .*

**Proof** Let  $\{\lambda_n\}_{n \geq 1} \subseteq (0, \lambda^*)$  be such that  $\lambda_n \uparrow \lambda^*$ . We know that  $\lambda_n \in \mathcal{L}$  for all  $n \in \mathbb{N}$  and so we can find  $u_n = u_{\lambda_n} \in S_{\lambda_n} \subseteq \text{int } C_+$  ( $n \in \mathbb{N}$ ) increasing (see Proposition 9).

Let  $\hat{\varphi}_{\lambda_n}(\cdot)$  be the functional from the proof of Proposition 11, with  $u_\mu = u_{n-1}$ ,  $u_\mu = u_{n+1}$  ( $n \geq 2$ ). Then we have

$$\begin{aligned} \hat{\varphi}_{\lambda_n}(u_n) &\leq \hat{\varphi}_{\lambda_n}(u_{n-1}) \\ &= \int_\Omega G(Du_{n-1})dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta]u_{n-1}^p dz - \int_\Omega [\vartheta(u_{n-1}) + \lambda_n u_{n-1}^{q-1} \\ &\quad + f(z, u_{n-1} + \beta u_{n-1}^{p-1})]u_{n-1} dz \\ &\leq \int_\Omega G(Du_{n-1})dz + \frac{1}{p} \int_\Omega [\xi(z) + \beta]u_{n-1}^p dz - \int_\Omega [\vartheta(u_{n-1}) + \lambda_{n-1} u_{n-1}^{q-1} \\ &\quad + f(z, u_{n-1}) + \beta u_{n-1}^{p-1}]u_{n-1} dz \\ &\leq \int_\Omega (a(Du_{n-1}), Du_{n+1})dz + \int_\Omega \xi(z)u_{n-1}^p dz - \int_\Omega [\vartheta(u_{n-1}) + \lambda_{n-1} u_{n-1}^{q-1} \\ &\quad + f(z, u_{n-1})]u_{n-1} dz \text{ (see (3) and recall that } \beta > \|\xi\|_\infty) \\ &= 0 \text{ (since } u_{n-1} \in S_{\lambda_{n-1}}). \end{aligned} \tag{70}$$

Also, we have

$$\langle A(u_n), h \rangle + \int_\Omega [\xi(z) + \beta]u_n^{p-1} h dz = \int_\Omega d_{\lambda_n}(z, u_n) h dz$$

for all  $h \in W_0^{1,p}(\Omega)$  and all  $n \in \mathbb{N}$ . (71)

Using (70), (71) and reasoning as in the proof of Proposition 11 (see Claim 1), we obtain that

$$\{u_n\}_{n \geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

From this, as in the proof of Proposition 11, exploiting the  $(S)_+$  property of  $A(\cdot)$ , we obtain

$$u_n \rightarrow u_* \text{ in } W_0^{1,p}(\Omega). \tag{72}$$

Passing to the limit as  $n \rightarrow \infty$  in (71) and using (72), we have

$$u_* \in S_{\lambda_*} \subseteq \text{int } C_+ \text{ and so } \lambda^* \in \mathcal{L}.$$

The proof is now complete.  $\square$

This proposition implies that

$$\mathcal{L} = (0, \lambda^*].$$

Summarizing the situation for problem  $(P_\lambda)$ , we can state the following bifurcation-type result.

**Theorem 13** *If hypotheses  $H(a)$ ,  $H(\xi)$ ,  $H(\vartheta)$ ,  $H(f)$  hold, then there exists  $\lambda^* > 0$  such that*

(a) *for all  $\lambda \in (0, \lambda^*)$  problem  $(P_\lambda)$  has at least two positive solutions*

$$u_0, \hat{u} \in \text{int } C_+, \quad u_0 \neq \hat{u};$$

(b) *for  $\lambda = \lambda^*$  problem  $(P_\lambda)$  has at least one positive solution  $u_* \in \text{int } C_+$ ;*

(c) *for all  $\lambda > \lambda^*$  problem  $(P_\lambda)$  has no positive solutions.*

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## Compliance with ethical standards

**Conflict of interest** The authors confirm that there is no conflict of interest.

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