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Infinitely many smooth nodal solutions for Orlicz Robin problems

Anouar Bahrouni [a], Hlel Missaoui [a], Vicenţiu D. Rădulescu [b,c,d,*]

[a] Mathematics Department, Faculty of Sciences, University of Monastir, 5019 Monastir, Tunisia
[b] Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland
[c] Brno University of Technology, Faculty of Electrical Engineering and Communication, Technická 3058/10, Brno 61600, Czech Republic
[d] Department of Mathematics, University of Craiova, Street A.I. Cuza No. 13, 200585 Craiova, Romania

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A B S T R A C T

In this note, we study a Robin problem driven by the Orlicz g-Laplace operator. In particular, by using a regularity result and Kajikiya's theorem, we prove that the problem has a whole sequence of distinct smooth nodal solutions converging to the trivial one. The analysis is developed in the most general abstract setting that corresponds to Orlicz–Sobolev function spaces.

1. Introduction

In this paper, we are interested in the existence and multiplicity of smooth nodal (that is, sign-changing) solutions for the following nonlinear Robin boundary problem:

$$\begin{cases} -\operatorname{div}\left(a(|\nabla u(x)|)\nabla u(x)\right) + a(|u(x)|)u(x) = f(x, u(x)), & x \in \Omega \\ a(|\nabla u(x)|)\dfrac{\partial u(x)}{d\nu} + b(x)|u(x)|^{p-2}u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 3)$ is a smooth bounded domain, $\Delta_g u := \operatorname{div}\left(a(|\nabla u|)\nabla u\right)$ is the Orlicz g-Laplace operator, $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, ν is the unit exterior vector on $\partial\Omega$, $p > 0$, $b \in C^{1,\theta}(\partial\Omega)$ with $\theta \in (0, 1)$ and $\inf_{x\in\partial\Omega} b(x) > 0$.

* Corresponding author at: Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland.

E-mail addresses: bahrounianouar@yahoo.fr (A. Bahrouni), hlel.missaoui@fsm.rnu.tn (H. Missaoui), radulescu@inf.ucv.ro (V.D. Rădulescu).

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Very recently, the authors in [1] proved the existence of one smooth sign-changing solution of problem (P). The multiplicity question of smooth nodal solutions has been treated by many authors, see for instance [2–4]. None of the aforementioned works study the multiplicity of smooth nodal solution for the Orlicz–Sobolev problems (with Dirichlet, Neumann, or Robin boundary conditions). Hence, a natural question is whether or not there exist multiple smooth nodal solutions. Based on regularity results obtained in [1,5], and Kajikiya's theorem [6], we show that problem (P) has infinitely many smooth nodal solutions.

Before stating our main result, we need the following class of hypotheses on the functions $a:(0,+\infty)\to(0,+\infty)$ and the N-function (see [7] for details) $G(t):=\int_0^t g(s)ds$, where $g(t):=a(|t|)t$ if $t\neq 0$ and $g(t)=0$ if $t=0$.

(H_g) (g_1) : $a(t)\in C^1(0,+\infty)$, $a(t)>0$ and $a(t)$ is an increasing function for $t>0$.
(g_2) : $1<p<g^-:=\inf_{t>0}\frac{g(t)t}{G(t)}\leq g^+:=\sup_{t>0}\frac{g(t)t}{G(t)}<d$,
(g_3) : $0<g^--1=a^-:=\inf_{t>0}\frac{g'(t)t}{g(t)}\leq g^+-1=a^+:=\sup_{t>0}\frac{g'(t)t}{g(t)}$.
(g_4) : $t\mapsto G\left(\sqrt{t}\right)$ is convex on $[0,+\infty)$, $\int_0^\delta\left(\frac{t}{G(t)}\right)^{\frac{1}{d-1}}dt<\infty$ and $\int_\beta^{+\infty}\left(\frac{t}{G(t)}\right)^{\frac{1}{d-1}}dt=\infty$, for some constants $\beta,\delta>0$.

We assume that $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function, $f(x,.)$ is odd and $f(x,0)=0$, for a.a. $x\in\Omega$, and satisfies the following class of assumptions:

(H_f) (f_1) There exist an odd increasing homomorphism $h\in C^1(\mathbb{R},\mathbb{R})$, and a positive function $\widehat{a}(t)\in L^\infty(\Omega)$ such that $|f(x,t)|\leq\widehat{a}(x)\left(1+h(|t|)\right)$, for all $t\in\mathbb{R}$, $x\in\overline{\Omega}$,

$$1<g^+<h^-:=\inf_{t>0}\frac{h(t)t}{H(t)}\leq h^+:=\sup_{t>0}\frac{h(t)t}{H(t)}\leq\frac{g_*^-}{g^-},$$
$$1<h^--1:=\inf_{t>0}\frac{h'(t)t}{h(t)}\leq h^+-1:=\sup_{t>0}\frac{h'(t)t}{h(t)}$$

and

$$\lim_{t\to+\infty}\frac{G(kt)}{H(t)}=\lim_{t\to+\infty}\frac{H(kt)}{G_*(t)}=0,\ \text{for all } k>0$$

where $H(t):=\int_0^t h(s)\ ds$ is an N-function, $g_*^-:=\frac{dg^-}{d-g^-}$, and G_* is defined in Section 2.
(f_2) $\lim_{t\to\pm\infty}\frac{F(x,t)}{|t|^{g^+}}=+\infty$, uniformly in $x\in\Omega$, where $F(x,t)=\int_0^t f(x,s)ds$.
(f_3) There is an odd increasing homomorphism $q\in C^1(\mathbb{R},\mathbb{R})$, and constants $c_0\geq 0$, $\delta\geq 0$ such that

$$c_0q(t)t\leq f(x,t)t\leq q^+F(x,t),\ \text{for a.a. } x\in\Omega \text{ and all } 0<|t|\leq\delta,$$

$$1<q^-:=\inf_{t>0}\frac{q(t)t}{Q(t)}\leq q^+:=\sup_{t>0}\frac{q(t)t}{Q(t)}<p<g^-,$$
$$1<q^--1:=\inf_{t>0}\frac{q'(t)t}{q(t)}\leq q^+-1:=\sup_{t>0}\frac{q'(t)t}{q(t)}$$

and

$$\lim_{t\to+\infty}\frac{Q(kt)}{G(t)}=0,\ \text{for all } k>0,$$

where $Q(t):=\int_0^t q(s)ds$ is an N-function.
(f_4) There exist $\eta_-<0$ and $\eta_+>0$ such that

$$f(x,\eta_+)<0<f(x,\eta_-),\quad\text{for a.a. } x\in\Omega.$$

Now, we can state our main result

Theorem 1.1. *Suppose that hypotheses (H_f) and (H_g) are satisfied, then problem (P) has a sequence $\{u_n\}_{n\in\mathbb{N}} \subset C^1(\overline{\Omega}) \cap W^{1,G}(\Omega)$ of distinct nodal solutions such that $u_n \to 0$ in $C^1(\overline{\Omega})$ as $n \to \infty$.*

To the best of our knowledge, this is the first work deal with the existence of infinitely many smooth nodal solutions for problem (P). The main theorem of this note extends the result obtained in [1]. We would like to mention that condition (g_4) is weaker than the assumption assumed in the aforementioned reference. The new assumption (g_4) leads us to prove some important propositions (Propositions 3.1 and 3.2) to keep all the results obtained in [1,5] valid for problem (P).

2. Preliminarie and main result

In this section, we present the main space for the study of problem (P) and some notions needed in the sequel. The assumptions made on a and G ensure that G is an even N-function (see [7]). Moreover, G and its conjugate N-function $\tilde{G}$ satisfy the well-known $\triangle_2$-condition. Therefore, we can define the Orlicz space $L^G(\Omega)$ as the vectorial space of all measurable functions $u : \Omega \to \mathbb{R}$ such that $\rho(u) = \int_\Omega G(|u(x)|)\, dx < \infty$.

Definition 2.1. On the Orlicz space $L^G(\Omega)$ we define the Luxemburg norm by the formula

$$\|u\|_{(G)} = \inf\left\{\lambda > 0 \ : \ \rho(\frac{u}{\lambda}) \leq 1\right\}.$$

In our case, the space $L^G(\Omega)$ is a separable reflexive Banach space under the above Luxemburg norm. Now, from the Orlicz space $L^G(\Omega)$, we define the Orlicz–Sobolev space $W^{1,G}(\Omega)$ by

$$W^{1,G}(\Omega) := \left\{u \in L^G(\Omega) : \ \frac{\partial u}{\partial x_i} \in L^G(\Omega),\ i = 1, \dots, d\right\}.$$

Here, the space $W^{1,G}(\Omega)$ is a Banach space and it inherits the separability and reflexivity from the space $L^G(\Omega)$ with respect to the norm

$$\|u\| = \inf\left\{\lambda > 0 \ : \ \mathcal{K}(\frac{u}{\lambda}) \leq 1\right\},$$

where

$$\mathcal{K}(u) = \int_\Omega G(|\nabla u(x)|)dx + \int_\Omega G(|u(x)|)\, dx. \tag{2.1}$$

Next, we mention the following optimal fractional Sobolev inequality introduced by A. Cianchi [8]. The optimal N-function for embedding theorem is defined by

$$G_*(t) := G(M^{-1}(t)), \quad \text{for all } t \geq 0, \tag{2.2}$$

where

$$M(t) := \left(\int_0^t \left(\frac{s}{G(s)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}}, \quad \text{for all } t \geq 0. \tag{2.3}$$

The optimal embedding is the next result [8].

Theorem 2.2. *Under the assumptions $(g_1) - (g_4)$, the continuous embedding $W^{1,G}(\Omega) \hookrightarrow L^{G_*}(\Omega)$ holds, where G_* is defined in (2.2). Moreover, for any N-function B, the embedding $W^{1,G}(\Omega) \hookrightarrow L^B(\Omega)$ is compact if and only if $\lim_{t\to+\infty} \frac{B(kt)}{G_*(t)} = 0$, for all $k > 0$.*

The above result is optimal in the sense that if the embedding holds for an N-function B, then the space $L^{G_*}(\Omega)$ is continuously embedded into $L^B(\Omega)$.

In the following lemma, we give some properties of the N-function and the relationship between the norm of the Orlicz–Sobolev space and its module.

Lemma 2.3. *(see [9]) Let $B(t) := \int_0^{|t|} b(s)ds$ be an N-function such that $b \in C^1(0, +\infty)$ and*

$$1 < b^- := \inf_{t>0} \frac{b(t)t}{B(t)} \leq b^+ := \sup_{t>0} \frac{b(t)t}{B(t)} < +\infty.$$

Then

(1) $\min\left\{t^{b^-}, t^{b^+}\right\} B(z) \leq B(tz) \leq \max\left\{t^{b^-}, t^{b^+}\right\} B(z)$, *for all $t, z \geq 0$.*
(2) $\min\left\{\|u\|_{(B)}^{b^-}, \|u\|_{(B)}^{b^+}\right\} \leq \rho(u) \leq \max\left\{\|u\|_{(B)}^{b^-}, \|u\|_{(B)}^{b^+}\right\}$, *for all $u \in L^B(\Omega)$.*
(3) $\min\left\{\|u\|^{b^-}, \|u\|^{b^+}\right\} \leq \int_\Omega B(|\nabla u(x)|)dx + \int_\Omega B(|u(x)|)\, dx \leq \max\left\{\|u\|^{b^-}, \|u\|^{b^+}\right\}$, *for all $u \in W^{1,B}(\Omega)$.*

Let $u, v : \Omega \to \mathbb{R}$ be two measurable functions such that $u(x) \leq v(x)$ for a.a. $x \in \Omega$, then we introduce the order interval $[u, v] = \left\{y \in W^{1,G}(\Omega) : u(x) \leq y(x) \leq v(x) \text{ for a.a. } x \in \Omega\right\}$. Recall that $C^1(\overline{\Omega})$ is an ordered Banach space with a positive order cone $C^1(\overline{\Omega})_+ = \left\{u \in C^1(\overline{\Omega}),\ u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\right\}$. This cone has a nonempty interior given by $\mathrm{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega})_+,\ u(x) > 0 \text{ for all } x \in \overline{\Omega}\right\}$.

3. Proof of the main result

First, we give some important properties of the optimal N-function G_* which will be useful in the proof of Theorem 1.1.

Proposition 3.1. *Under the assumptions $(g_1) - (g_2)$ and (g_4) the following inequality holds*

$$\min\left\{t^{g_*^-}, t^{g_*^+}\right\} G_*(z) \leq G_*(tz) \leq \max\left\{t^{g_*^-}, t^{g_*^+}\right\} G_*(z), \quad \textit{for all } t, z \geq 0,$$

where $g_*^- := \frac{dg^-}{d-g^-}$ and $g_*^+ := \frac{dg^+}{d-g^+}$.

Proof. According to the definition of $M(t)$ (see (2.3)), for all $t > 0$ and $z \geq 0$, we have

$$M(tz) = \left(\int_0^{tz} \left(\frac{s}{G(s)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}} = t^{\frac{d-1}{d}} \left(\int_0^{z} \left(\frac{ts}{G(ts)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}}.$$

Using Lemma 2.3-(1), for all $0 < t \leq 1$ and $z \geq 0$, we obtain

$$\begin{aligned} M(tz) &\leq t^{\frac{d-1}{d}} \left(\int_0^{z} \left(\frac{ts}{t^{g^+}G(s)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}} = t^{\left(\frac{d-1}{d} - \frac{g^+-1}{d}\right)} \left(\int_0^{z} \left(\frac{s}{G(s)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}} \\ &= t^{\frac{d-g^+}{d}} M(z) \end{aligned}$$

and

$$\begin{aligned} M(tz) &\geq t^{\frac{d-1}{d}} \left(\int_0^{z} \left(\frac{ts}{t^{g^-}G(s)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}} = t^{\left(\frac{d-1}{d} - \frac{g^--1}{d}\right)} \left(\int_0^{z} \left(\frac{s}{G(s)}\right)^{\frac{1}{d-1}} ds\right)^{\frac{d-1}{d}} \\ &= t^{\frac{d-g^-}{d}} M(z). \end{aligned}$$

Thus,

$$t^{\frac{d-g^-}{d}} M(z) \leq M(tz) \leq t^{\frac{d-g^+}{d}} M(z), \text{ for all } 0 \leq t \leq 1 \text{ and all } z \geq 0. \tag{3.4}$$

By the same way, we get

$$t^{\frac{d-g^+}{d}} M(z) \leq M(tz) \leq t^{\frac{d-g^-}{d}} M(z), \text{ for all } t > 1 \text{ and all } z \geq 0. \tag{3.5}$$

By (3.4) and (3.5), it follows that

$$\zeta_0(t) M(z) \leq M(tz) \leq \zeta_1(t) M(z), \quad \text{for all } t, z \geq 0, \tag{3.6}$$

where

$$\zeta_0(t) = \min\left\{ t^{\frac{d-g^-}{d}}, t^{\frac{d-g^+}{d}} \right\} \text{ and } \zeta_1(t) = \max\left\{ t^{\frac{d-g^-}{d}}, t^{\frac{d-g^+}{d}} \right\}. \tag{3.7}$$

Putting in the inequality (3.6) $\tau = M(z)$ and $\kappa = \zeta_0(t)$ that is, $z = M^{-1}(\tau)$ and $t = \zeta_0^{-1}(\kappa)$, we get $\kappa\tau \leq M(\zeta_0^{-1}(\kappa) M^{-1}(\tau))$. Since M^{-1} is non-decreasing, we infer that

$$M^{-1}(\kappa\tau) \leq \zeta_0^{-1}(\kappa) M^{-1}(\tau), \quad \text{for all } \kappa, \tau > 0. \tag{3.8}$$

Similarly, putting in (3.6) $\tau = M(z)$ and $\kappa = \zeta_1(t)$ that is, $z = M^{-1}(\tau)$ and $t = \zeta_1^{-1}(\kappa)$, we obtain

$$\zeta_1^{-1}(\kappa) M^{-1}(\tau) \leq M^{-1}(\kappa\tau), \quad \text{for all } \kappa, \tau > 0. \tag{3.9}$$

From (3.7), (3.8) and (3.9), it yields that

$$\min\left\{ t^{\frac{d}{d-g^-}}, t^{\frac{d}{d-g^+}} \right\} M^{-1}(z) \leq M^{-1}(tz) \leq \max\left\{ t^{\frac{d}{d-g^-}}, t^{\frac{d}{d-g^+}} \right\} M^{-1}(z), \quad \text{for all } t, z \geq 0. \tag{3.10}$$

From Lemma 2.3-(1), we deduce that

$$\min\left\{ t^{g_*^-}, t^{g_*^+} \right\} G_*(z) \leq G_*(tz) \leq \max\left\{ t^{g_*^-}, t^{g_*^+} \right\} G_*(z), \quad \text{for all } t, z \geq 0$$

where $g_*^- = \frac{dg^-}{d-g^-}$ and $g_*^+ = \frac{dg^+}{d-g^+}$. This ends the proof. $\square$

The function G_* inherits the $\triangle_2$-condition from the function G. Indeed, we have the following property.

Proposition 3.2. *Under the assumptions $(g_1) - (g_2)$ and (g_4) the following inequality holds*

$$g_*^- \leq \frac{g_*(t)t}{G_*(t)} \leq g_*^+, \quad \textit{for all } t > 0, \ \textit{where } G_*(t) = \int_0^t g_*(s)ds.$$

G_* satisfies the $\triangle_2$-condition and

$$\min\left\{ \|u\|_{(G_*)}^{g_*^-}, \|u\|_{(G_*)}^{g_*^+} \right\} \leq \int_\Omega G_*(u)dx \leq \max\left\{ \|u\|_{(G_*)}^{g_*^-}, \|u\|_{(G_*)}^{g_*^+} \right\}, \ \textit{for all } u \in L^{G_*}(\Omega).$$

Proof. From Proposition 3.1, we have

$$t^{g_*^-} G_*(z) \leq G_*(tz) \leq t^{g_*^+} G_*(z), \quad \text{for all } t, z > 1, \tag{3.11}$$

and

$$1^{g_*^+} G_*(z) \leq G_*(z) \leq 1^{g_*^-} G_*(z), \quad \text{for all } z > 0. \tag{3.12}$$

Putting together (3.11) and (3.12), we find

$$\frac{t^{g_*^-} - 1^{g_*^-}}{t-1} G_*(z) \leq \frac{G_*(tz) - G_*(z)}{t-1} \leq \frac{t^{g_*^+} - 1^{g_*^+}}{t-1} G_*(z), \quad \text{for all } t, z > 1. \tag{3.13}$$

Passing to the limit as $t \to 1$ in (3.13), we deduce that

$$g_*^- \leq \frac{g_*(z)z}{G_*(z)} \leq g_*^+, \quad \text{for all } z > 0, \text{ where } G_*(z) = \int_0^z g_*(s)ds.$$

Hence, G_* satisfies the $\triangle_2$-condition and

$$\min\left\{\|u\|_{(G_*)}^{g_*^-}, \|u\|_{(G_*)}^{g_*^+}\right\} \leq \int_\Omega G_*(u)dx \leq \max\left\{\|u\|_{(G_*)}^{g_*^-}, \|u\|_{(G_*)}^{g_*^+}\right\}, \text{ for all } u \in L^{G_*}(\Omega).$$

Thus the proof. $\square$

Remark 3.3. From Propositions 3.1, 3.2 and Theorem 2.2, all the results obtained in [1,5] are valid for problem (P) with the new assumption (g_4). In particular, from [1, Theorem2.20], we have

$$\emptyset \neq S_+ \subset \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \quad \text{and } \emptyset \neq S_- \subset -\operatorname{int}\left(C^1(\overline{\Omega})_+\right). \tag{3.14}$$

Here, $S_+ := \{u : \ u \text{ is a non-negative solution of (P)}\}$ and $S_- := \{u : \ u \text{ is a non-positive solution of (P)}\}$. Moreover, by [1, Proposition 4.5], we have that problem (P) admits a smallest positive solution $u^* \in \operatorname{int}\left(C^1(\overline{\Omega})_+\right) \cap [0, \eta_+]$ and a biggest negative solution $v^* \in -\operatorname{int}\left(C^1(\overline{\Omega})_+\right) \cap [\eta_-, 0]$.

Proof of Theorem 1.1. Let $\mu > \max\{-\eta_-, \eta_+\}$. Then $[\eta_-, \eta_+] \subset [-\mu, \mu]$. Let $\xi(\cdot) \in C(\mathbb{R})$ be an even function such that

$$0 \leq \xi(t) \leq 1, \ \xi\big|_{[\eta_-, \eta_+]} = 1 \quad \text{and } \operatorname{supp}(\xi) \subset [-\mu, \mu]. \tag{3.15}$$

Using the function $\xi(\cdot)$, we define a Carathèodory function $\widehat{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\widehat{f}(x,t) := \xi(t)f(x,t) + (1 - \xi(t))q(t), \quad \text{for all } x \in \Omega \ \text{ and all } t \in \mathbb{R} \tag{3.16}$$

where $q(t)$ is defined in (f_3). From (3.15)–(3.16) and hypothesis (f_1), we have

$$\widehat{f}(x,.)\big|_{[\eta_-, \eta_+]} = f(x,.)\big|_{[\eta_-, \eta_+]} \quad \text{for all } x \in \Omega, \tag{3.17}$$

$$\left|\widehat{f}(x,t)\right| \leq \widehat{C}\left(1 + q(|t|)\right) \quad \text{for a.a. } x \in \Omega, \text{ all } t \in \mathbb{R} \text{ where } \widehat{C} > 0. \tag{3.18}$$

Now, we consider the following Robin problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u(x)|)\nabla u(x)) + a(|u(x)|)u(x) = \widehat{f}(x, u(x)), & x \in \Omega \\ a(|\nabla u(x)|)\frac{\partial u(x)}{d\nu} + b(x)|u(x)|^{p-2}u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{R}$$

Let $\widehat{F}(x,t) = \int_0^t \widehat{f}(x,s)ds$ and define the energy $J : W^{1,G}(\Omega) \to \mathbb{R}$ by

$$J(u) := \mathcal{K}(u) + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - \int_\Omega \widehat{F}(x,u)dx \ \text{ for all } u \in W^{1,G}(\Omega).$$

We know that $J \in C^1\left(W^{1,G}(\Omega), \mathbb{R}\right)$. Moreover, J is even and coercive (see (3.17) and recall that $q^+ < g^-$). It follows, by [10, Proposition 5.1.15, p. 369], that J is bounded from below and satisfies the Palais–Smale condition. So, J satisfies hypothesis (A_1). It is remain to verify that J satisfies hypothesis (A_2) of [6, Theorem1].

To this end, we consider $V \subset W^{1,G}(\Omega)$ as a finite-dimensional subspace. So, there exists $\varrho \in (0,1)$ small such that

$$u \in V, \ \|u\| \leq \varrho \quad \Rightarrow \quad |u(x)| \leq \delta \text{ for a.a. } x \in \Omega, \tag{3.19}$$

where $\delta > 0$ is defined in hypothesis (f_3). Let $u \in V$ such that $\|u\| = \varrho$. From (3.17)–(3.19), hypothesis (f_3), Theorem 2.2 and Lemma 2.3, we deduce that

$$\begin{aligned} J(u) &\leq \mathcal{K}(u) + \frac{1}{p}\int_{\partial\Omega} b(x)|u|^p d\gamma - C_0\frac{q^-}{q^+}\int_{\Omega} Q(u(x))dx \\ &\leq \|u\|^{g^-} + \frac{1}{p}C_b\int_{\partial\Omega}|u|^p d\gamma - C_0\frac{q^-}{q^+}\min\left\{\|u\|_{(Q)}^{q^-}, \|u\|_{(Q)}^{q^+}\right\} \\ &\leq \|u\|^{g^-} + \frac{1}{p}C_bC_1\|u\|^p - C_0C_2\frac{q^-}{q^+}\|u\|^{q^+} \text{ (since all norms are equivalent on } V), \end{aligned} \tag{3.20}$$

for some $C_b, C_1, C_2 > 0$. Since $q^+ < p < g^-$, we can choose $\varrho \in (0,1)$ even smaller if necessary such that

$$J(u) < 0 \text{ for all } u \in V \text{ with } \|u\| = \varrho. \tag{3.21}$$

It follows that $\sup\{J(u) : \ u \in V, \ \|u\| = \varrho\} < 0$. So, we conclude that J satisfies hypotheses $(A_1)-(A_2)$ of Theorem 1 of Kajikiya [6]. Then, there exists $\{u_n\}_{n\in\mathbb{N}} \subset K_J := \left\{u \in W^{1,G}(\Omega), \ J'(u) = 0\right\}$ such that

$$u_n \to 0 \text{ in } W^{1,G}(\Omega). \tag{3.22}$$

From [5], there exist $\alpha \in (0,1)$ and $C_3 > 0$ such that

$$u_n \in C^{1,\alpha}(\overline{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}(\overline{\Omega})} \leq C_3, \text{ for all } n \in \mathbb{N}. \tag{3.23}$$

The compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ (see [11]) and (3.22)–(3.23) imply that $u_n \to 0$ in $C^1(\overline{\Omega})$. Thus, there exists $n_0 \in \mathbb{N}$ such that $u_n \in [\eta_-, \eta_+] \cap [v^*, u^*] \cap C^1(\overline{\Omega})$, for all $n \geq n_0$. It follows, by (3.17), that $\{u_n\}_{n\geq n_0}$ are nodal solutions of (P). This ends the proof of Theorem 1.1. □

Data availability

No data was used for the research described in the article.

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