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Axially symmetric solutions of the Schrödinger–Poisson system with zero mass potential in $\mathbb{R}^2$ ☆

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A B S T R A C T

In this paper, we consider the following planar Schrödinger–Poisson system

$$\begin{cases} -\Delta u + \phi u = K(x) f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases}$$

where $K \in \mathcal{C}(\mathbb{R}^2, (0,\infty))$ and $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$. By constructing a new energy estimate inequality and applying some new tricks, we prove that the above system has an axially symmetric solution which has a special minimax characterization. Our result extends previous results in the literature.

1. Introduction

This paper is concerned with the following planar Schrödinger–Poisson system:

$$\begin{cases} -\Delta u + \phi u = K(x) f(u), & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where K and f satisfy the following basic assumptions:

(K1) $K \in \mathcal{C}(\mathbb{R}^2, (0,\infty))$, $K(x) = K(|x_1|, |x_2|), \forall x \in \mathbb{R}^2$ and $\liminf_{|x|\to\infty} K(x) \in (0,\infty)$;

(F1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$, there exist $\mathcal{C}_0 > 0$ and $p \in (2,\infty)$ such that $|f(u)| \le \mathcal{C}_0 \left(1 + |u|^{p-1}\right), \forall u \in \mathbb{R}$;

(F2) $f(u) = o(|u|)$ as $u \to 0$.

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System (1.1) is a special form of the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \mu\phi u = f(x,u), & x \in \mathbb{R}^N, \\ \Delta\phi = u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

with $\mu \in \mathbb{R}\backslash\{0\}$, $V \in \mathcal{C}(\mathbb{R}^N,(0,\infty))$ and $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R},\mathbb{R})$, which derives from quantum mechanics and semiconductor theory (see e.g. [1]). As we know, the solution ϕ of the Poisson equation in (1.2) can be solved by $\phi = \Gamma_N * u^2$, where $\Gamma_N(x) = \begin{cases} \frac{1}{2\pi}\ln|x|, & N=2, \\ \frac{1}{N(2-N)\omega_N}|x|^{2-N}, & N \geq 3, \end{cases}$ and ω_N is the volume of the unit N-ball. With this formal inversion, (1.2) is converted into a single equation $-\Delta u + V(x)u + \mu(\Gamma_N * u^2)u = K(x)f(u), x \in \mathbb{R}^N$. Based on variational methods, the existence of nontrivial solutions and ground state solutions to (1.2) has been investigated extensively, see [2–6]. However, most of them are concerned either with the case $\inf_{x\in\mathbb{R}^N} V(x) > 0$ or if both $V(x)$ and $K(x)$ vanish at infinity, that is, $\liminf_{|x|\to\infty} V(x) = \liminf_{|x|\to\infty} K(x) = 0$. A natural question is what happens if $\liminf_{|x|\to\infty} V(x) = 0$ but $\liminf_{|x|\to\infty} K(x) > 0$ in (1.2). To the best of our knowledge, there is no related result in this case. This is one of the basic research motivations of this paper.

On the other hand, unlike the case of $N = 3$, it is more difficult to study (1.2) with $N = 2$ due to the presence of $\Gamma_2 = \frac{1}{2\pi}\ln|x|$ which is sign-changing and is neither bounded from above nor from below, see [7–11]. Let us point out that the methods applied in the aforementioned papers do not work for (1.1). In this paper, we will obtain the existence of axially symmetric solutions to (1.1), and give a positive answer to the above question. The fact that $V(x) \equiv 0$ but $\liminf_{|x|\to\infty} K(x) > 0$ enforces the implementation of new ideas and tricks.

To state our result, we need the following monotonicity assumptions:

(K2) $K \in \mathcal{C}^1(\mathbb{R}^2,\mathbb{R})$, $\nabla K(x)\cdot x \leq 0$ and $t \mapsto 4K(tx) - \nabla K(tx)\cdot(tx)$ is nonincreasing on $(0,\infty)$ for every $x \in \mathbb{R}^2$;

(F3) the function $\dfrac{f(u)u - F(u)}{u^3}$ is nondecreasing on both $(-\infty,0)$ and $(0,\infty)$.

Our result is as follows.

Theorem 1.1. *Assume that K and f satisfy* (K1), (K2) *and* (F1)–(F3). *Then problem* (1.1) *has an axially symmetric solution satisfying*

$$\Phi(\bar{u}) = \inf_{u\in\mathcal{M}} \Phi(u) = \inf_{u\in E\backslash\{0\}} \max_{t>0} \Phi(t^2 u_t) \text{ with } \mathcal{M} := \{u \in E\backslash\{0\} : J(u) = 0\},$$

where $u_t(x) = u(tx)$, the definitions of E and J are given by (2.4) and (3.7).

Throughout this paper, the norms of $H^1(\mathbb{R}^2)$ and $L^i(\mathbb{R}^2)$ are denoted by $\|\cdot\|_{H^1}$ and $\|\cdot\|_i$ for $1 \leq i \leq \infty$. For any $u \in H^1(\mathbb{R}^2)$ and $t > 0$, $u_t(x) := u(tx)$. We shall denote various positive constants by C_i, $i = 1,2,\ldots$.

2. Variational framework and preliminaries

For measurable functions $u, v : \mathbb{R}^2 \to \mathbb{R}$, we define the following symmetric bilinear forms

$$(u,v) \mapsto A_1(u,v) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(2+|x-y|)\, u(x)v(y)\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

$$(u,v) \mapsto A_2(u,v) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln\left(1+\frac{2}{|x-y|}\right) u(x)v(y)\mathrm{d}x\mathrm{d}y \tag{2.2}$$

and $(u,v) \mapsto A_0(u,v) = A_1(u,v) - A_2(u,v) = \int_{\mathbb{R}^2} \phi_u(x)uv\mathrm{d}x$. Using the Hardy–Littlewood–Sobolev inequality (see [12]), one has

$$|A_2(u,v)| \le \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}|u(x)v(y)|\mathrm{d}x\mathrm{d}y \le C_1\|u\|_{4/3}\|v\|_{4/3} \tag{2.3}$$

Under (K1), we study problem (1.1) in the function space

$$E := X \cap H^1_{as} \ \text{ with the norm } \|u\|_E := \left(\|u\|^2_{H^1} + \|u\|^2_*\right)^{\frac{1}{2}}, \tag{2.4}$$

where $\|u\|^2_* = \int_{\mathbb{R}^2} \ln(2+|x|)u^2(x)\mathrm{d}x$, $H^1_{as} = \left\{u \in H^1(\mathbb{R}^2) : \ u(x_1,x_2) = u(|x_1|,|x_2|), \forall x \in \mathbb{R}^2\right\}$ and $X = \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \ln(2+|x|)u^2\mathrm{d}x < \infty\right\}$. The equation $\Delta\phi = u^2$ can be solved by $\phi_u = \frac{1}{2\pi}\ln|x| * u^2$ for any $u \in H^1(\mathbb{R}^2)$. Then the energy function of (1.1) can be defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2} \phi_u(x)u^2\mathrm{d}x - \int_{\mathbb{R}^2} K(x)F(u)\mathrm{d}x, \tag{2.5}$$

A standard argument shows that the embedding $X \hookrightarrow L^s(\mathbb{R}^2)$ is compact for $s \in [2,\infty)$ and $\Phi \in \mathcal{C}^1(X,\mathbb{R})$, see [10]. As in [10], we have the following auxiliary properties.

Lemma 2.1. *Assume that* (K1), (F1) *and* (F2) *hold. If u is a critical point of Φ restricted to E, then u is a critical point of Φ on X.*

Lemma 2.2. *Under* (K1)*, there hold $A_1(u^2,v^2) \ge \frac{1}{8\pi}\|u\|_2^2\|v\|_*^2$ and $\gamma\|u\|^2_{H^1} \le 2\|\nabla u\|_2^2 + A_1(u^2,u^2)$ for all $u,v \in E$.*

3. Proof of Theorem 1.1

In this section, we always assume that (K1)–(K2) and (F1)–(F3) hold.

Lemma 3.1. *For all $x \in \mathbb{R}^2, t > 0$ and $u \in \mathbb{R}$, there holds*

$$g(x,t,u) := \frac{1}{t^2}K(t^{-1}x)F(t^2u) - K(x)F(u) + \frac{1-t^4}{2}K(x)[f(u)u - F(u)] - \frac{1-t^4}{4}\nabla K(x)\cdot xF(u) \ge 0. \tag{3.1}$$

Proof. Using (F1) and (F2), we observe that (3.1) holds for $u = 0$. For any fixed $x \in \mathbb{R}^2$ and $u \ne 0$, we have

$$\begin{aligned}\frac{\mathrm{d}(g(x,t,u))}{\mathrm{d}t} &= 2K(t^{-1}x)t^3u^3\left[\frac{f(t^2u)t^2u - F(t^2u)}{(t^2u)^3} - \frac{f(u)u - F(u)}{u^3}\right] \\ &\quad + 2t^3\left[K(t^{-1}x) - K(x)\right][f(u)u - 3F(u)] \\ &\quad + t^3u^3\nabla K(t^{-1}x)\cdot(t^{-1}x)\left[\frac{F(u)}{u^3} - \frac{F(t^2u)}{(t^2u)^3}\right] \\ &\quad + t^3F(u)\left\{\left[4K(t^{-1}x) - \nabla K(t^{-1}x)\cdot(t^{-1}x)\right] - [4K(x) - \nabla K(x)\cdot x]\right\} \\ &:= G_1(t) + G_2(t) + G_3(t) + G_4(t).\end{aligned} \tag{3.2}$$

By (K1) and (F3), we have $G_1(t)\begin{cases} \ge 0, & t \ge 1, \\ \le 0, & 0 < t < 1.\end{cases}$ Note that (F1)–(F3) imply

$$\frac{1-t^4}{2}f(u)u + \frac{t^4-3}{2}F(u) + \frac{1}{t^2}F(t^2u) \ge 0, \quad \forall\, t > 0,\ u \in \mathbb{R}. \tag{3.3}$$

Letting $t \to 0$ in (3.3), we have

$$f(u)u - 3F(u) \geq 0, \qquad \forall\, u \in \mathbb{R}. \tag{3.4}$$

Note that (K2) yields

$$t \mapsto K(t^{-1}x) \text{ is nondereasing on } (0, \infty). \tag{3.5}$$

Then (3.4) and (3.5) imply $G_2(t) \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1. \end{cases}$ Using (3.4), it is checked easily that

$$\frac{F(u)}{u^3} \text{ is nondereasing on } (-\infty, 0) \cup (0, +\infty). \tag{3.6}$$

Since $\nabla K(x)\cdot x \leq 0$ for all $x \in \mathbb{R}^2$, then (3.6) leads to $G_3(t) \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1. \end{cases}$ Since $F(u) \geq 0$ for all $u \in \mathbb{R}$, then (K2) implies $G_4(t) \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1. \end{cases}$ Combining (3.2) with the properties of G_1–G_4, we obtain for any fixed $x \in \mathbb{R}^2$ and $u \neq 0$, $\frac{\mathrm{d}(g(x,t,u))}{\mathrm{d}t} \begin{cases} \geq 0, & t \geq 1, \\ \leq 0, & 0 < t < 1, \end{cases}$ which implies that $g(x, t, u) \geq g(x, 1, u) = 0$ for all $x \in \mathbb{R}^2$, $t > 0$ and $u \in \mathbb{R}$. □

Next, we construct an energy estimate inequality related to $\Phi(u)$, $\Phi(t^2 u_t)$ and $J(u)$, where

$$\begin{aligned} J(u) = \left.\frac{\mathrm{d}\Phi(t^2 u_t)}{\mathrm{d}t}\right|_{t=1} &= 2\|\nabla u\|_2^2 + \int_{\mathbb{R}^2} \phi_u(x) u^2 \mathrm{d}x - \frac{1}{8\pi}\|u\|_2^4 \\ &\quad - \int_{\mathbb{R}^2} \{2K(x)[f(u)u - F(u)] - \nabla K(x)\cdot x F(u)\}\,\mathrm{d}x, \qquad \forall\, u \in E. \end{aligned} \tag{3.7}$$

Lemma 3.2. *The following two inequalities hold:*

$$\Phi(u) \geq \Phi(t^2 u_t) + \frac{1 - t^4}{4} J(u) + \frac{1 - t^4 + 4t^4 \ln t}{32\pi}\|u\|_2^4, \qquad \forall\, u \in E, \quad t > 0. \tag{3.8}$$

and

$$\Phi(u) \geq \frac{1}{4} J(u) + \frac{1}{32\pi}\|u\|_2^4, \qquad \forall\, u \in E. \tag{3.9}$$

Proof. From (2.5), (3.1) and (3.7), we deduce that

$$\begin{aligned} &\Phi(u) - \Phi(t^2 u_t) \\ &= \frac{1 - t^4}{2}\|\nabla u\|_2^2 + \frac{1 - t^4}{4}\int_{\mathbb{R}^2} \phi_u(x) u^2 \mathrm{d}x + \frac{t^4 \ln t}{8\pi}\|u\|_2^4 + \int_{\mathbb{R}^2} \left[\frac{1}{t^2} K(t^{-1}x) F(t^2 u) - K(x) F(u)\right] \mathrm{d}x \\ &= \frac{1 - t^4}{4} J(u) + \frac{1 - t^4 + 4t^4 \ln t}{32\pi}\|u\|_2^4 + \int_{\mathbb{R}^2} \left\{\frac{1}{t^2} K(t^{-1}x) F(t^2 u) - K(x) F(u)\right. \\ &\qquad \left. + \frac{1 - t^4}{2} K(x)\left[f(u)u - F(u)\right] - \frac{1 - t^4}{4} \nabla K(x)\cdot x F(u)\right\} \mathrm{d}x \\ &\geq \frac{1 - t^4}{4} J(u) + \frac{1 - t^4 + 4t^4 \ln t}{32\pi}\|u\|_2^4, \qquad \forall\, u \in E, \quad t > 0. \end{aligned}$$

This shows that (3.8) holds. Moreover, it follows from (K2), (2.5), (3.4) and (3.7) that

$$\begin{aligned} \Phi(u) - \frac{1}{4} J(u) &= \frac{1}{32\pi}\|u\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2} \{2K(x)\left[f(u)u - 3F(u)\right] - \nabla K(x)\cdot x F(u)\}\,\mathrm{d}x \\ &\geq \frac{1}{32\pi}\|u\|_2^4, \qquad \forall\, u \in E, \end{aligned}$$

hence (3.9) holds. □

A standard computation shows that $1 - t^4 + 4t^4 \ln t \geq 0$ for all $t > 0$. Then Lemma 3.2 implies:

Corollary 3.3. *$\Phi(u) = \max_{t>0} \Phi(t^2 u_t)$ for all $u \in \mathcal{M}$.*

Lemma 3.4. *For any $u \in E \setminus \{0\}$, there exists a constant $t_u > 0$ such that $t_u^2 u_{t_u} \in \mathcal{M}$.*

Proof. Fix $u \in E\setminus\{0\}$ and define a function $\eta(t) := \Phi(t^2 u_t)$ on $(0, \infty)$. By (2.5) and (3.7), we have $\eta'(t) = 0 \Leftrightarrow \quad J(t^2 u_t) = 0 \quad \Leftrightarrow \quad t^2 u_t \in \mathcal{M}, \quad \forall\, t > 0$. By (K2), one has

$$-2K(x) \leq \nabla K(x) \cdot x \leq 2K(x), \qquad \forall\, x \in \mathbb{R}^2. \tag{3.10}$$

Then by (F3), (3.6) and (3.10), we deduce that for all $0 < t \leq 1$,

$$\frac{\eta'(t)}{t^3} \geq 2\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^2} \phi_u(x) u^2 \mathrm{d}x - \frac{4\ln t + 1}{8\pi}\|u\|_2^4 - 2\|K\|_\infty \int_{\mathbb{R}^2} f(u) u \mathrm{d}x,$$

and for all $t > 1$,

$$\frac{\eta'(t)}{t^3} \leq 2\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^2} \phi_u(x) u^2 \mathrm{d}x - \frac{4\ln t + 1}{8\pi}\|u\|_2^4 + 2\|K\|_\infty \int_{\mathbb{R}^2} F(u) \mathrm{d}x.$$

Then the above two inequalities lead to $\eta'(t) > 0$ for $0 < t \leq 1$ small and $\eta'(t) < 0$ for $t > 1$ large. Therefore there exists $t_u > 0$ such that $\eta'(t_u) = 0$ and $t_u^2 u_{t_u} \in \mathcal{M}$. □

The following lemma is a direct conclusion of Corollary 3.3 and Lemma 3.4.

Lemma 3.5. *$\inf_{u\in\mathcal{M}} \Phi(u) := m = \inf_{u\in E\setminus\{0\}} \max_{t>0} \Phi(t^2 u_t)$.*

Lemma 3.6. *$m = \inf_{u\in\mathcal{M}} \Phi(u) > 0$.*

Proof. Using $J(u) = 0$ for $u \in \mathcal{M}$, a standard argument shows that there exists $\varrho > 0$ such that $\|u\|_{H^1} \geq \varrho, \forall\, u \in \mathcal{M}$. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. We distinguish the two cases:

Case (1) $\inf_{n\in\mathbb{N}} \|u_n\|_2 := \varrho_1 > 0$. Then (3.9) implies $m + o(1) = \Phi(u_n) \geq \frac{1}{32\pi}\|u_n\|_2^4 \geq \frac{1}{32\pi}\varrho_1^4$.

Case (2) $\inf_{n\in\mathbb{N}} \|u_n\|_2 = 0$. Passing to a subsequence, we can assume that $\|u_n\|_2 \to 0$ and $\|\nabla u_n\|_2 \geq \frac{\varrho}{2}$ for all $n \in \mathbb{N}$. Jointly with (2.3) and the Gagliardo–Nirenberg inequality, we have

$$A_2(u_n^2, u_n^2) \leq C_1\|u_n\|_{8/3}^4 \leq C_2\|u_n\|_2^3\|\nabla u_n\|_2, \|u_n\|_p^p \leq C_3\|u_n\|_2^2\|\nabla u_n\|_2^{p-2}, \frac{|\ln(\|\nabla u_n\|_2)|}{\|\nabla u_n\|_2^2} \leq C_4. \tag{3.11}$$

Let $t_n = \|\nabla u_n\|_2^{-1/2}$. Since $J(u_n) = 0$, it follows from (F1), (F2), Corollary 3.3 and (3.11) that

$$\begin{aligned}
m + o(1) &= \Phi(u_n) \geq \Phi(t_n^2 (u_n)_{t_n}) \\
&\geq \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{t_n^4}{4}A_2(u_n^2, u_n^2) - \frac{t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 - \frac{\|K\|_\infty}{t_n^2}\int_{\mathbb{R}^2}\left[|t_n^2 u_n|^2 + C_5|t_n^2 u_n|^p\right]\mathrm{d}x \\
&\geq \frac{t_n^4}{2}\|\nabla u_n\|_2^2 - \frac{C_2}{4}t_n^4\|u_n\|_2^3\|\nabla u_n\|_2 - \frac{t_n^4 \ln t_n}{8\pi}\|u_n\|_2^4 \\
&\quad - t_n^2\|K\|_\infty\|u_n\|_2^2 - C_6 t_n^{2p-2}\|u_n\|_2^2\|\nabla u_n\|_2^{p-2} \\
&= \frac{1}{2} - \frac{C_2\|u_n\|_2^3}{4\|\nabla u_n\|_2} + \frac{\ln(\|\nabla u_n\|_2)}{16\pi\|\nabla u_n\|_2^2}\|u_n\|_2^4 - \frac{\|K\|_\infty\|u_n\|_2^2}{\|\nabla u_n\|_2} - \frac{C_6\|u_n\|_2^2}{\|\nabla u_n\|_2} = \frac{1}{2} + o(1).
\end{aligned} \tag{3.12}$$

Cases (1) and (2) show that $m = \inf_{u\in\mathcal{M}} \Phi(u) > 0$. □

Lemma 3.7. *m is achieved. Moreover, the minimizer is a critical point of Φ in E.*

Proof. Let $\{u_n\} \subset \mathcal{M}$ be such that $\Phi(u_n) \to m$. Then (3.9) yields $m + o(1) = \Phi(u_n) - \frac{1}{4}J(u_n) \geq \frac{1}{32\pi}\|u_n\|_2^4$. This shows that $\{\|u_n\|_2\}$ is bounded. Now, we prove that $\{\|\nabla u_n\|_2\}$ is also bounded. Arguing by contradiction, suppose that $\|\nabla u_n\|_2 \to \infty$. Let $t_n = (2\sqrt{m}/\|\nabla u_n\|_2)^{1/2}$. Since $t_n \to 0$, we have $t_n^4 \ln t_n \to 0$. Arguing as in (3.12), we can derive a contradiction, and so $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. It is easy to check that $\delta_0 := \limsup_{n\to\infty} \|u_n\|_2 > 0$ and $A_1(u_n^2, u_n^2)$ is bounded. Jointly with Lemma 2.2, we have $\{\|u_n\|_*\}$ is bounded, and so $\{u_n\}$ is bounded in E. We may thus assume, passing to a subsequence, that $u_n \rightharpoonup \bar{u}$ in $E, u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s \in [2, \infty), u_n \to \bar{u}$ a.e. on $\mathbb{R}^2$. A standard argument shows that $J(\bar{u}) \leq \liminf_{n\to\infty} J(u_n) = 0$. Note that $\bar{u} \neq 0$ due to $\delta_0 > 0$. In view of Lemma 3.4, there exists $\bar{t} > 0$ such that $\bar{t}^2\bar{u}_{\bar{t}} \in \mathcal{M}$, and so $\Phi(\bar{t}^2\bar{u}_{\bar{t}}) \geq m$. Thus, it follows from (2.5), (3.7), (3.8), Fatou's lemma and Lebesgue's dominated convergence theorem that

$$\begin{aligned} m &= \lim_{n\to\infty} \left[\Phi(u_n) - \frac{1}{4}J(u_n)\right] \\ &= \lim_{n\to\infty} \left\{\frac{1}{32\pi}\|u_n\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2} \{2K(x)[f(u_n)u_n - 3F(u_n)] - \nabla K(x)\cdot xF(u_n)\}\,\mathrm{d}x\right\} \\ &\geq \frac{1}{32\pi}\|\bar{u}\|_2^4 + \frac{1}{4}\int_{\mathbb{R}^2} \{2K(x)[f(\bar{u})\bar{u} - 3F(\bar{u})] - \nabla K(x)\cdot xF(\bar{u})\}\,\mathrm{d}x \\ &= \Phi(\bar{u}) - \frac{1}{4}J(\bar{u}) \geq \Phi\left(\bar{t}^2\bar{u}_{\bar{t}}\right) - \frac{\bar{t}^4}{4}J(\bar{u}) \geq m - \frac{\bar{t}^4}{4}J(\bar{u}) \geq m, \end{aligned}$$

which implies that $J(\bar{u}) = 0$ and $\Phi(\bar{u}) = m$. Similarly to the proofs of [10, Lemma 4.11], we can deduce the last conclusion. □

Theorem 1.1 is a direct consequence of Lemmas 2.1, 3.5 and 3.7.

CRediT authorship contribution statement

Lixi Wen: Conceptualization, Formal analysis, Investigation, Methodology, Resources, Software, Validation, Writing - original draft. **Sitong Chen:** Funding acquisition, Project administration, Data curation , Writing - review & editing. **Vicenţiu D. Rădulescu:** Supervision, Visualization.

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