Singular elliptic problems with convection term in anisotropic media

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Abstract. We present some recent existence and uniqueness results for elliptic boundary value problems involving singular nonlinearities that generalize the Lane-Emden-Fowler equation. The following types of problems are considered: (i) singular problems with sublinear nonlinearity and two parameters; (ii) combined effects of asymptotically linear and singular nonlinearities in bifurcation problems; (iii) bifurcation for a class of singular elliptic problems with subquadratic convection term. In some concrete situations we also establish the asymptotic behavior of the solution around the bifurcation point. Our approach relies on maximum principle combined with various techniques for elliptic equations.

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MOTIVATION AND PREVIOUS RESULTS

Singular elliptic problems have been intensively studied in the last decades. These kind of problems are closely related to the study of blow-up boundary solutions associated to elliptic problems. To be more specific, let us consider the following example

$$\begin{cases} \Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $p > 1$. It is known that this problem has a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ see [13]. With the change of variable $v = u^{-1}$, the problem (1) becomes

$$\begin{cases} -\Delta v = v^{2-p} - \dfrac{2}{v}|\nabla v|^2 & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The above equation contains both singular nonlinearities (like v^{-1} or v^{2-p}, if $p > 2$) and a convection term (denoted by $|\nabla v|^2$). These nonlinearities make more difficult to handle

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problems like (2). Our purpose in this paper is to give an overview on some old and new results in this direction. We recall the pioneering paper of Crandall, Rabinowtz and Tartar [16] that contains one of the first existence results for singular elliptic problems. More exactly, in [16] the following problem has been considered

$$\begin{cases} -\Delta u + au = u^{-\alpha} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

This is a generalization of the Lane-Emden-Fowler equation that corresponds to the case $a = 0$. It is proved in [16] that problem (3) has a unique solution, for any $\alpha > 0$.

Singular elliptic equations arise in the study of non-Newtonian fluids, chemical heterogenous catalysts, in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction-diffusion processes where the condition $u \geq 0$ is viewed as the density of a reactant and the region where $u = 0$ is called the *dead core*, where no reaction takes place (see [2] for the study of a single, irreversible steady-state reaction).

Problems of this type are also encountered in glacial advance, in transport of coal slurries down conveyor belts and in several other geophysical and industrial contents (see [4] for the case of the incompressible flow of a uniform stream past a semi-infinite flat plate at zero incidence). For more details we also refer to [8, 14, 26, 29] and the references therein.

Many authors considered the problem

$$\begin{cases} -\Delta u + K(x)u^{-\alpha} = \lambda u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\lambda \geq 0$ and $\alpha, p \in (0,1)$. For $K \equiv -1$, it was proved in Coclite and Palmieri [15] that (4) has at least one solution for all $\lambda \geq 0$ and $0 < p < 1$. In turn, if $p \geq 1$, there exists $\lambda^* > 0$ such that (4) has a solution for $0 < \lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

On the other hand, if $K \equiv 1$ and $\lambda = p = 1$, the problem (4) was considered in [10] where it is shown that (4) has no solution, provided that $0 < \alpha < 1$ and $\lambda_1 \geq 1$ (that is, if Ω is "small"), where λ_1 denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$. In Shi and Yao [30] it is proved that for $\lambda > 0$ sufficiently large, problem (4) has at least one solution $u_\lambda \in C^1(\overline{\Omega})$ and

$$c_1 d(x) \leq u_\lambda(x) \leq c_2 d(x),$$

for any $x \in \Omega$ and for some constants $c_1, c_2 > 0$ independent of x.

Problems related to multiplicity and uniqueness become difficult even in simple cases. In [28] it is studied the existence of radial symmetric solutions to the problem

$$\begin{cases} \Delta u + \lambda(u^p - u^{-\alpha}) = 0 & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $\alpha > 0$, $0 < p < 1$, $\lambda > 0$, and B_1 is the unit ball in $\mathbb{R}^N$. Using a bifurcation theorem of Crandall and Rabinowitz, it has been shown in [28] that there exists $\lambda_1 > \lambda_0 > 0$ such that the above problem has no solutions for $\lambda < \lambda_0$, exactly one solution for $\lambda = \lambda_0$ or $\lambda > \lambda_1$, and two solutions for $\lambda_0 < \lambda \le \lambda_1$.

Our purpose in this survey paper is to present various existence, and non–existence results for several classes of singular elliptic problems. We also take into account bifurcation nonlinear problems and establish the precise rate decay of the solution in some concrete situations. We intend to reflect the "competition" between different quantities, such as: sublinear or superlinear nonlinearities, singular nonlinear terms (like $u^{-\alpha}$, for $\alpha > 0$), convection nonlinearities (like $|\nabla u|^a$, with $0 < a \le 2$), as well as sign–changing potentials.

A SINGULAR PROBLEM WITH SUBLINEAR NONLINEARITY

Consider the following boundary value problem with two parameters:

$$\begin{cases} -\Delta u + K(x)g(u) = \lambda f(x,u) + \mu h(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \ge 2)$, $K, h \in C^{0,\gamma}(\overline{\Omega})$, with $h > 0$ on $\overline{\Omega}$, and λ, μ are positive real numbers. We suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is positive on $\overline{\Omega} \times (0,\infty)$. We also assume that f is non–decreasing with respect to the second variable and is sublinear, that is,

$(f1)$ the mapping $(0,\infty) \ni s \longmapsto \dfrac{f(x,s)}{s}$ is non–increasing for all $x \in \overline{\Omega}$;

$(f2)$ $\displaystyle\lim_{s\downarrow 0} \frac{f(x,s)}{s} = +\infty$ and $\displaystyle\lim_{s\to\infty} \frac{f(x,s)}{s} = 0$, uniformly for $x \in \overline{\Omega}$.

We assume that $g \in C^1(0,\infty)$ is a nonnegative and nonincreasing function satisfying

$(g1)$ $\displaystyle\lim_{s\to 0^+} g(s) = +\infty$;

$(g2)$ $\displaystyle\int_0^1 g(s)ds < +\infty$.

The above conditions $(g1)$ and $(g2)$ are fulfilled by singular nonlinearities like $g(u) = u^{-\alpha}$, with $\alpha \in (0,1)$. This case includes the generalized Lane-Emden equations (see [35]). Obviously, hypothesis $(g2)$ implies the following Keller-Osserman type condition around the origin:

$(g3)$ $\displaystyle\int_0^1 \left(\int_0^t g(s)ds\right)^{-1/2} dt < \infty.$

As proved by Bénilan, Brezis and Crandall [4], condition $(g3)$ is equivalent to the *property of compact support*, that is, for every $h \in L^1(\mathbb{R}^N)$ with compact support, there exists a unique $u \in W^{1,1}(\mathbb{R}^N)$ with compact support such that $\Delta u \in L^1(\mathbb{R}^N)$ and $-\Delta u + g(u) = h$, a.e. in $\mathbb{R}^N$. That it is why it is natural to try to find solutions in the class

$$\mathscr{E} = \{ u \in C^2(\Omega) \cap C(\overline{\Omega});\ \Delta u \in L^1(\Omega) \}.$$

Our analysis developed in [18] showed that the sign of the extremal values of K plays an significant role in the study of the existence of a classical solution to problem (5). Define

$$\mathscr{E} := \{u \in C^2(\Omega);\ \Delta u \in L^1(\Omega)\}.$$

We will show that (5) has a solution in $\mathscr{E}$ for λ and μ belonging to a certain range. A very useful tool in our approach is the following existence result which is due to Shi and Yao [30] and concerns the problem

$$\begin{cases} -\Delta u = \Psi(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $\psi \in C^{2,\gamma}(\partial\Omega)$ $(0<\gamma<1)$ and $\Psi : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ is a Hölder continuous function with exponent γ on each compact subset of $\overline{\Omega} \times (0,\infty)$. Note that Ψ is not defined in the origin with respect to the second variable so Ψ may be singular at that point. We have

Lemma 1. (see [30]). *Assume that Ψ satisfies the following assumptions*

$(A1)$ $\quad \limsup_{s\to+\infty}\left(s^{-1}\max_{x\in\overline{\Omega}}\Psi(x,s)\right) < \lambda_1;$

$(A2)$ $\quad$ *for each $t>0$, there exists a constant $D(t)>0$ such that*

$$\Psi(x,r) - \Psi(x,s) \geq -D(t)(r-s), \quad \textit{for } x \in \overline{\Omega} \ \textit{ and } \ r \geq s \geq t;$$

$(A3)$ $\quad$ *there exist $\eta_0 > 0$ and an open subset $\Omega_0 \subset \Omega$ such that*

$$\min_{x\in\overline{\Omega}}\Psi(x,s) \geq 0 \quad \textit{for } x \in (0,\eta_0),$$

and

$$\lim_{s\to 0^+}\frac{\Psi(x,s)}{s} = +\infty \quad \textit{uniformly for } x \in \Omega_0.$$

Then, the problem (6) has at least one positive solution $u \in C^{2,\gamma}(G) \cap C(\overline{\Omega})$, for any compact set $G \subset \Omega \cup \{x \in \partial\Omega;\ \psi(x) > 0\}$.

The main difficulty in the treatment of (5) is the lack of the usual maximal principle between super and sub-solutions, due to the singular character of the equation. To overcome it, we state the following comparison principle which improves Lemma 3 in Shi and Yao [30]. The proof was given in [22] and uses some ideas from [30], that goes back to the pioneering work of Brezis and Kamin [6].

Lemma 2. (see [22]). *Let $\Psi : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ be a continuous function such that the mapping $(0,\infty) \ni s \longmapsto \dfrac{\Psi(x,s)}{s}$ is strictly decreasing at each $x \in \Omega$. Assume that there exists v, $w \in C^2(\Omega) \cap C(\overline{\Omega})$ such that*

(a) $\quad \Delta w + \Psi(x,w) \leq 0 \leq \Delta v + \Psi(x,v)$ *in* Ω;

(b) $\quad v, w > 0$ *in* Ω *and* $v \leq w$ *on* $\partial\Omega$;

(c) $\quad \Delta v \in L^1(\Omega)$ *or* $\Delta w \in L^1(\Omega)$.

Then $v \leq w$ in Ω.

Notice that $\Psi_{\lambda,\mu}(x,s) = -K(x)g(u) + \lambda f(x,s) + \mu h(x)$, $(x,s) \in \overline{\Omega} \times (0,\infty)$ satisfies the hypotheses in Lemma 1 and Lemma 2 provided $K \le 0$ in $\overline{\Omega}$. In this case we have

Theorem 1. *Assume that $K \le 0$, f satisfies conditions $(f1)-(f2)$ and g satisfies $(g1)-(g2)$. Then problem* (5) *has a unique solution $u_{\lambda,\mu}$ in $\mathscr{E}$, for any λ, $\mu > 0$. Moreover, $u_{\lambda,\mu}$ is increasing with respect to λ and μ.*

Let us now consider the case where $K > 0$ in $\overline{\Omega}$. Our next result shows the importance of condition $(g2)$.

Theorem 2. *Assume that $K > 0$ and f satisfies $(f1)-(f2)$. If $\int_0^1 g(s)ds = +\infty$, then problem* (5) *has no classical solution, for any λ, $\mu > 0$.*

If g satisfy $(g2)$ the we have the following result

Theorem 3. *Assume that $K > 0$, f satisfies $(f1)-(f2)$, and g satisfies $(g1)-(g2)$. Then there exists $\lambda_*, \mu_* > 0$ such that:*
– problem (5) *has at least one solution in $\mathscr{E}$ either if $\lambda > \lambda_*$ or if $\mu > \mu_*$.*
– problem (5) *has no solution in $\mathscr{E}$ if $\lambda < \lambda_*$ and $\mu < \mu_*$.*
Moreover, if either $\lambda > \lambda_*$ or if $\mu > \mu_*$, then problem (5) *has a maximal solution in $\mathscr{E}$ which is increasing with respect to λ and μ.*

The diagram of dependence on λ and μ in Theorem 3 is depicted in Figure 1 below.

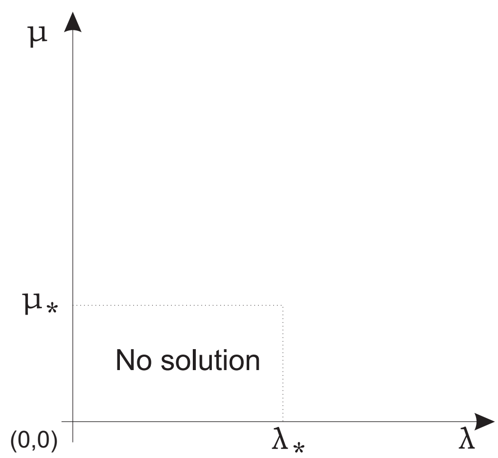

FIGURE 1. The dependence on λ and μ in Theorem 3

Sketch of the Proof. We split the proof in several steps.
Step I. *Existence of the solutions of (5) for λ and μ large.*
Let us remark first that $\Phi_{\lambda,\mu}(x,s) = \lambda f(x,s) + \mu h(x)$, define for all $(x,s) \in \overline{\Omega} \times [0,\infty)$ satisfies the hypotheses in Lemma 1 and Lemma 2. Hence, the exists a unique solution

$U_{\lambda,\mu} \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta U_{\lambda,\mu} = \lambda f(x, U_{\lambda,\mu}) + \mu h(x) & \text{in } \Omega, \\ U_{\lambda,\mu} > 0 & \text{in } \Omega, \\ U_{\lambda,\mu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

Obvious, $U_{\lambda,\mu}$ is a super-solution of (5) for all $\lambda, \mu > 0$. The main point is to find a sub-solution of (5). To this aim we consider the one dimensional problem

$$\begin{cases} h''(t) = g(h(t)), \quad \text{for all } t > 0, \\ h > 0, \quad \text{in } (0, \infty), \\ h(0) = 0. \end{cases} \tag{8}$$

Since $\int_0^1 g(s)ds < \infty$, we deduce that h' can be extended in origin by taking $h'(0) = 0$. Hence $h \in C^2(0,\infty) \cap C^1[0,\infty)$. We fix $\mu > 0$ and we prove that if $\lambda > 0$ is large enough, there exists $M > 0$ such that $\underline{u}_{\lambda,\mu} = Mh(\varphi_1)$ is a sub-solution of (5). Using Lemma 2 we get $\underline{u}_{\lambda,\mu} \le U_{\lambda,\mu}$ in Ω and by standard elliptic arguments there exists a solution $u_{\lambda,\mu} \in C^2(\Omega) \cap C(\overline{\Omega})$ of (5). Since $h \in C^1[0,+\infty)$, we deduce that $u_{\lambda,\mu} \in \mathscr{E}$. In the same manner we fix $\lambda > 0$ and we deduce the existence of a solution in $\mathscr{E}$ of problem (5) provided $\mu > 0$ is large enough.

Step II. *Nonexistence for λ, μ small.*

Using the assumptions of f there exists $m > 0$ such that

$$f(x,s) + h(x) - p(d(x))g(s) < ms, \quad \text{for all } (x,s) \in \Omega \times (0,+\infty).$$

We claim that problem (5) has no classical solution for $0 < \lambda, \mu \le \min\{1, \lambda_1/2m\}$. Indeed, if u_0 would be a classical solution of (5) with $0 < \lambda, \mu \le \min\{1, \lambda_1/2m\}$, then, according to the above inequality, u_0 is a sub-solution of

$$\begin{cases} -\Delta u = \dfrac{\lambda_1}{2} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Obviously, φ_1 is a super-solution of (9) and by Lemma 2 we get $u_0 \le \varphi_1$ in Ω. Thus, by standard elliptic arguments, problem (9) has a solution $u \in C^2(\overline{\Omega})$. Multiplying by φ_1 in (9) and then integrating over Ω we have $-\int_\Omega \varphi_1 \Delta u dx = \frac{\lambda_1}{2} \int_\Omega u \varphi_1 dx$, that is, $-\int_\Omega u \Delta \varphi_1 dx = \frac{\lambda_1}{2} \int_\Omega u \varphi_1 dx$. The above equality yields $\int_\Omega u \varphi_1 dx = 0$, but this is clearly a contradiction, since u and φ_1 are both positive on Ω. It follows that (5) has no classical solutions for $0 < \lambda, \mu \le \min\{1, \lambda_1/2m\}$.

Step III. *Existence of a maximal solution of (5).*

We show that if (5) has a solution $u_{\lambda,\mu} \in \mathscr{E}$, then it has a maximal solution that belongs to $\mathscr{E}$. Let $\lambda, \mu > 0$ be such that (5) has a solution $u_{\lambda,\mu} \in \mathscr{E}$. If $U_{\lambda,\mu}$ is the solution of (7), by Lemma 2 we have $u_{\lambda,\mu} \le U_{\lambda,\mu}$ in $\overline{\Omega}$. For any $j \ge 1$, denote $\Omega_j = \left\{ x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \frac{1}{j} \right\}$. Let $u_0 = U_{\lambda,\mu}$ and u_j be the solution of

$$\begin{cases} -\Delta \zeta + g(u_{j-1}) = \lambda f(x, u_{j-1}) + \mu h(x) & \text{in } \Omega_j, \\ \zeta = u_{j-1} & \text{in } \Omega \setminus \Omega_j. \end{cases}$$

Using the fact that the mapping $\Phi_{\lambda,\mu} = \lambda f(x,s) + \mu x(x)$, $(x,s) \in \overline{\Omega} \times (0,+\infty)$ is non-decreasing with respect to the second variable, we get $u_{\lambda,\mu} \le u_j \le u_{j-1} \le u_0$ in $\overline{\Omega}$. Define $\widehat{u}_{\lambda,\mu}(x) = \lim_{j\to\infty} u_j(x)$ for all $x \in \overline{\Omega}$. By standard elliptic arguments (see [23]) it follows that $\widehat{u}_{\lambda,\mu}$ is a solution of (5). It is clear that $\widehat{u}_{\lambda,\mu}$ is the maximal solution of (5). Moreover, since $u_{\lambda,\mu} \le \widehat{u}_{\lambda,\mu}$ in $\overline{\Omega}$, we have $\widehat{u}_{\lambda,\mu} \in \mathscr{E}$.

Step IV. *Dependence on λ and μ.*

We first prove the dependence on λ of the maximal solution of (5). For this purpose, define

$$A := \{\lambda > 0;\ (5) \text{ has at least a solution in } \mathscr{E}, \text{ for all } \mu > 0\}.$$

Let $\lambda_* = \inf A$. From the previous steps we have $A \ne \emptyset$ and $\lambda_* > 0$. Using Lemma 2 we deduce that $(\lambda^*,+\infty)$. To prove the dependence on μ we argue in the same manner by defining

$$B := \{\mu > 0;\ (5) \text{ has at least a solution in } \mathscr{E}, \text{ for all } \lambda > 0\}$$

and $\mu_* = \inf B$.

The proof of Theorem 3 is now complete. □

The following result give partial answers in the case where the potential $K(x)$ changes sign.

Theorem 4. *Assume that K changes the sign in Ω, f satisfies $(f1)-(f2)$ and g verifies $(g1)-(g2)$. Then there exist λ_* and $\mu_* > 0$ such that problem (5) has at least one solution $u_{\lambda,\mu} \in \mathscr{E}$, provided that either $\lambda > \lambda_*$ or $\mu > \mu_*$. Moreover, for $\lambda > \lambda_*$ or $\mu > \mu_*$, $u_{\lambda,\mu}$ is increasing with respect to λ and μ.*

COMBINED EFFECTS OF ASYMPTOTICALLY LINEAR AND SINGULAR NONLINEARITIES IN BIFURCATION PROBLEMS OF LANE-EMDEN-FOWLER TYPE

This section is devoted to the study of the following singular elliptic problem with asymptotic nonlinearities.

$$\begin{cases} -\Delta u = \lambda f(u) + a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

where $a \in C^{0,\gamma}(\overline{\Omega})$, $a \ge 0$, $a \not\equiv 0$ in $\overline{\Omega}$, The main feature here is that the smooth term $f(x,u)$ is asymptotically linear. In other words, we drop out the assumption $(f2)$ on f and we require in turn that f fulfils the following assumption

$(f3)$ there exists $m := \lim_{s\to\infty} \frac{f(x,s)}{s} \in (0,+\infty)$, uniformly for $x \in \overline{\Omega}$.

We also require the following growth condition of g near the origin

$(g3)$ there exists $c, \eta > 0$ and $\alpha \in (0,1)$ such that $g(s) \le cs^{-\alpha}$, $\forall s \in (0,\eta)$.

A careful examination of (10) reveals the fact that the singular term $g(u)$ is not significant. Actually, the conclusions are close to those established in [27, Theorem A], where an elliptic problem associated to an asymptotically linear function is studied.

Let λ_1 be the first Dirichlet eigenvalue of $(-\Delta)$ in Ω and $\lambda^* := \lambda_1/m$. Set $a_* := \min_{x\in\overline{\Omega}} a(x)$ and $d(x) := \mathrm{dist}\,(x, \partial\Omega)$.

Theorem 5. *Assume that conditions $(f1)$, $(f3)$, $(g1)$, and $(g3)$ are fulfilled. Then the following hold.*

(i) *If $\lambda \geq \lambda^*$, then problem* (10) *has no solutions in $\mathscr{E}$.*

(ii) *If $a_* > 0$ (resp. $a_* = 0$) then problem* (10) *has a unique solution $u_\lambda \in \mathscr{E}$ for all $-\infty < \lambda < \lambda^*$ (resp. $0 < \lambda < \lambda^*$) with the properties:*
(ii1) *u_λ is strictly increasing with respect to λ;*
(ii2) *there exist two positive constants c_1, $c_2 > 0$ depending on λ such that $c_1\, d(x) \leq u_\lambda(x) \leq c_2\, d(x)$, for all $x \in \Omega$;*
(ii3) $\lim_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, *uniformly on compact subsets of* Ω.

The bifurcation diagram is depicted in Figure 2.

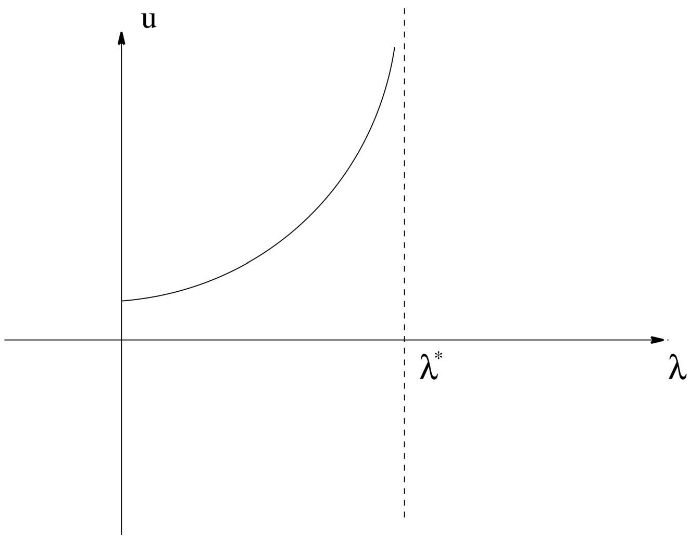

FIGURE 2. The bifurcation in Theorem 5.

Proof. The first part of the proof relies on standard arguments based on the maximum principle (see [12] for details). The proof of (ii1) follows from Lemma 2 while the proof of (ii2) use in a definite manner the growth assumption $(g3)$ on g. The most interesting part of the proof concerns (ii3) and, due to the special character of our problem, we will be able to show that, in this case, L^2–boundedness implies H_0^1–boundedness! We refer to [27] for a related problem and further results.

Let $u_\lambda \in \mathscr{E}$ be the unique solution of (10) for $0 < \lambda < \lambda^*$. We prove that $\lim\limits_{\lambda \nearrow \lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. Suppose the contrary. Since $(u_\lambda)_{0<\lambda<\lambda^*}$ is a sequence of nonnegative super–harmonic functions in Ω then, by Theorem 4.1.9 in [24], there exists a subsequence of $(u_\lambda)_{\lambda<\lambda^*}$ [still denoted by $(u_\lambda)_{\lambda<\lambda^*}$] which is convergent in $L^1_{\text{loc}}(\Omega)$.

We first prove that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. We argue by contradiction. Suppose that $(u_\lambda)_{\lambda<\lambda^*}$ is not bounded in $L^2(\Omega)$. Thus, passing eventually at a subsequence we have $u_\lambda = M(\lambda)w_\lambda$, where

$$M(\lambda) = ||u_\lambda||_{L^2(\Omega)} \to \infty \quad \text{as } \lambda \nearrow \lambda^* \text{ and } w_\lambda \in L^2(\Omega), \, \|w_\lambda\|_{L^2(\Omega)} = 1. \tag{11}$$

Using $(f1)$, $(g3)$ and the monotonicity assumption on g, we deduce the existence of $A, B, C, D > 0$ $(A > m)$ such that

$$f(t) \le At + B, \quad g(t) \le Ct^{-\alpha} + D, \quad \text{for all } t > 0. \tag{12}$$

This implies

$$\frac{1}{M(\lambda)}(\lambda f(u_\lambda) + a(x)g(u_\lambda)) \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*$$

that is,

$$-\Delta w_\lambda \to 0 \quad \text{in } L^1_{\text{loc}}(\Omega) \text{ as } \lambda \nearrow \lambda^*. \tag{13}$$

By Green's first identity, we have

$$\int_\Omega \nabla w_\lambda \cdot \nabla\phi \, dx = -\int_\Omega \phi \, \Delta w_\lambda \, dx = -\int_{\operatorname{Supp}\phi} \phi \, \Delta w_\lambda \, dx \quad \forall \phi \in C_0^\infty(\Omega). \tag{14}$$

Using (13) we derive that

$$\begin{aligned} \left| \int_{\operatorname{Supp}\phi} \phi \, \Delta w_\lambda \, dx \right| &\le \int_{\operatorname{Supp}\phi} |\phi||\Delta w_\lambda| \, dx \\ &\le \|\phi\|_{L^\infty} \int_{\operatorname{Supp}\phi} |\Delta w_\lambda| \, dx \to 0 \quad \text{as } \lambda \nearrow \lambda^*. \end{aligned} \tag{15}$$

Combining (14) and (15), we arrive at

$$\int_\Omega \nabla w_\lambda \cdot \nabla\phi \, dx \to 0 \ \text{ as } \lambda \nearrow \lambda^*, \quad \forall \phi \in C_0^\infty(\Omega). \tag{16}$$

By definition, the sequence $(w_\lambda)_{0<\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$.

We claim that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$. Indeed, using (12) and Hölder's inequality, we have

$$\begin{aligned}
\int_\Omega |\nabla w_\lambda|^2 &= -\int_\Omega w_\lambda \Delta w_\lambda = \frac{-1}{M(\lambda)}\int_\Omega w_\lambda \Delta u_\lambda \\
&= \frac{1}{M(\lambda)}\int_\Omega [\lambda w_\lambda f(u_\lambda) + a(x)g(u_\lambda)w_\lambda] \\
&\leq \frac{\lambda}{M(\lambda)}\int_\Omega w_\lambda (Au_\lambda + B) + \frac{||a||_\infty}{M(\lambda)}\int_\Omega w_\lambda (Cu_\lambda^{-\alpha} + D) \\
&= \lambda A \int_\Omega w_\lambda^2 + \frac{||a||_\infty C}{M(\lambda)^{1+\alpha}}\int_\Omega w_\lambda^{1-\alpha} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)}\int_\Omega w_\lambda \\
&\leq \lambda^* A + \frac{||a||_\infty C}{M(\lambda)^{1+\alpha}}|\Omega|^{(1+\alpha)/2} + \frac{\lambda B + \|a\|_\infty D}{M(\lambda)}|\Omega|^{1/2}.
\end{aligned}$$

From the above estimates, it is easy to see that $(w_\lambda)_{\lambda<\lambda^*}$ is bounded in $H_0^1(\Omega)$, so the claim is proved. Then, there exists $w \in H_0^1(\Omega)$ such that (up to a subsequence)

$$w_\lambda \rightharpoonup w \quad \text{weakly in } \; H_0^1(\Omega) \; \text{ as } \lambda \nearrow \lambda^* \tag{17}$$

and, since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$,

$$w_\lambda \to w \quad \text{strongly in } \; L^2(\Omega) \; \text{ as } \lambda \nearrow \lambda^*. \tag{18}$$

On the one hand, by (11) and (18), we derive that $\|w\|_{L^2(\Omega)} = 1$. Furthermore, using (16) and (17), we infer that

$$\int_\Omega \nabla w \cdot \nabla \phi \, dx = 0, \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

Since $w \in H_0^1(\Omega)$, using the above relation and the definition of $H_0^1(\Omega)$, we get $w = 0$. This contradiction shows that $(u_\lambda)_{\lambda<\lambda^*}$ is bounded in $L^2(\Omega)$. As above for w_λ, we can derive that u_λ is bounded in $H_0^1(\Omega)$. So, there exists $u^* \in H_0^1(\Omega)$ such that, up to a subsequence,

$$\begin{cases}
u_\lambda \rightharpoonup u^* & \text{weakly in } H_0^1(\Omega) \text{ as } \lambda \nearrow \lambda^*, \\
u_\lambda \to u^* & \text{strongly in } L^2(\Omega) \text{ as } \lambda \nearrow \lambda^*, \\
u_\lambda \to u^* & \text{a.e. in } \Omega \text{ as } \lambda \nearrow \lambda^*.
\end{cases} \tag{19}$$

Now we can proceed to obtain a contradiction. Multiplying by φ_1 in (10) and integrating over Ω we have

$$-\int_\Omega \varphi_1 \Delta u_\lambda = \lambda \int_\Omega f(u_\lambda)\varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1, \quad \text{for all } \; 0 < \lambda < \lambda^*. \tag{20}$$

On the other hand, by $(f1)$ it follows that $f(u_\lambda) \geq mu_\lambda$ in Ω, for all $0 < \lambda < \lambda^*$. Combining this with (20) we obtain

$$\lambda_1 \int_\Omega u_\lambda \varphi_1 \geq \lambda m \int_\Omega u_\lambda \varphi_1 + \int_\Omega a(x)g(u_\lambda)\varphi_1, \quad \text{for all } \; 0 < \lambda < \lambda^*. \tag{21}$$

Notice that by $(g1)$, (19) and the monotonicity of u_λ with respect to λ we can apply the Lebesgue convergence theorem to find

$$\int_\Omega a(x)g(u_\lambda)\varphi_1\,dx \to \int_\Omega a(x)g(u^*)\varphi_1\,dx \quad \text{as } \lambda \nearrow \lambda_1.$$

Passing to the limit in (21) as $\lambda \nearrow \lambda^*$, and using (19), we obtain

$$\lambda_1 \int_\Omega u^*\varphi_1 \geq \lambda_1 \int_\Omega u^*\varphi_1 + \int_\Omega a(x)g(u^*)\varphi_1.$$

Hence $\int_\Omega a(x)g(u^*)\varphi_1 = 0$, which is a contradiction. Therefore $\lim_{\lambda\nearrow\lambda^*} u_\lambda = +\infty$, uniformly on compact subsets of Ω. This concludes the proof. □

BIFURCATION AND ASYMPTOTICS FOR A SINGULAR ELLIPTIC EQUATION WITH CONVECTION TERM

In the present section we continue the bifurcation analysis for a large class of semilinear elliptic equations with singular nonlinearity and Dirichlet boundary condition.

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with a smooth boundary. We are concerned in this section with singular elliptic problems of the following type

$$\begin{cases} -\Delta u = K(x)g(u) + \lambda|\nabla u|^a + \mu f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

where $K \in C^{0,\gamma}(\overline{\Omega})$, $0 < a \leq 2$, $0 < \mu$ and $\lambda \in \mathbb{R}$. Throughout this section we suppose that $f : \overline{\Omega} \times [0,\infty) \to [0,\infty)$ is a Hölder continuous function which is nondecreasing with respect to the second variable and is positive on $\overline{\Omega} \times (0,\infty)$. We assume that $g \in C^1(0,+\infty)$ is positive and fulfils the assumption $(g1)$.

The main feature of this section is the presence of the convection term $|\nabla u|^a$ combined with the potential K that may be negative.

Problems of this type arise in the study of guided modes of an electromagnetic field in a nonlinear medium, satisfying adequate constitutive hypotheses. The following two examples illustrate situations of this type: (i) if $f(u) = u^3(1+\gamma u^2)^{-1}$ $(\gamma > 0)$ then problem (22) describes the variation of the dielectric constant of gas vapors where a laser beam propagates (see [31, 32]); (ii) nonlinearities of the type $f(u) = (1-e^{-\gamma u^2})u$ arise in the context of laser beams in plasmas (see [33]). If $f(u) = e^{u/(1+\varepsilon u)}$ $(\varepsilon > 0)$ then the corresponding equation describes the temperature dependence of the reaction rate for exothermic reactions obeying the simple Arrhenius rate law in circumstances in which the heat flow is purely conductive (see [5, 34]). In this context the parameter ε is a dimensionless ambient temperature and the parameter λ is a dimensionless heat evolution rate. The corresponding equation

$$-\Delta u = g(u) + \lambda|\nabla u|^p + \mu e^{u/(1+\varepsilon u)} \qquad \text{in } \Omega$$

represents heat balance with reactant consumption ignored, where u is a dimensionless temperature excess. The Dirichlet boundary condition $u = 0$ on $\partial\Omega$ is an isothermal condition and, in this case, it describes the exchange of heat at the surface of the reactant by Newtonian cooling.

Our general setting includes some simple prototype models from boundary-layer theory of viscous fluids (see [35]). If $\lambda = 0$ and $f \equiv 0$, (22) is called the Lane-Emden-Fowler equation. Problems of this type, as well as the associated evolution equations, describe naturally certain physical phenomena. For example, super-diffusivity equations of this type have been proposed by de Gennes [17] as a model for long range Van der Waals interactions in thin films spreading on solid surfaces. This equation also appears in the study of cellular automata and interacting particle systems with self-organized criticality (see [9]), as well as to describe the flow over an impermeable plate (see [7, 8]). We also point out that, due to the meaning of the unknowns (concentrations, populations, etc.), only the positive solutions are relevant in most cases.

As remarked in [11, 25], the requirement that the nonlinearity grows at most quadratically in $|\nabla u|$ is natural in order to apply the maximum principle.

In our first result we assume $\lambda = -1$ and $K < 0$ in Ω. In this case, the existence of a solution to (22) is close related to the decay rate around its singularity. In that sense, we prove the following nonexistence result.

Theorem 6. *Assume that $K < 0$ in $\overline{\Omega}$, $\mu = -1$, f satisfies $(f1) - (f2)$ and g verify $(g1)$. If $\int_0^1 g(s)ds = +\infty$, then problem (22) has no classical solutions.*

We now assume that $\int_0^1 g(s)ds < +\infty$, that is, g fulfils $(g2)$. We show in this case that (22) has at least one solution provided $\mu > 0$ is large enough.

Theorem 7. *Assume that $K < 0$ in $\overline{\Omega}$, $\mu = -1$, f satisfies $(f1) - (f2)$ and g verify $(g1) - (g2)$.*
Then, there exists $\mu^* > 0$ such that (22) has at least one classical solution if $\mu > \mu^*$ and no solution exists if $\mu < \mu^*$.

The results are different if $K > 0$ in $\overline{\Omega}$. We first consider $\lambda = 1$ and $0 < a \le 1$. In this case the study of existence is close related to the asymptotic behavior of the nonlinear term $f(x,u)$. To this aim we need the following assumptions on f :

$(f3)$ there exists $c > 0$ such that $f(x,s) \ge cs$ for all $(x,s) \in \overline{\Omega} \times [0,\infty)$;

$(f4)$ the mapping $(0,\infty) \ni s \longmapsto \dfrac{f(x,s)}{s}$ is nondecreasing for all $x \in \overline{\Omega}$,

Theorem 8. *Assume that $K < 0$ in $\overline{\Omega}$, $\lambda = 1$ and $0 < a \le 1$. Then the following properties hold true.*

(i) *If f satisfies either $(f3)$ or $(f4)$, then there exists $\mu^* > 0$ such that problem (22) has at least one classical solution for $\mu < \mu^*$ and no solutions exist if $\mu > \mu^*$.*

(ii) *If $0 < a < 1$ and f satisfies $(f1) - (f2)$, then problem (22) has at least one solution for all $\mu \ge 0$.*

We now analyze the case $\mu = 1$. Our framework is related to the sublinear case, described by assumptions $(f1)$ and $(f2)$.

Theorem 9. *Assume that $K < 0$ in $\overline{\Omega}$, $\mu = 1$ and f satisfies assumptions $(f1)$ and $(f2)$. Then the following properties hold true.*

(i) *If $0 < a < 1$, then problem (22) has at least one classical solution for all $\lambda \in \mathbb{R}$.*

(ii) *If $1 \leq a \leq 2$, then there exists $\lambda^* \in (0,\infty]$ such that problem (22) has at least one classical solution for $-\infty < \lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$. Moreover, if $1 < a \leq 2$, then λ^* is finite.*

Related to the above result we raise the following **open problem:** if $K \equiv 1$, $a = 1$ and $\mu = 1$, is λ^* a finite number?

Theorem 9 shows the importance of the convection term $\lambda|\nabla u|^a$ in the singular problem (22). Indeed, according to Theorem 1 and for any $\mu > 0$, the boundary value problem

$$\begin{cases} -\Delta u = u^{-\alpha} + \lambda|\nabla u|^a + \mu u^\beta & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{23}$$

has a unique solution, provided that $\lambda = 0$ and α, $\beta \in (0,1)$. Theorem 9 shows that if λ is not necessarily 0, then the following situations may occur : (i) problem (23) has solutions if $a \in (0,1)$ and for all $\lambda \in \mathbb{R}$; (ii) if $a \in (1,2)$ then there exists $\lambda^* > 0$ such that problem (23) has a solution for any $-\infty < \lambda < \lambda^*$ and no solution exists if $\lambda > \lambda^*$.

Sketch of the proof. (i) We shall discuss separately the cases $\lambda > 0$ and $\lambda \leq 0$.

Case $\lambda > 0$. Since f is sublinear, there exists $\zeta \in C^2(\overline{\Omega})$ such that

$$\begin{cases} -\Delta\zeta = f(x,\zeta) & \text{in } \Omega, \\ \zeta > 0 & \text{in } \Omega, \\ \zeta = 0 & \text{on } \partial\Omega. \end{cases} \tag{24}$$

Obviously, ζ is a sub-solution for (22) since $\lambda \geq 0$. The main point is to find a super-solution $\overline{u}_\lambda$ of (22) such that $\zeta \leq \overline{u}_\lambda$ in Ω. By the results in [3], there exists $H : (0,\eta] \to [0,\infty)$ such that

$$\begin{cases} H''(t) = -g(H(t)), & \text{for all } 0 < t < \eta, \\ H(0) = 0, \\ H > 0 \quad \text{in } (0,\eta]. \end{cases} \tag{25}$$

Since H is concave, there exists $H'(0+) \in (0,+\infty]$. By taking $\eta > 0$ small enough, we can assume that $H' > 0$ in $(0,\eta]$, so H is increasing on $[0,\eta]$. We are looking for a super-solution of the form $\overline{u}_\lambda = MH(c\varphi_1)$, where $M, c > 0$ and $c\varphi_1 \leq \eta$ in Ω. Since $0 < a < 1$, we can prove that $\underline{u}_\lambda$ defined above is a super-solution of (22), provided $M > 1$ is large enough.

Case $\lambda \leq 0$. We fix $\nu > 0$ and let $u_\nu \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of (22) with $\lambda = \nu$. Then u_ν is a super-solution of (22) for all $\lambda \leq 0$. Set $m := \inf_{(x,s)\in\overline{\Omega}\times(0,\infty)} \Big(g(s) + f(x,s)\Big)$. Since $\lim_{s\searrow 0} g(s) = +\infty$ and the mapping $(0,\infty) \ni s \longmapsto \min_{x\in\overline{\Omega}} f(x,s)$ is positive and nondecreasing, we deduce that m is positive. Consider the problem

$$\begin{cases} -\Delta v = m + \lambda |\nabla v|^a & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{26}$$

If $\lambda = 0$, the existence of a solution to (26) is clearly understood. Assume that $\lambda < 0$. Then, 0 is a sub-solution of (26) while $C\varphi_1$ is a super-solution, for $C > 0$ large enough. Hence, (26) has at least one solution $v \in C^2(\Omega) \cap C(\overline{\Omega})$ and $v > 0$ in Ω. It only remains to remark that v is sub-solution of (22) and $-\Delta v \le m \le -\Delta u_\nu$ in Ω, which gives $v \le u_\nu$ in Ω. Again by sub- super-solution method we conclude that (22) has at least one classical solution $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$.
(ii) The proof follows the same steps as above. The main difference is that in this case we are able to show that $\overline{u}_\lambda = MH(c\varphi_1)$ is a super-solution of (22) only for small values of $\lambda > 0$ since $1 \le a \le 2$. In order to prove the nonexistence of a classical solution to (22) we use the following result which is due to Alaa and Pierre (see [1]).

Lemma 3. (see [1]). *If $a > 1$, then there exists a real number $\bar{\sigma} > 0$ such that the problem*

$$\begin{cases} -\Delta v \ge |\nabla u|^a + \sigma & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{27}$$

has no solutions for $\sigma > \bar{\sigma}$.

We give in what follows a complete description in the special case $f \equiv 1$ and $a = 2$. More precisely, we consider the problem

$$\begin{cases} -\Delta u = g(u) + \lambda |\nabla u|^2 + \mu & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

A key role in this case will be played by the asymptotic behavior of the singular term g. In the statement of the next result we remark some similarities with Theorem 5. Let $\lim_{s\to\infty} g(s) = a \in [0,\infty)$.

Theorem 10. *The following properties hold true.*

(i) *Problem* (28) *has solution if and only if $\lambda(a+\mu) < \lambda_1$.*

(ii) *Assume $\mu > 0$ is fixed, g is decreasing and let $\lambda^* := \lambda_1/(a+\mu)$. Then problem* (28) *has a unique solution u_λ for all $\lambda < \lambda^*$ and the sequence $(u_\lambda)_{\lambda<\lambda^*}$ is increasing with respect to λ. Moreover, if* $\limsup_{s\searrow 0} s^\alpha g(s) < +\infty$, *for some $\alpha \in (0,1)$, then the sequence of solutions $(u_\lambda)_{0<\lambda<\lambda^*}$ has the following properties:*

(ii1) *for all $0 < \lambda < \lambda^*$ there exist two positive constants c_1, c_2 depending on λ such that $c_1\, d(x) \le u_\lambda \le c_2\, d(x)$ in Ω;*

(ii2) $u_\lambda \in C^{1,1-\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;

(ii3) $u_\lambda \longrightarrow +\infty$ *as $\lambda \nearrow \lambda^*$, uniformly on compact subsets of Ω.*

We refer to [21] and [22] for complete proofs and further details.

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