List of scientific or artistic achievements which present a major contribution to the development of a specific discipline

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Information contained herein should clearly refer to two different periods, i.e., the period prior to the award of the PhD degree and the period between the conferment of the PhD degree and the award of the post-doctoral degree of doctor habilitated.

1 Information on scientific and artistic achievements set out in art. 219 para 1. point 2 of the Act

1.1 Cycle of scientific articles related thematically

In the case of co-authored works, it is recommended that the applicant and his/her collaborators present a declaration of their substantive (expressed NOT as a percentage) contribution to every work [e.g. author of a research hypothesis, research initiator, performed specific research (e.g. performed specific experiments, designed and compiled questionnaires etc.), performed result analysis, prepared a manuscript of an article and other]. The author's contribution, including the applicant's contribution, should be described in detail so as to make it possible to precisely assess his/her contribution and role in the creation of each of the works.

The following twelve articles have been prepared and published after obtaining my Ph.D. in June 1995. These papers focus on the theme:

Local and nonlocal problems in nonlinear analysis

[A1] V. Ambrosio, V.D. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, *Journal de Mathématiques Pures et Appliquées* **142** (2020), 101-145.

In September 2016, on the occasion of a conference organized in Perugia, G. Mingione [1] proposed a new research direction in relationship with the "double phase" problems, introduced by J. Ball [6] and V. Zhikov [38]. Such models arise in nonlinear elasticity and composite materials. G. Mingione proposed to extend the qualitative and quantitative analysis of double phase problems to the cases of *anisotropic* double phase problems and *nonlocal* double phase problems. In this sense, he suggested to establish not only fine regularity properties for the local case (as it is also developed in papers [A3] and [A4]) but also to extend the double phase framework to the anisotropic setting, as it is given in this paper.

We study concentration and multiplicity properties of solutions for fractional problems with unbalanced growth. Such patterns are connected with the analysis of nonlinear problems and stationary waves for models arising in mathematical physics (fractional quantum mechanics in the study of particles on stochastic fields, fractional superdiffusion, fractional white-noise limit). The main result in this paper establishes both the multiplicity and concentration of positive solutions. This is done in the case of small perturbations of the parameter associated with the potential that satisfies local conditions in the sense of M. del Pino and P. Felmer [11]. The multiplicity property is expressed in terms of the Ljusternik-Schnirelmann category, while the concentration is developed in connection with the set where the potential attains its minimum.

We exchanged many ideas on various versions of this paper starting with our meeting at the conference in Urbino [2] and we continued our discussions on sketches of proofs during numerous on-line conversations. We have equally contributed to the research developed in this paper and I assess my contribution as 50%.

[A2] S. Chen, V.D. Rădulescu, X. Tang, L. Wen, Planar Kirchhoff equations with critical exponential growth and trapping potential, *Mathematische Zeitschrift* **302** (2022), 1061-1089.

This research was initiated during my visit at the Central South University in December 2019. The final version has been prepared in November 2021, when L. Wen visited me at the University of Craiova. Problems of this type have been intensively studied after the seminal contributions of J.-L. Lions [21] and S. Pohozaev [28]. The *critical case* for the entire Euclidean space in the *two-dimensional* setting is less studied and this corresponds to the *exponential growth* of Trudinger-Moser type for the reaction. As established by J. Moser [27] and N. Trudinger [37], this kind of nonlinearity has the *maximal growth* that can be treated variationally in $H^1(\mathbb{R}^2)$. In our case, due to the lack of compactness, it is more difficult to rule out the concentration phenomena and the vanishing phenomena of Cerami sequences. Another feature of this paper consists in the presence of a *Rabinowitz-type trapping potential* in the source term, in the sense defined by P. Rabinowitz [30].

By developing some new analytical approaches and techniques, we prove the existence of nontrivial solutions and least energy solutions. Moreover, *without any monotonicity conditions* on the nonlinearity f governing the reaction, we give the mountain pass characterization of the least energy solution by constructing a fine path. In the *nonlocal* case considered in this paper, we adapted an idea inspired by the method introduced by L. Jeanjean and K. Tanaka [18] in the framework of nonlinear *scalar field* equations. Another contribution in this paper is that we remove the common restriction on $\liminf_{t\to+\infty} tf(t)/e^{\alpha_0 t^2}$, which is crucial in the literature to overcome the loss of compactness caused by the critical exponential nonlinearity.

We had numerous on-line meetings in order to elaborate the final version of this paper. All authors were involved in writing and proofreading. I assess my contribution to this paper as 30%.

[A3] Y. Fang, V.D. Rădulescu, C. Zhang, Regularity of solutions to degenerate fully nonlinear elliptic equations with variable exponent, *Bulletin of the London Mathematical Society* **53** (2021), 1863-1878.

The study of regularity properties of solutions for problems with nonstandard growth has a central interest in nonlinear analysis. In [29], I co-organized with G. Mingione a special issue devoted to recent developments to this field. Some relevant advances in regularity theory are synthesized in our joint Highly Cited paper [25] (cf. Web of Science).

This paper complements the regularity theory for double phase problems and its purpose is twofold: (i) the analysis is developed in the anisotropic case, corresponding to the presence of several variable exponents; (ii) the regularity of solutions is established in the framework of degenerate fully nonlinear elliptic equations. This research was initiated during my visit at the Harbin Institute of Technology in December 2019 and is motivated by the authors' common interest in the study of various classes of double phase equations. The problem studied in this paper features an inhomogeneous degenerate term modelled on the double phase integrand with variable exponents

$$H(x,\xi) = |\xi|^{p(x)} + a(x)|\xi|^{q(x)}, \quad q(x) \ge p(x) > 1, \text{ for all } x \in \Omega,$$

where $\Omega \subset \mathbb{R}^n$ $(n \ge 2)$ is a bounded domain. This hypothesis on the integrand extends to the *anisotropic* setting the condition introduced in the basic papers of P. Marcellini [23, 24] in the *isotropic* case as a prototype of energy functionals satisfying nonstandard growth conditions of (p,q)-type.

This paper considers the fully nonlinear elliptic equations with variable exponent nonhomogeneous degeneracy of the form

$$\left[|Du|^{p(x)} + a(x)|Du|^{q(x)}\right] F(D^2u) = f(x) \text{ in } \Omega,$$

where $F(\cdot)$ is a uniformly (λ, Λ)-elliptic operator. By making use of geometric tangential methods and combing a refined improvement-of-flatness approach with compactness and scaling techniques, we establish a sharp regularity result, in the sense that the viscosity solutions of this problem are locally of class $C^{1,\alpha}(\Omega)$. This property extends to the anisotropic case and complements a recent result of C. De Filippis [10] who proved the local Hölder continuity of the gradient of viscosity solutions to degenerate equations of the type

$$[|Dw|^p + a(x)|Dw|^q]\, F(D^2w) = f(x) \in L^\infty(\Omega) \text{ in } \Omega.$$

I took part, together with my co-authors, in all activities of this work. I have participated in devising the general plan, organizing the structure of the paper, actually proving and writing down the main results. I assess my contribution as 1/3.

[A4] Y. Fang, V.D. Rădulescu, C. Zhang, X. Zhang, Gradient estimates for multi-phase problems in Campanato spaces, *Indiana University Mathematics Journal* **71** (2022), 1079-1099.

The idea of this paper is the following. Let Ω be a domain in $\mathbb{R}^N$ $(N \ge 2)$ and consider the system

$$\operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F) \text{ in } \Omega.$$

If $p = 2$, then the system is linear and it follows by the Calderón-Zygmund theory that if $F \in [L^q_{loc}(\Omega)]^m$ for some $q > p$, then $Du \in [L^q_{loc}(\Omega)]^m$. This result was extended to the nonlinear degenerate case by E. DiBenedetto and J. Manfredi [12] who obtained gradient estimates in $[BMO_{loc}(\Omega)]^{Nm}$. In this paper, we are concerned with the case of *multi-phase* problems. The key features of this paper are the following:
(i) the problem under consideration is characterized by the fact that both ellipticity and growth switch between three different types of polynomial according to the position, which describes a feature of *strongly anisotropic materials*;
(ii) the results obtained in this paper are different from the BMO-type estimates for the usual p-Laplacian equation due to E. DiBenedetto and J. Manfredi. The new gradient estimate obtained in this paper is deduced in *Campanato spaces.* These function spaces are due to S. Campanato [9] and they are Banach spaces which extend the notion of functions of bounded mean oscillation due to F. John and L. Nirenberg [19], allowing to describe situations where the oscillation of the function in a ball is proportional to some power of the radius other than the dimension.

Several versions of the paper have been written during the preparation of this work. The final version was obtained after more than one year and I contributed equally with my co-authors in designing the main steps of the proofs, developing the main abstract tools, typesetting and discussing several versions of the paper. I assess my contribution as 25%.

[A5] L. Jeanjean, V.D. Rădulescu, Nonhomogeneous quasilinear elliptic problems: linear and sublinear cases, *Journal d'Analyse Mathématique* **146** (2022), no. 1, 327-350.

In March 2013, I visited the University of Franche-Comté at the invitation of L. Jeanjean. Before arriving in Besançon, I read the papers [33, 34, 35] by C. Stuart, who was the Ph.D. advisor of L. Jeanjean at EPFL. I was very interested in Stuart's papers and I proposed to dr. Jeanjean to develop a joint collaboration in order to extend the contributions of C. Stuart. The original idea starts from the linear case discussed in Example 1 of Brezis [7, pp. 291-293], which can be extended to suitable linear perturbations of the Laplace operator. Stuart [33] generalized this result by replacing the usual Laplace operator with a very general *nonhomogeneous differential operator* of the type

$$\text{div}\left[\gamma\left(\frac{u^2+|\nabla u|^2}{2}\right)\right],$$

where γ is a suitable continuous function satisfying hypotheses (g1)–(g3) in [33]. However, the right-hand side of problem (1.2) in [33, p. 328] remains *linear*, as in the case discussed by H. Brezis. Our idea was to substantially extend the research developed by C. Stuart in a *nonlinear* setting.

In this paper, we analyze two distinct situations. We first discuss the *sublinear case* and our analysis is done under more relaxed assumptions on the differential operator as those introduced by C. Stuart. In this framework, we establish that if the reaction has a sublinear decay at $+\infty$ and at most a linear growth near the origin, then the problem has solutions in the case of *high perturbations* of the reaction. At the same time, no solutions exist for *small perturbations* of the right-hand side. Next, we discuss the case where the nonlinearity is no longer linear, but it has a *linear growth*. Here, the main challenge is to establish the existence of two non-negative solutions. A fine argument in the proof relies on the notion of *localizing the Palais-Smale sequence*.

The paper had several versions and I contributed actively in all of them, starting with the formulation of the problem, designating the main difficulties, improving the proofs, typesetting and discussing the final version. Throughout this period, we received an enthusiastic support from C. Stuart who sent us many useful comments. The final version was read by H. Brezis who suggested several improvements. I assess my contribution as 50%.

[A6] C. Ji, V.D. Rădulescu, Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation, *Calculus of Variations and Partial Differential Equations* **59** (2020), no. 4, Paper No. 115, 28 pp.

In December 2018, I met C. Ji on the occasion of a conference organized in Heilongjiang Institute of Technology in Harbin. We started an active collaboration and we were both interested in problems dealing with the magnetic Laplace operator, as those studied by M. Esteban and P.L. Lions [14]. Since we had a background in the qualitative analysis of solutions for various classes of Schrödinger-type equations, we decided to study together the case of nonlinear *magnetic Schrödinger equations* with *lack of compactness*. In the absence of the magnetic field, there is an impressive literature concerning the qualitative analysis of bound state solutions, starting with the pioneering paper by A. Floer and A. Weinstein [15] in the one-dimensional case. The first concentration properties in the absence of the magnetic field and in a subcritical setting are due to M. Del Pino and P. Felmer [11]. In this paper, we are interested in multiplicity and concentration properties of solutions in the *magnetic* case. Another feature of this paper is that the reaction is assumed yo be *only continuous*, so the arguments developed by C. Alves, G. Figueiredo and M. Furtado [5] in the smooth setting fail. A useful compactness result via translations is established in Proposition 4.1 of our paper for functions in a suitable Nehari manifold. Finally, by combining variational methods, Nehari analysis, and the Ljusternik-Schnirelmann theory, we prove multiplicity and concentration properties of solutions for small values of the positive parameter that describes the magnetic differential operator. Other arguments used in the proof are the diamagnetic inequality of E. Lieb and M. Loss [20, Theorem 7.21] and the penalization approach introduced by M. Del Pino and P. Felmer [11].

I took part, together with my co-author, in all activities on this work. We established together the general plan and the structure of the paper and we coordinated about the main steps of the proof. Several on-line meetings were very useful to produce the final version of this paper. I assess my contribution as 50%.

[A7] D. Qin, V.D. Rădulescu, X. Tang, Ground states and geometrically distinct solutions for periodic Choquard-Pekar equations, *Journal of Differential Equations* **275** (2021), 652-683.

In December 2018 I was invited by X. Tang to give a talk at the Central South University in Changsha. On this occasion I met D. Qin and we started to discuss together research problems of common interest. A few months before my visit in China, I was invited by E. Mitidieri to collaborate with P. Pucci for organizing a special issue of *Nonlinear Analysis* devoted to the study of *nonlocal* problems. I refer to [3] for this volume. Combining the opportunity to give a talk in Changsha with my recent work for this special issue, I decided to collaborate with D. Qin and X. Tang for the study of a particular class of nonlocal equations. We decided to develop the analysis of ground state solutions to the *Choquard-Pekar equation* in a *periodic non-autonomous* setting. A solid motivation in applied sciences is that the proposed subject is in close connection with models coupling the Schrödinger equation of quantum physics together with non-relativistic Newtonian gravity.

In this paper, we first obtain a sufficient condition for the existence of *infinitely many pairs* of geometrically distinct solutions. A key feature of this result is that we do not need the Cauchy-Schwarz type hypothesis used by N. Ackerman [4] order to establish the existence of infinitely many solutions. Instead, we use a weaker assumption, which also weakens the standard Ambrosetti-Rabinowitz growth condition. The proof is based on topological and variational arguments. We first construct the linking structure of the associated energy functional, which enables us to find the corresponding Cerami sequences. Next, we apply a concentration-compactness argument in order to obtain the existence of nontrivial solutions. The proof of multiplicity results is carried out via the Ljusternik-Schnirelmann theory and deformation arguments.

This is a Highly Cited Paper, according with Web of Science. The final version of this paper was done in December 2019, when I visited again the Central South University. In the meantime, we collaborated intensively to develop various parts of this paper. This was done both by on-line discussions and by email messages. I assess my contribution to this paper as 1/3.

[A8] N.S. Papageorgiou, V.D. Rădulescu, D. Repovš, Positive solutions for nonlinear Neumann problems with singular terms and convection, *Journal de Mathématiques Pures et Appliquées* **136** (2020), 1-21.

My interest in the study of singular problems started almost 20 years ago. In 2008, I co-authored a book [16] on this subject, while in 2009 and 2015 I prepared special issues of *J. Math. Anal. Appl.* and *Nonlinear Analysis* devoted to the analysis of singular and degenerate phenomena in nonlinear analysis and mathematical physics, cf. [31, 32]. This paper is the fruit of my constant interest to singular elliptic equations. In [13] we studied the qualitative analysis of positive solutions for the Lane-Emden-Fowler equation with convection and singular potential. This study has been developed in the framework of a Dirichlet boundary condition. In coordination with my co-authors, we decided to extend these results to the case of Neumann elliptic equations with *convection* and *singular nonlinearity.* We were interested in the study of competing effects of a singular reaction and a suitable perturbation, which also depends on the gradient (convection term). The presence of the gradient in the perturbation excludes from consideration a variational approach. The particular structure of the problem implies a different approach, which relies essentially on the Leray-Schauder alternative principle, cf. A. Granas and J. Dugundji [17, p. 124]. Using also suitable truncation and comparison techniques combined with nonlinear regularity theory, we obtain the existence of positive smooth solutions.

We exchanged ideas and sketches of proofs during our personal meetings in 2016–2018 but also on the occasion of numerous call phones or on-line meetings. We were also involved in writing and proofreading. I assess my contribution as 40%.

[A9] N.S. Papageorgiou, A. Pudelko, V.D. Rădulescu, Non-autonomous (p, q)–equations with unbalanced growth, *Mathematische Annalen* (2022). https://doi.org/10.1007/s00208-022-02381-0

Following the new research directions proposed by G. Mingione in [1] and after our close collaboration for the coordination of the special issue [26] devoted to variational problems with nonstandard growth conditions and nonuniformly elliptic operators, we have developed in this paper an analysis of unbalanced double phase problems in the *non-autonoumous case*. The problem studied in this paper is motivated by numerous models arising in mathematical physics, for instance, the approach of the Born-Infeld equation that appears in electromagnetism.

We are concerned with a nonlinear Dirichlet problem with a source term described by a differential operator with a power-type nonhomogeneous term. At the same time, the potential that describes the differential operator satisfies general regularity assumptions and it belongs to the p-Muckenhoupt class. Accordingly, the thorough spectral and qualitative analysis contained in this paper are developed in Musielak-Orlicz-Sobolev spaces. The corresponding energy functional is a *non-autonomous variational integral* that satisfies nonstandard growth conditions of (p, q)-type, following the terminology introduced in the pioneering papers of Marcellini [23, 24]. Our analysis covers both the *coercive resonant* case and the *noncoercive (asymptotic resonance or nonresonance)* case. An important role in this analysis is played by the spectral analysis of the weighted Laplace-type operator. The main results of this paper establish existence and multiplicity properties for non-autonomous problems problems in which there is resonance asymptotically at $\pm\infty$.

The study initiated in November 2019 when N. Papageorgiou visited me in Craiova and it continued during several on-line meetings. The final version was elaborated in October 2021 when all three authors met in Kraków and discussed all the arguments developed in this paper. I contributed actively in the description of the problem, organization of the paper, main steps of proofs, and the reading and corrections of the final version. I assess my contribution to this paper as 40%.

[A10] M. Yang, V.D. Rădulescu, X. Zhou, Critical Stein-Weiss elliptic systems: symmetry, regularity and asymptotic properties of solutions, *Calculus of Variations and Partial Differential Equations* **61** (2022), issue 3, Article 109, 38 pp.

The Hardy-Littlewood inequality is a basic tool for the treatment of singular elliptic problems. A pertinent weighted interpolation inequality was established in higher dimensions by L. Caffarelli, R. Kohn and L. Nirenberg [8]. In the special case of power weights the optimal result is due to E. Stein and G. Weiss [36], who established a *general weighted Hardy-Littlewood-Sobolev inequality.* In December 2019, I was invited by M. Yang to give a talk at the Zhejiang Normal University. In our discussions with X. Zhou, we decided to analyze the role of the Stein-Weiss inequality in the understanding of some qualitative and asymptotic properties of solutions to a class of weighted local-nonlocal systems with critical exponents. We started our approach after observing that the best constant of a certain differential inequality is related to a nonlocal Hartree equation, which is a special case of the weighted Choquard equation with Stein-Weiss potential.

We are concerned in this paper with three nonlocal elliptic systems with *weighted Stein-Weiss convolution* part. The first one does not have a variational structure and we obtain symmetry properties for the positive solutions via moving plane arguments, which can be easily applied to more complicated equations without maximum principles. Next, we are interested in a class of nonlocal systems with variational structure and Stein-Weiss potential. We consider the subcritical and critical cases and we establish several regularity, symmetry and asymptotic properties of solutions. Finally, we describe a class of Hamiltonian systems with Stein-Weiss potential and we obtain the symmetry of positive solutions. The proofs use several refined tools, such as a Pohozaev-type identity, singular integral analysis, and iterative arguments.

The paper opens perspectives in the analysis of elliptic systems with nonlocal structure, singular potential, and weighted Stein-Weiss convolution part. The analysis can be extended to problems without a variational structure. We discussed several versions of the paper and the authors had equal contributions in designing the final version of this paper. I assess my contribution as 1/3.

[A11] J. Zhang, W. Zhang, V.D. Rădulescu, Double phase problems with competing potentials: concentration and multiplication of ground states, *Mathematische Zeitschrift* **301** (2022), 4037-4078.

I met for the first time J. Zhang and W. Zhang when I visited the Central South University in Changsha in December 2018. They attended my talk and after that we started to discuss several problems of common interest. In December 2019, I visited again Changsha and they invited me to gave a talk at the Hunan University of Technology and Business. On this occasion, I invited them to Craiova and we started to discuss about concentrating properties of solutions. In such a way we established the general structure of this paper and we started to create the *interface* between double phase equations, multiplication and concentration of solutions, and associated combined effects.

The problem studied in this paper combines the multiple effects generated by two variable potentials (in the absorption and in the reaction terms), which imply more complex phenomena to locate the concentration positions. In such a way, the main concentration phenomenon established in this paper creates a bridge between the *global maximum* point of the solution versus the global maximum of the reaction potential and the *global minimum* of the absorption potential. Since the problem is considered in the entire Euclidean space, the corresponding Palais-Smale sequences do not have the compactness property. The notion of *Ljusternik-Schnirelmann category* plays an important role in the multiplicity properties established in this paper. The lack of compactness is overcome by using the Lions compactness lemma [22].

We had numerous exchanges of opinions concerning all the steps in the preparation of this paper. The final version was achieved in February 2022, during the one-year visit of my co-authors in Romania. We had equal parts in the elaboration of this paper, typesetting, final reading and corrections. I assess my contribution as 1/3.

[A12] S. Zeng, V.D. Rădulescu, P. Winkert, Double phase implicit obstacle problems with convection and multivalued mixed boundary value conditions, *SIAM Journal on Mathematical Analysis* **54** (2022), 1898-1926.

I started to study nonsmooth problems during the preparation of my Ph.D. thesis at the University of Craiova. A key role in the understanding of such problems is due to the late Prof. P.D. Panagiotopoulos, who created the theory of hemivariational inequalities and who introduced me to nonsmooth equations, in strong relationship with problems arising in civil engineering.

This paper has been generated by an incipient discussion that I had with S. Zeng in October 2019 in Kraków. In coordination with P. Winkert and in accordance with our common interest for double phase equations, we decided to study *implicit problems* in this new abstract setting. Our approach is nonsmooth and it involves a multivalued mixed boundary condition, as well as the presence of a nonlinear reaction depending on the gradient. The notion of Clarke subdifferential plays a key role in the understanding of the associated energy functional. Due to the *unbalanced structure* of the problem, the associated energy density changes its ellipticity and growth properties according to the point in the domain. The problem is driven by a nonhomogeneous differential operator with different isotropic growth, which generates a double phase associated energy. Due to these features, we are concerned in this paper with the combined effects of a nonstandard operator with unbalanced growth, a convection nonlinearity, three multivalued terms, and an implicit obstacle constraint. The proofs rely on the Kakutani–Ky Fan fixed point theorem for multivalued operators in combination with tools from nonsmooth analysis and theory of pseudomonotone operators.

We coordinated in an efficient manner for the final version of the paper and we had several fruitful on-line meetings. I assess my contribution to this paper as 1/3.

2 Information on scientific or artistic activity

2.1 Publications not including articles mentioned in 1.1

Articles prepared after the award of the PhD degree in June 1995 (selection)

[B1] V.D. Rădulescu, "Sur l'équation multigroupe stationnaire de la diffusion des neutrons", *C.R. Acad. Sci. Paris, Ser. I* **323** (1996), 765-768.

[B2] P.D. Panagiotopoulos, V.D. Rădulescu,"Perturbations of hemivariational inequalities with constraints and applications", *J. Global Optimiz.* **12** (1998), 285-297.

[B3] M. Fundos, P.D. Panagiotopoulos, V.D. Rădulescu, "Existence theorems of Hartmann-Stampacchia type for hemivariational inequalities and applications", *J. Global Optimiz.* **15** (1999), 41-54.

[B4] F. Cîrstea, V.D. Rădulescu, "Existence and uniqueness of positive solutions to a semilinear elliptic problem in $\mathbb{R}^N$", *J. Math. Anal. Appl.* **229** (1999), 417-425.

[B5] M. Bocea, P.D. Panagiotopoulos, V.D. Rădulescu, "A perturbation result for a double eigenvalue hemivariational inequality and applications", *J. Global Optimiz.* **14** (1999), 137-156.

[B6] M. Degiovanni, V.D. Rădulescu, "Perturbations of nonsmooth symmetric nonlinear eigenvalue problems", *C.R. Acad. Sci. Paris* **329** (1999), 281-286.

[B7] V.D. Rădulescu, "Perturbations of hemivariational inequalities with constraints", *Revue Roum. Math. Pures Appl.* **44** (1999), 455-461.

[B8] F. Gazzola, V.D. Rădulescu, "A nonsmooth critical point theory approach to some nonlinear elliptic equations in unbounded domains", *Differential and Integral Equations* **13** (2000), 47-60.

[B9] F. Cîrstea, V.D. Rădulescu, "Existence and nonexistence results for quasilinear problems with nonlinear boundary condition", *J. Math. Anal. Appl.* **244** (2000), 169-183.

[B10] M. Degiovanni, M. Marzocchi, V.D. Rădulescu, "Multiple solutions of hemivariational inequalities with area-type term", *Calculus of Variations and PDE* **10** (2000), 355-387.

[B11] F. Cîrstea, V.D. Rădulescu, "Multiple solutions of degenerate perturbed elliptic problems involving a subcritical Sobolev exponent", *Topol. Meth. Nonlin. Anal.* **15** (2000), 281-298.

[B12] F. Cîrstea, V.D. Rădulescu, "On a double bifurcation quasilinear problem arising in the study of anisotropic continuous media", *Proc. Edinburgh Math. Soc.* **44** (2001), 527-548.

[B13] V.D. Rădulescu, "Perturbations of eigenvalue problems with constraints for hemivariational inequalities", *From Convexity to Nonconvexity, volume dedicated to the memory of Prof. G. Fichera*,

Nonconvex Optim. Appl., 55, Kluwer Acad. Publ., Dordrecht, 2001 (R. Gilbert, P. Pardalos, Eds.), 243-253.

[B14] F. Cîrstea, D. Motreanu, V.D. Rădulescu, "Weak solutions of quasilinear problems with nonlinear boundary conditions", *Nonlinear Analysis, T.M.A.* **43** (2001), 623-636.

[B15] F. Cîrstea, V.D. Rădulescu, "Blow-up solutions for semilinear elliptic problems", *Nonlinear Analysis, T.M.A.* **48** (2002), 541-554.

[B16] F. Cîrstea, V.D. Rădulescu, "Existence and uniqueness of blow-up solutions for a class of logistic equations", *Commun. Contemp. Math.* **4** (2002), 559-586.

[B17] V.D. Rădulescu, D. Smets, M. Willem, "Hardy-Sobolev inequalities with remainder terms", *Topol. Meth. Nonlin. Anal.* **20** (2002), 145-149.

[B18] F. Cîrstea, V.D. Rădulescu, "Uniqueness of the blow-up boundary solution of logistic equations with absorption", *C. R. Acad. Sci. Paris, Ser. I* **335** (2002), 447-452.

[B19] F. Cîrstea, V.D. Rădulescu, "Entire solutions blowing-up at infinity for semilinear elliptic systems", *J. Math. Pures Appliquées* **81** (2002), 827-846.

[B20] F. Cîrstea, V.D. Rădulescu, "Solutions with boundary blow-up for a class of nonlinear elliptic problems", *Houston J. Math.* **29** (2003), 821-829.

[B21] V.D. Rădulescu, M. Willem, "Elliptic systems involving finite Radon measures", *Differential and Integral Equations* **16** (2003), 221-229.

[B22] F. Cîrstea, V.D. Rădulescu, "Asymptotics for the blow-up boundary solution of the logistic equation with absorption", *C. R. Acad. Sci. Paris, Ser. I* **336** (2003), 231-236.

[B23] M. Ghergu, V.D. Rădulescu, "Sublinear singular elliptic problems with two parameters", *J Differential Equations* **195** (2003), 520-536.

[B24] V.D. Rădulescu, D. Smets, "Critical singular problems on infinite cones", *Nonlinear Analysis, T.M.A.* **54** (2003), 1153-1164.

[B25] I. Ionescu, V.D. Rădulescu, "Nonlinear eigenvalue problems arising in earthquake initiation", *Adv. Differential Equations* **8** (2003), 769-786.

[B26] M. Ghergu, V.D. Rădulescu, "Bifurcation and asymptotics for the Lane-Emden-Fowler equation", *C. R. Acad. Sci. Paris, Ser. I* **337** (2003), 259-264.

[B27] M. Ghergu, V.D. Rădulescu, "Explosive solutions of semilinear elliptic systems with gradient term", *RACSAM Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **97** (2003), 437-445.

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Articles prepared prior to the award of the PhD degree in June 1995 (selection)

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[D2] P. Mironescu, V.D. Rădulescu, "A bifurcation problem associated to a convex, asymptotically linear function", *C.R. Acad. Sci. Paris, Ser. I* **316** (1993), 667-672.

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[D4] C. Lefter, V.D. Rădulescu, "On the Ginzburg-Landau energy with weight", *C.R. Acad. Sci. Paris, Ser. I* **319** (1994), 843-848.

[D5] V.D. Rădulescu, "Mountain pass type theorems for non-differentiable convex functions", *Revue Roum. Math. Pures Appl.* **39** (1994), 53-62.

[D6] P. Mironescu, V.D. Rădulescu, "On a duality theorem", *Studii Cerc. Mat.* **46** (1994), 393-396.

[D7] V.D. Rădulescu, "A Lusternik-Schnirelman type theorem for locally Lipschitz functionals with applications to multivalued periodic problems", *Proc. Japan Acad.* **71A** (1995), 164-167.

[D8] P. Mironescu, V.D. Rădulescu, "A multiplicity theorem for locally Lipschitz periodic functionals", *J. Math. Anal. Appl.* **195** (1995), 621-637.

[D9] V.D. Rădulescu, "Nontrivial solutions for a multivalued problem with strong resonance", *Glasgow Math. Journal* **38** (1996), 53-61.

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[D11] P. Mironescu, V.D. Rădulescu, "The study of a bifurcation problem associated to an asymptotically linear function", *Nonlinear Analysis, T.M.A.* **26** (1996), 857-875.

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[E2] V.D. Rădulescu, *Partial Differential Equations*, Craiova University Press, 1999.

[E3] D. Motreanu, V.D. Rădulescu, *Variational and Nonvariational Methods in Nonlinear Analysis and Boundary Value Problems*, Nonconvex Optimization and Its Applications, Vol. 67, Kluwer Academic Publishers, Dordrecht, 388 pp., 2003.

[E4] C. Niculescu, V.D. Rădulescu (Editors), *Mathematical Analysis and Applications: International Conference on Mathematical Analysis and Applications*, Craiova (Romania), 23-24 September 2005, AIP Conference Proceedings Volume 835, American Institute of Physics, 176 pp., 2006.

[E5] V.D. Rădulescu, *Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations*, Contemporary Mathematics and Its Applications, vol. 6, Hindawi Publ. Corp., 210 pp., 2008.

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[E7] T.-L. Rădulescu, V.D. Rădulescu, T. Andreescu, *Problems in Real Analysis: Advanced Calculus on the Real Axis*, Springer, New York, xx+452 pp., 2009.

[E8] A. Kristály, V.D. Rădulescu, C. Varga, *Variational Principles in Mathematical Physics, Geometry and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral Problems*, Encyclopedia of Mathematics (No. 136), Cambridge University Press, Cambridge, 384 pp., 2010.

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[E15] V.D. Rădulescu, A. Sequeira, V. Solonnikov (Editors), *Recent Advances in PDEs and Applications*, Contemporary Mathematics Series, American Mathematical Society, Vol. 666, 404 pp., 2016.

[E16] G. Molica Bisci, V.D. Rădulescu, R. Servadei, *Variational Methods for Nonlocal Fractional Problems*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, Vol. 162, 400 pp., 2016.

[E17] G. Kassay, V.D. Rădulescu, *Equilibrium Problems and Applications*, Mathematics in Science and Engineering, Academic Press, Elsevier, Oxford, 440 pp., 2018.

[E18] N. Papageorgiou, V.D. Rădulescu, D. Repovš, *Nonlinear Analysis–Theory and Methods*, Springer Monographs in Mathematics, Springer-Verlag, Cham, 577 pp., 2019.

2.2 Presentations given at scientific conferences

Information on presentations given at national or international scientific or arts conferences, including a list of lectures delivered upon invitation and plenary lectures

2.2.1 Invited lecture series (selection)

1. *Comparison principles and applications to nonlinear elliptic equations I*, Critical Point Theory and Applications Summer School Cluj-Napoca, Romania, 9-13 July 2007 (talk given on July 9)

2. *Comparison principles and applications to nonlinear elliptic equations II*, Critical Point Theory and Applications Summer School Cluj-Napoca, Romania, 9-13 July 2007 (talk given on July 10)

3. *An overview of some research problems in applied nonlinear analysis*, Meeting on Mathematics, University of Reggio Calabria, 24 January 2013

4. *Combined effects in nonlinear elliptic problems*, Seminars on Nonlinear Analysis, Reggio Calabria, 22 October 2013

5. *Subcritical Emden-Fowler equations*, International School on Computational Commutative Algebra and Algebraic Geometry, Messina, 21-26 October 2013

6. *A qualitative property for a class on elliptic problems on the Sierpinski gasket*, Mini-symposium Recent Trends in Nonlinear Analysis and its Applications, 8th European Conference on Elliptic and Parabolic Problems, Gaeta, 26-30 May 2014

7. *Variational analysis on fractals*, Recent Trends on Nonlinear Phenomena, Reggio Calabria, 5-7 November 2014

8. *Variational principles for equilibrium problems and applications to inequality problems and fixed point theory*, Equilibrium and Optimization Methodology in Finance and Economics, King Saud University, Riyadh, Saudi Arabia, 9-11 November 2015

9. *Low- and high-energy solutions of nonlinear elliptic problems*, Centro Lamberto Cesari, Università degli Studi di Perugia, 16 September 2016

10. *Alcuni modelli matematici nelle scienze applicate: singolarità, frattali e fluidi non-newtoniani*, Accademia delle Scienze dell'Umbria, 10 January 2018

11. *Double phase problems with variable growth and Singular phenomena in nonlinear elliptic equations*, BiUrb Recent advances in variational methods, Università di Urbino Carlo Bo, 30-31 May 2019

12. *Anisotropic problems with multi-phase and mixed regime*, Webinar Series on Nonlinear Differential Problems, Progetti di Rilevante Interesse Nazionale, University of Messina, 9 April 2021

13. *Singular elliptic problems and beyond*, Webinar Series on Nonlinear Differential Problems, Progetti di Rilevante Interesse Nazionale, University of Messina, 23 April 2021

14. *Some striking results in the analysis of singular elliptic problems*, Workshop on PDEs and Applications, organized in the framework of the Research Project GNAMPA 2020 "Equazioni alle derivate parziali: problemi e modelli", University of Perugia, 18 June 2021

15. *Stein-Weiss inequalities, Choquard problems, and beyond*, International Forum on Nonlinear Analysis and Its Applications, Civil Aviation University of China, Tianjin, 6 November 2021

16. *Some mathematical models in applied sciences: singularities, fractals and non-Newtonian fluids*, 6th International USERN Congress: Science to Society, organized by the Universal Scientific Education and Research Network, Istanbul, 6-13 November 2021

17. *Isotropic and anisotropic equations with unbalanced growth*, International Workshop Recent Developments in PDEs and Applications, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia, 16-17 March 2022

18. *Hardy-Littlewood-Sobolev, Stein-Weiss, and beyond*, International Workshop "Advances in Nonlinear Analysis and PDEs", Shandong University of Technology and Business, Qingdao, 28-29 May 2022

19. *Anisotropic double phase problems with mixed regime in the applied sciences*, International Workshop Mathematical Modeling of Self-Organizations in Medicine, Biology and Ecology, Palermo, 29 May - 3 June 2022

20. *Ambrosetti-Prodi double phase problems with Robin boundary condition*, International Workshop on Nonlinear Analysis and PDEs, East China University of Science and Technology, Shanghai, 27 August 2022

2.2.2 Invited talks (selection)

1. *Bifurcation and asymptotics for Dirichlet boundary value problems with singular nonlinearity*, Fifth European Conference on Elliptic and Parabolic Problems: A special tribute to the work of Haim Brezis, Gaeta, 30 May - 3 June 2004

2. *Combined effects of singular nonlinearities and convection terms in the generalized Lane-Emden-Fowler equation*, AIMS' Sixth International Conference on Dynamical Systems, Differential Equations and Applications, Poitiers, 25-28 June 2006

3. *Problèmes singuliers en dynamiques des populations*, Conférence Francophone sur la Modélisation Mathématique en Biologie et en Médecine, Craiova, 12-14 July 2006

4. *Concentration of the spectrum for nonhomogeneous differential operators*, Liouville Theorems and Detours, INdAM Conference, Cortona, 18-25 May 2008

5. *Problèmes de bifurcation et de valeurs propres associés aux opérateurs differentiels non homogènes*, Premier Séminaire Roumain-Tunisien en Mathématiques (The 4-th Workshop Series on Mathematics), Institute of Mathematics Simion Stoilow of the Romanian Academy, Bucharest, 18-19 November 2009

6. *Perturbation effects in the analysis of nonlinear elliptic equations*, International Conference on Partial Differential Equations and Applications - in honour of Professor Philippe G. Ciarlet's 70th birthday, City University, Hong Kong, 5-8 December 2008 (Plenary Speaker)

7. *Qualitative analysis of some problems in the theory of non-Newtonian fluids*, Partial Differential Equations in Mathematical Physics and their Numerical Approximation, Levico Terme (Trento), 5-9 September 2011

8. *Degenerate phenomena in nonlinear eigenvalue problems*, Lectures on Partial Differential Equations, International Conference in honour of Professor Patrizia Pucci's 60th birthday, University of Perugia, 28-31 May 2012

9. *Variational analysis on the Sierpinski gasket*, Recent Advances in PDEs and Applications, Levico Terme, Trento, Italy, 17-21 February 2014

10. *Nonlinear elliptic problems on the Sierpinski gasket*, Recent Trends in Nonlinear Partial Differential Equations and Applications Celebrating Enzo Mitidieri's 60th Birthday, University of Trieste, 28-30 May 2014

11. *A characterization property for a class of fractional equations*, Third Conference on Recent Trends in Nonlinear Phenomena, University of Perugia, 28-30 September 2016

12. *Two classical results with lack of monotonicity*, James Serrin: from His Legacy to the New Frontiers, University of Perugia, 30 January - 3 February 2017

13. *Nonhomogeneous problems with singular weights*, Fourth Conference on Recent Trends in Nonlinear Phenomena, University of Messina, 18-20 September 2017
14. *Small and high perturbations of nonhomogeneous elliptic problems*, Two Nonlinear Days in Perugia on the occasion of Patrizia Pucci's 65th birthday, University of Perugia, 11-12 January 2018
15. *Equilibrium problems in the applied sciences*, Fourth Conference on Mathematical Sciences and Applications, King Saud University, Riyadh, 11-12 April 2018
16. *Singular elliptic problems and beyond*, ICMC Summer Meeting on Differential Equations, Sao Carlos, Brazil, 2 February 2021
17. *Nonstandard phenomena in the study of anisotropic problems*, Methods of Nonlinear Analysis in Differential and Integral Equations, Ignacy Łukasiewicz Rzeszów University of Technology, 15 May 2021
18. *Singular and degenerate elliptic problems: new results and some perspectives*, International Conference "Qualitative Properties of Nonlinear Partial Differential Equations", dedicated to Professor Ildefonso Diaz on the occasion of his 70th anniversary, Universidad Complutense de Madrid, 13-15 July 2021
19. *Singular and double phase problems: new results and some perspectives*, Nonlinear Elliptic PDEs in Ancona, 3 September 2021
20. *Double phase problems with variable growth*, 5th Conference on Mathematical Sciences and Applications, The Saudi Association for Mathematical Sciences (SAMS) and King Abdullah University of Science and Technology (KAUST), 17-18 November 2021
21. *New perspectives on anisotropic Stein-Weiss inequalities and Choquard problems*, Plenary Speaker, XIV Summer Workshop in Mathematics, Universidade de Brasilia, 17-21 January 2022
22. *Anisotropic double phase problems with mixed regime*, One Day in Double Phase Problems in Ancona, Università Politecnica delle Marche, INdAM–GNAMPA Project 2020 "Studio di problemi frazionari nonlocali tramite tecniche variazionali", 28 January 2022
23. *Elliptic equations driven by the Stuart differential operator: new results and perspectives*, ICMC Summer Meeting on Differential Equations - Chapter 2022, Universidade de Sao Paulo, 31 January - 2 February 2022
24. *New phenomena in the analysis of double phase problems*, Keynote Speaker, International Conference "Nonlinear Differential Problems", University of Palermo, 31 May 2022
25. *Elliptic equations driven by the Stuart differential operator and some perspectives*, Summer School in Nonlinear Analysis with a Special Tribute to Patrizia Pucci, Università degli Studi della Tuscia, Viterbo, 20-24 June 2022
26. *Two striking results in the analysis of double phase problems*, International Conference on Numerical Analysis, Numerical Modeling, Approximation Theory, Tiberiu Popovici Institute of Numerical Analysis, Cluj Napoca, 26-28 October 2022

27. *New results and perspectives in the analysis of double phase problems*, Recent and New Perspectives in Nonlinear Analysis, Università degli Studi di Urbino, 3-4 November 2022

Prior to the award of the Ph.D.:

1. *Bifurcation problems with linear growth*, Summer School organized by Universidad Complutense from Madrid, Almeria, June 1993

2.2.3 Seminar talks (selection)

1. *Bifurcation phenomena associated to degenerate or singular elliptic equations*, Oxford PDE Seminar, University of Oxford, 14 November 2011
2. *Singular phenomena in nonlinear elliptic equations*, Università Cattolica del Sacro Cuore, Brescia, 19 June 2012
3. *Nonlinear eigenvalue problems for nonhomegenous differential operators*, January 22, 2014, in the framework of the Program "Free Boundary Problems and Related Topics", Isaac Newton Institute for Mathematical Sciences, Cambridge, 6 January - 4 July 2014
4. *Singular phenomena in nonlinear elliptic equations*, Stockholms Matematikcentrum, Stockholm University, 27 January 2016
5. *Effets multiples dans l'étude des problèmes elliptiques singuliers*, Journée d'Équations aux Dérivées Partielles, Université de Tunis, 16 March 2016
6. *Combined effects in nonlinear elliptic singular problems*, Journée d'Équations aux Dérivées Partielles, Kairouan, Tunisia, 18 March 2016
7. *Maximum principle and Keller-Osserman theorem revisited*, Seminar of Functional Analysis, AGH University of Science and Technology, Krakow, 8 November 2017
8. *How much monotonicity is necessary in nonlinear PDEs?*, Faculty of Mathematics and Applied Physics, Rzeszów University of Technology, Rzeszów, 17 November 2017
9. *Nonlinear eigenvalue problems: old and new*, Seminar of the Chair of Optimization and Control, Jagiellonian University, Krakow, 23 November 2017
10. *Nonstandard phenomena in nonlinear elliptic PDEs*, Stockholms Matematikcentrum, Stockholm University, 8 February 2018
11. *Anisotropic double phase problems and perspectives*, Seminar of the Department of Mathematics, King Saud University, Riyadh, 3 April 2019
12. *Double phase problems with variable growth*, Dipartimento di Matematica, Università di Pisa, 2 May 2019
13. *Probleèmes à double phase et exposant variable*, Séminaire d'Analyse, LAMFA, Université de Picardi Jules Verne, Amiens, 20 May 2019
14. *Lane-Emden problems with variable double phase and multiple regime*, Seminar of Functional Analysis, AGH University of Science and Technology, Krakow, 16 October 2019

15. *Anisotropic problems with unbalanced growth and mixed regime*, Seminar of the Chair of Nonlinear Analysis, Rzeszów University of Technology, Rzeszów, 18 October 2019

16. *Singularity and nonlinearity: old and new*, Civil Aviation University of China, Tianjin, 21 January 2021

17. *Nonstandard phenomena in the study of double phase problems*, Monday's Nonstandard Seminar & Seminar of Differential Equations, University of Warsaw, 22 February 2021

18. *Singular phenomena in nonlinear elliptic equations*, Guangxi University, Nanning, 9 April 2021

19. *Double phase problems and a strange discontinuity property of the spectrum*, Zhejiang Normal University, Jinhua, 21 May 2021

20. *Nonstandard elliptic equations and new qualitative properties*, China University of Geosciences, Wuhan, 27 May 2021

21. *Some perspectives in the study of singular elliptic problems*, Central South University, Changsha, 28 July 2021

22. *Nonlinear elliptic equations: new results and some perspectives*, Seminário de Análise, Universidade de Brasilia, 27 August 2021

23. *Elliptic double-phase problems: new results and some perspectives*, Brno University of Technology, 2 September 2021

24. *Nonlinear elliptic equations: singular phenomena and double phase problems*, Three Gorges University, Yichang, Hubei, 21 September 2021

25. *Elliptic double phase problems and some perspectives*, Institute of Mathematics, Lodz University of Technology, 12 October 2021

26. *Two striking results in the study of double phase problems*, Seminar of Functional Analysis, AGH University of Science and Technology, Krakow, 13 October 2021

27. *Stein-Weiss inequalities, Choquard problems, and beyond*, Zhejiang Normal University, Jinhua, 21 October 2021

28. *New phenomena in the study of nonlinear elliptic equations*, Lanzhou University, 19 November 2021

29. *Double phase problems: new results and perspectives*, University of Electronic Science and Technology of China, Chengdu, 8 January 2022

30. *Hardy-Littlewood-Sobolev, Stein-Weiss, and applications to Choquard problems*, Guangzhou University, 10 January 2022

31. *Nonstandard phenomena in the study of double phase problems*, Seminar of the Department of Differential Equations, AGH University of Science and Technology, Krakow, 10 May 2022

32. *Double phase problems: concentration of the spectrum and equations with mixed regime*, Honghe University, 26 May 2022

33. *Non-autonomous double phase problems with unbalanced growth*, Three Gorges University, Yichang, Hubei, 15 June 2022

34. *New phenomena in the study of double phase problems*, Seminário de EDP e Matemática Aplicada, Universidade Federal Fluminense, Rio de Janeiro, 15 June 2022

35. *Ambrosetti-Prodi problems for Robin (p,q)-equations*, Central South University, Changsha, 20 June 2022

36. *A new differential operator: difficulties and perspectives*, Central South University, Changsha, 11 July 2022

37. *Resonant non-autonomous double phase Dirichlet equations*, Zhejiang Normal University, Jinhua, 30 August 2022

38. *Double-phase elliptic equations: concentration of the spectrum and problems with mixed regime*, Brno University of Technology, 31 August 2022

39. *Introduction to double phase problems: new results and some perspectives*, Nanchang University, 6 September 2022

40. *Introduction to double phase problems: new results and some perspectives*, Baotou Teachers' College, 7 September 2022

41. *Ambrosetti-Prodi double phase problems with Robin boundary condition*, Seminar of Functional Analysis, AGH University of Science and Technology, Krakow, 12 October 2022

Prior to the award of the Ph.D.:

1. *Réseaux d'Abrikosov dans le système de Ginzburg-Landau*, Séminaire du Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie (Paris 6), June 1995

2. *Ginzburg-Landau vortices and renormalized energy with weight*, Seminar of the Department of Mathematics, University of Uppsala, October 1995

2.2.4 Original lectures for graduate students

1. *Functional Analysis* (10 hours, Central European University, Budapest, September 2002)

2. *Nonlinear Analysis and Mathematical Physics* (52 hours, École Normale Supérieure, Bucharest, Academic year 2005-2006)

3. *Applied Functional Analysis and Partial Differential Equations* (48 hours, École Normale Supérieure, Bucharest, Academic year 2010-2011)

4. *Comparison Principles and Critical Point Methods in Nonlinear Analysis*, Mini-courses in Mathematical Analysis, University of Padova, 18-22 June 2012

5. *Singular Phenomena in Nonlinear Elliptic Equations*, Mini-courses in Partial Differential Equations, Women in Mathematics Summer School, ICTP, Trieste, 27 May - 1 June 2013

6. Between 2002 and 2014 I organized the *Ateliers d'Écriture Scientifique* at the Doctoral School of the Université de Picardie "Jules Verne", Amiens

2.3 Research visits (selection)

Information on internships completed in scientific or artistic institutions, also abroad, including the place, time and duration of internships and its character

2.3.1 Extended research visits

1. 15 December 1997 - 15 February 1998: Universities of Sussex and Oxford, with a Royal Society Research Fellowship
2. 1998-2001 (4 months every year): PAST Visiting Professor at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (now, Paris Sorbonne University)
3. 1 September - 30 November 2002: Université de Savoie–Chambéry with a CNRS research visiting position (Poste Rouge)
4. 6 January - 4 July 2014: Isaac Newton Institute, Cambridge, Programme *Free Boundary Problems and Related Topics* (G.-Q. Chen, H. Shahgholian, J.-L. Vázquez, organizers)

Prior to the award of the Ph.D.:

1. January 1992 – July 1994 (6 months every year): PhD Scholarship at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (now Paris Sorbonne University), funded by the European Community
2. January 1995 – July 1995: PhD Scholarship at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (now, Paris Sorbonne University), funded by the French Government

2.3.2 Short term research visits

1. Politecnico di Milano (March 1996, with a CNR research grant)
2. Freie Universität in Berlin (two weeks in May 1996)
3. Aristotle University in Thessaloniki (June 1996)
4. Leiden University (October and November 1996)
5. Università Cattolica di Brescia (March 1997, with a CNR research grant)
6. Aristotle University in Thessaloniki (15 May - 15 June 1997)
7. Université Catholique de Louvain (Belgium) in November 1998
8. University of Perugia (Nov. 15 - Dec. 15, 1999, with a CNR research grant)
9. Université Catholique de Louvain (Belgium) in October 2001
10. Université de Picardie "Jules Verne", Amiens (February 2002)
11. Politecnico di Milano (June - July 2002, with a GNAMPA–INdAM Visiting Professor position)
12. Central-European University, Budapest (10 days in September 2002)

13. Université de Picardie “Jules Verne”, Amiens (February 2003)
14. Université de Tunis El Manar (two weeks in April 2003)
15. Institut Elie Cartan, Université Henri Poincaré (Nancy I) (May 2003)
16. Mathematisches Institut, Basel Universität (two weeks in June 2003)
17. Université de Perpignan (July 2003)
18. Université de Picardie “Jules Verne”, Amiens (February 2004)
19. Université de Savoie–Chambéry (two weeks in March 2004)
20. Université de Tunis El Manar (two weeks in April 2004)
21. Université Catholique de Louvain (Belgium) in November 2004
22. Université de Picardie “Jules Verne”, Amiens (February 2005)
23. Universidad Complutense de Madrid (one week in March 2005)
24. City University of Hong Kong (two weeks in April 2005)
25. Université de Tunis El Manar (two weeks in May 2005)
26. Université de Franche Comté and Université de Limoges (two weeks in November 2005)
27. Université de Picardie “Jules Verne”, Amiens (February 2006)
28. Université de Tunis El Manar (one week in May 2006)
29. Université de Poitiers (June 2006)
30. Université de Savoie (two weeks in August 2006)
31. Central European University in Budapest (one week in September 2006)
32. Université de Picardie “Jules Verne”, Amiens (one week in October 2006)
33. University of Perugia (November 2006, with a GNAMPA–INdAM Visiting Professor position)
34. Université de Picardie “Jules Verne”, Amiens (February 2007)
35. Université de Tunis El Manar (one week in March 2007)
36. Université de Haute Alsace (May 2007)
37. Université de La Rochelle (one week in July 2007)
38. Approximation and Wavelets, Bilateral Workshop Romania-Germany, October 1-4, 2007, Königswinter, Germany
39. Université Catholique de Louvain (December 2007)
40. Université de Picardie “Jules Verne”, Amiens (February 2008)

41. Université de Tunis El Manar (two weeks in March 2008)
42. Université de Limoges (May 2008)
43. Université de Tours (June 2008)
44. University of Perugia (two weeks in July 2008) with a GNAMPA–INdAM Visiting Professor position
45. Visiting Professor, Institute of Mathematics, Physics and Mechanics, University of Ljubljana (July–September 2008)
46. University of Cagliari (two weeks in October 2008)
47. Scuola Normale Superiore di Pisa (one week in October 2008)
48. City University of Hong Kong (one week in December 2008)
49. Université de Picardie "Jules Verne", Amiens (February 2009)
50. Université de Tunis El Manar (one week in April 2009)
51. University of Rzeszów (one week in May 2009)
52. University of Ljubljana (one week in May 2009)
53. Université Pierre et Marie Curie Paris VI (one week in August 2009)
54. Université de La Rochelle (one week in September 2009)
55. Université de Picardie "Jules Verne", Amiens (February 2010)
56. University of Messina, Italy (one week in April 2010)
57. Université de Tunis El Manar (one week in January 2011)
58. Université de Picardie "Jules Verne", Amiens (May 2011)
59. University of Oxford (one week in November 2011)
60. University of Monastir (one week in March 2012)
61. Université de Poitiers (one week in March 2012)
62. University of Perugia (15 May–15 June 2012) with a GNAMPA–INdAM Visiting Professor position
63. Université de Picardie "Jules Verne", Amiens (November 2012)
64. Universities of Catania and Reggio Calabria (two weeks in January 2013)
65. Université de Besançon (March 2013)
66. Université de Poitiers (April 2013)

67. ICTP Trieste (one week in May 2013)
68. King Abdulaziz University, Jeddah, Saudi Arabia (one week in September 2013)
69. Universities of Reggio Calabria and Messina (one week in October 2013)
70. Université de Picardie “Jules Verne”, Amiens (November 2013)
71. University of Ljubljana (one week in January 2014)
72. Université Cadi Ayyad, Marrakech (one week in March 2014)
73. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in April 2014)
74. University of Pisa (one week in May 2014)
75. Recent Trends in Nonlinear Partial Differential Equations and Applications Celebrating Enzo Mitidieri’s 60th Birthday, University of Trieste, 28-30 May 2014
76. Université de Picardie “Jules Verne”, Amiens (November 2014)
77. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in December 2014)
78. University of Perugia (one week in January 2015)
79. Senior Research Fellow, City University of Hong Kong (February 2015)
80. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in April 2015)
81. King Saud University, Riyadh, Saudi Arabia (one week in May 2015)
82. Université de Pau (two weeks in October 2015)
83. King Saud University, Riyadh, Saudi Arabia (one week in November 2015)
84. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in December 2015)
85. University of Stockholm (one week in January 2016)
86. Université de Tunis (one week in March 2016)
87. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in April 2016)
88. University of Perugia (one week in September 2016)
89. King Saud University, Riyadh, Saudi Arabia (one week in October 2016)
90. Université de Picardie “Jules Verne”, Amiens (November 2016)
91. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in December 2016)
92. University of Perugia (one week in January 2017)
93. University of Perugia (one week in January 2018)
94. University of Stockholm (one week in February 2018)

95. King Saud University, Riyadh, Saudi Arabia (one week in April 2018)
96. Harbin Engineering University, China (three weeks in November 2018)
97. Université de Picardie "Jules Verne", Amiens (one week in December 2018)
98. University of Trieste (three weeks in March 2019)
99. King Saud University, Riyadh, Saudi Arabia (one week in April 2019)
100. University of Pisa, Italy (one week in April 2019)
101. Università di Urbino Carlo Bo, Urbino, Italy (one week in May 2019)
102. Université de Picardie "Jules Verne", Amiens (one week in May 2019)
103. Central South University, Changsha, China (one month in November 2019)
104. Brno University of Technology (one week in September 2021 and September 2022)

2.4 Participation in research projects

Information on participation in the works of research teams realizing projects through national and international competitions, including the projects which have been completed and projects in progress, and information on the function performed in the team.

2.4.1 Participation as principal investigator of research projects

1. 1998: *Applications of partial differential equations to nonsmooth mechanics and high energies physics*, grant funded by the Romanian Research Agency CNCSIS (No. CNCSIS 9/1998/A/1)
2. 1999-2001: *Systems governed by the mathematical physics equations, expert systems and applications to physics*, grant funded by the Romanian Research Agency CNCSIS (Nos. CNCSIS 79/1999/A/1, 1/2000/A/1, 340/2001/A/1)
3. 2003-2004: *Analysis of some classes of singular boundary problems in anisotropic media: existence, uniqueness and asymptotic behavior of the solutions*, grant funded by the Romanian Academy (No. GAR 12/2004).
4. 2003-2004: *Nonlinear analysis and applications in solid mechanics*, Vicenţiu Rădulescu and Mircea Sofonea (co-principal investigators). Program EGIDE-Brancusi between University of Craiova and University of Perpignan, France.
5. 2004-2006: *Nonlinearities and singularities in mathematical physics*, grant funded by the Romanian Research Agency CNCSIS (No. CNCSIS 308/2006).
6. 2005-2006: *Singular problems of Lane-Emden-Fowler type with convection*, grant funded by the Romanian Academy (No. GAR 80/2006).
7. 2005-2007: *Partial differential equations and applications*, Vicenţiu Rădulescu and Olivier Goubet (co-principal investigators). Program EGIDE-Brancusi (PAI 08915PG) between University of Craiova and Université de Picardie Jules Verne, Amiens, France.

8. 2007-2008: *Analysis and control of nonlinear differential systems*, grant funded by the Romanian Research Agency CNCSIS (No. CNCSIS 589/2008)

9. 2007-2008: *Degenerate and singular problems in nonlinear analysis*, grant funded by the Romanian Academy (No. GAR 315/2007)

10. 2007-2010: *Degenerate and singular models in mathematical physics*, grant funded by the Romanian Research Agency CNCSIS (No. CNCSIS ID-79/2007)

11. 2010-2013: *Differential systems in nonlinear analysis and applications*, grant funded by the Romanian Research Agency CNCS, the only grant in Mathematics approved in the competition "Complex Research Exploratory Projects" (No. PCCE-8/2010)

12. 2011-2016: *Qualitative and numerical analysis of nonlinear problems on fractals*, grant funded by the Romanian Research Agency CNCS, approved in the competition "Research Exploratory Projects" (No. PCE-47/2011)

13. 2014-2015: *Qualitative analysis of some degenerate and singular phenomena in nonlinear analysis*, Highly Cited Research Project No. 39-130-35-HiCi, King Abdulaziz University, Jeddah, Saudi Arabia

14. 2014-2017: *Information security assurance systems based on non-linear analysis models a informational flow*, grant CNCS "Advanced Research Exploratory Projects" (No. PN-II-PT-PCCA-2013-4-0614)

15. 2015-2016: *Nonsupersingular elliptic curves in nonlinear analysis and applications to asymmetric criptography*, Highly Cited Research Project No. 91-130-35-HiCi, King Abdulaziz University, Jeddah, Saudi Arabia

16. 2017-2020: *Qualitative and numerical analysis of some classes of anisotropic differential systems and applications*, grant funded by the Romanian Research Agency UEFISCDI and approved in the competition "Research Exploratory Projects" (No. PN-III-P4-ID-PCE-2016-0130)

17. 2017-2020: *Continuous and discrete systems in nonlinear analysis*, Research Project funded by the Slovenian Research Agency ARRS (J1-8131)

18. 2017-2020: *Analysis of continuous and discrete mathematical models in biology, chemistry and genetics*, Research Project funded by the Slovenian Research Agency ARRS (N1-0064), bilateral grant between Slovenia and Hungary

19. 2021-2024: *Nonlinearity and anisotropy*, grant funded by the Romanian Research Agency UEFISCDI and approved in the competition "Research Exploratory Projects" No. (PN-III-P4-ID-PCE-2020-0068)

2.4.2 Participation as member of research projects

1. Project P.I.C.S. between France and Romania (1999-2003); principal investigators: H. Brezis and M. Iosifescu

2. Project P.I.C.S. between France and Romania (2005-2009); principal investigators: D. Cioranescu and M. Iosifescu

3. European Project *Smart Systems: New Materials, Adaptive Systems and Their Nonlinearities. Modelling, Control and Numerical Simulation*; principal investigator: B. Miara

4. Grant 2-CEx06-11-18/2006 *Deterministic and stochastic differential methods in the study of some evolution models*; principal investigator: R. Purice

5. Grant J1-9643 (2007-2010, funded by the Slovenian Research Agency) *New methods in geometry and topology and their applications*; principal investigator: D. Repovš (Univ. Ljubljana)

6. Grant P1-0292 (2009-2014, funded by the Slovenian Research Agency) *Topology and geometry*; principal investigator: D. Repovš (Univ. Ljubljana)

7. Grant J1-7025 (2016-2018, funded by the Slovenian Research Agency) *Selected problems of nonlinear analysis*; principal investigator: D. Repovš (Univ. Ljubljana)

2.5 Peer reviews

Information on scientific and artistic activity works reviewed, in particular those published in international journals.

I have reviewed more than 200 papers for several journals, including: *J. Math. Pures Appl., Journal of Functional Analysis, Journal of Differential Equations, Transactions Amer. Math. Soc., Proceedings Amer. Math. Soc., Math. Z., Nonlinearity, Calc. Var. Partial Differential Equations, ZAMP, Proc. Royal Soc. London, Proc. A Royal Soc. Edinburgh, J. London Math. Soc., Bulletin London Math. Soc., Israel J. Math., J. Math. Anal. Appl., Nonlinear Analysis, Nonlinear Analysis RWA, Comptes Rendus Math., Topol. Meth. Nonlin. Anal., Opuscula Math., Math. Methods Appl. Sciences, NoDEA, Discrete and Continuous Dynamical Systems, Manuscripta Math., Comm. Contemp. Math., Advanced Nonlinear Studies, ESAIM: Control, Optimisation and Calculus of Variations, Applied Math. Letters, J. Optimiz. Theory Appl., Revista Matematica Complutense*, etc.

I have also reviewed several books submitted to Springer, De Gruyter Series in Nonlinear Analysis and Applications, Academic Press (Mathematics in Science and Engineering Book Series), etc.

2.5.1 Other reviewing activity

1. 1996 – present: 571 reviews written for Mathematical Reviews (MathSciNet)

2. 1995 – present: 1159 reviews written for Zentralblatt MATH

3. 2000 – present: reviewer of more than 20 Ph.D. theses defended in Romania, France, Italy, Australia, Saudi Arabia and Tunisia

4. My reviewing activity is complemented by my editorial activity. I am co-Editor-in-Chief and founder of the journal *Advances in Nonlinear Analysis* (2021 Impact Factor: 4.544; journal ranked 3/332 in the JCR category Mathematics and ranked 5/267 in the JCR category Applied Mathematics). I am also co-Editor-in-Chief of the journal *Boundary Value Problems* (2021 Impact Factor: 1.793) and Associated Editor of *The Journal of Geometric Analysis, Bulletin of Mathematical Sciences, Advisory Editor, Mathematical Methods in the Applied Sciences Asymptotic Analysis, Complex Variables and Elliptic Equations, Rendiconti del Circolo Matematico di*

Palermo, Opuscula Mathematica, Demonstratio Mathematica, Discrete and Continuous Dynamical Systems, Series S, Journal of Numerical Analysis and Approximation Theory, Analele Stiintifice ale Universitatii Ovidius Constanta, as well as of the book series *Mathematics in Science and Engineering–Academic Press* and *De Gruyter Series in Nonlinear Analysis and Applications*.

5. I have reviewed research projects for *Narodowe Centrum Nauki*, *Czech Science Foundation*, *Swiss National Science Foundation*, *Chilean Comisión Nacional de Investigación Cientifica y Tecnológica (CONICYT)*, and *Romanian Research Agency (UEFISCDI)*.

3 Information on cooperation with social and economic environment

In the period 2014-2017, I was the Principal Investigator of the Research Project *Security information systems based on nonlinear information flow analysis*. This project was financed by the Romanian Research Agency in the framework of the Program "Advanced Research Exploratory Projects–Information Technology and Communication" (Project No. PN-II-PT-PCCA-2013-4-0614). The budget of this project was of 1.250.000 RON $\sim$ 1.163.000 PLN. The web page of this research project may be found at

`http://stiinte.ucv.ro/digsig/en/index.html`

The partnership in this applied research project was the Romanian company SIVECO:

`http://www.siveco.ro/en`

This is a renowned Romania company that develops and exports software products and high value added consultancy projects to countries within the European Union, the Middle East, North Africa and the CIS area. The research activities in the framework of this project have been mainly developed in the Research Laboratory on Pure and Applied Nonlinear Analysis at the University of Craiova:

`https://sites.google.com/edu.ucv.ro/pana`

During my career, I was also involved in other projects strictly related with the industry and the applied sciences. For instance, I was the Project Manager of the grant *Analiza zveznih in diskretnih matematičnih modelov v biologiji, kemiji in genetiki (Analysis of continuous and discrete mathematical models in biology, chemistry and genetics)*. This is the grant N1-0064 between Slovenia (Institute of Mathematics, Physics and Mechanics, Ljubljana) and Hungary (Eötvös Loránd, Budapest) during the period 01.11.2017 - 31.10.2020 and the budget of this project is 91.553 EUR. The project was financed by the Hungarian National Research, Development and Innovation Office NKFIH – Nemzeti Kutatási, Fejlesztési és Innovációs Hivatal and the web page of this project can be found at
`http://www.imfm.si/raziskave-in-projekti/raziskovalni-projekti/n1-0064-analiza-zveznih-in-diskretnih-matematicnih-modelov-v-biologiji-kemiji-in-genetiki`

4 Scientometric information

Source: Web of Science. Acquired on: October 23, 2022

H-index: 50 (53, cf. MathSciNet)
Times Cited: 9,318 (9,485, cf. MathSciNet)
Times Cited Without self-citations: 8,133

Average per item: 19,95
Citing Articles: 4,094 (total)
3,781 (without self-citations)
Publications: 469
Highly Cited Papers: 27
Hot Papers: 1
In 2014, 2019, 2020 and 2021, I have been selected as *Highly Cited Researcher*.

4.1 Identifiers

MathSciNet ID: 143765
Web of Science: `https://www.webofscience.com/wos/author/record/A-1503-2012`
Orcid ID: `https://orcid.org/0000-0003-4615-5537`
Scopus ID: 35608668800
Google Scholar: `https://scholar.google.com/citations?user=Abvg2zEAAAAJ&hl=fr&oi=ao`
Slovenian research database: `https://cris.cobiss.net/ecris/si/en/researcher/32599`

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Summary of Professional Accomplishments

Vicenţiu D. Rădulescu

October 25, 2022

1 Diplomas and degrees

Diplomas, degrees conferred in specific areas of sciences and arts, including the name of the institution which conferred the degree, year of degree conferment, title of the Ph.D. dissertation

1. February 2003: Habilitation "à diriger des recherches" at the Université Pierre et Marie Curie–Paris 6 (Sorbonne Université): *Analyse de quelques problèmes aux limites elliptiques non linéaires.* Mémoire realized under the coordination of Professor Haim Brezis (member of the French Academy–Institut de France). Reviewers: Catherine Bandle, Otared Kavian and Michel Willem. The other members of the commission were Fabrice Bethuel, Doina Cioranescu and Laurent Véron.

2. June 1995: Ph.D. at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (Sorbonne Université) under the coordination of Professor Haim Brezis, with the thesis *Analysis of Some Problems Related to the Ginzburg-Landau Equation.* The commission was composed by Haim Brezis (president), Fabrice Béthuel, Thierry Cazenave, Doina Cioranescu, Alain Haraux, Frédéric Hélein and L.A. Peletier. For this thesis I received the highest academic distinction: *très honorable avec félicitations.*

3. December 1993: Ph. D. at the University of Craiova, Romania with the thesis *Applications of Operator Theory to Nonlinear Analysis.* Adviser: Constantin Niculescu.

2 Information on employment

Information on employment in research institutes or faculties/departments or school of arts

1. Positions at the AGH University of Science and Technology:

   since 01.10.2018 : profesor nadzwyczajny AGH;

   since 01.10.2019 : profesor uczelni;

   since 21.12.2020 for an indefinite period: adiunkt badawczy;

   15.11.2021–14.11.2026 : profesor uczelni w grupie pracowników badawczych

2. May 2007 – present: Professorial Fellow at the "Simion Stoilow" Institute of Mathematics of the Romanian Academy, Bucharest

3. Febr. 1990 – present: full professor (since 1998; previously, assistant, lecturer and associate professor) at the Department of Mathematics, University of Craiova, Romania

4. Sept. 1982 – Febr. 1990: professor at the Ioniţă Asan National College, Caracal, Romania

5. Honorary positions:
   – 2018–2023: Guest Professor, Harbin Engineering University
   – 2014–2021: Distinguished Adjunct Professor, King Abdulaziz University, Jeddah
   – 2014–2019: Honorary Director, Institute of Mathematics of the Heilongjiang Institute of Technology, Harbin
   – 2008: Distinguished Foreign Professor, University of Ljubljana

3 Description of the achievements

Description of the achievements, set out in art. 219 para 1 point 2 of the Act

3.1 Title of the habilitation thesis

Local and nonlocal problems in nonlinear analysis

3.2 Scientific articles constituting the habilitation thesis

The achievement consists of the following twelve papers:

[A1] V. Ambrosio, V.D. Rădulescu, Fractional double-phase patterns: concentration and multiplicity of solutions, *Journal de Mathématiques Pures et Appliquées* **142** (2020), 101-145.

[A2] S. Chen, V.D. Rădulescu, X. Tang, L. Wen, Planar Kirchhoff equations with critical exponential growth and trapping potential, *Mathematische Zeitschrift* **302** (2022), 1061-1089.

[A3] Y. Fang, V.D. Rădulescu, C. Zhang, Regularity of solutions to degenerate fully nonlinear elliptic equations with variable exponent, *Bulletin of the London Mathematical Society* **53** (2021), 1863-1878.

[A4] Y. Fang, V.D. Rădulescu, C. Zhang, X. Zhang, Gradient estimates for multi-phase problems in Campanato spaces, *Indiana University Mathematics Journal* **71** (2022), 1079-1099.

[A5] L. Jeanjean, V.D. Rădulescu, Nonhomogeneous quasilinear elliptic problems: linear and sublinear cases, *Journal d'Analyse Mathématique* **146** (2022), no. 1, 327-350.

[A6] C. Ji, V.D. Rădulescu, Multiplicity and concentration of solutions to the nonlinear magnetic Schrödinger equation, *Calculus of Variations and Partial Differential Equations* **59** (2020), no. 4, Paper No. 115, 28 pp.

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3.3 Outline of the research topic

Notations

In the sequel we denote by Ω a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with boundary $\partial\Omega$. We consider several classes of Laplace-type differential operators. The prototype is the p-Laplace operator Δ_p $(1 < p < \infty)$ but we also consider various weighted or nonhomogeneous differential operators. For instance, in the anisotropic setting, a particular role is played by the $p(x)$-Laplace operator, where p is a given real-valued function defined on Ω. The function spaces considered in the sequel are assumed to be defined over the field of real numbers. We mainly consider Lebesque spaces, Sobolev spaces, Hölder spaces, Musielak-Orlicz-Sobolev spaces, and various anisotropic or weighted spaces. We use the same notations as in H. Brezis [24]. A list of general notations (including the main function spaces) is provided in [24, pp. 583-584].

Abstract

This work deals essentially with "nonlinearity and anisotropy". We are concerned with the analysis of several local, nonlocal or local-nonlocal phenomena arising in the applied sciences.

In several circumstances, the problems are driven by various classes of unbalanced local or nonlocal operators, which generate *double phase* or *multi phase* energy functionals. I have expressed a constant interest in such type of problems during the last years, motivated by discussions with several leaders in the field. The founder of the mathematical analysis of solutions to double phase problems (particularly for regularity properties of solutions) is P. Marcellini. In his very recent paper [87], P. Marcellini writes: "Without pretending to give a full landscape of the researches on the nonlinear differential problems with general growth conditions that can be found in the mathematical literature, we have to mention Vicenţiu Rădulescu who, together with some collaborators, gave a series of multiplicity results and existence of nontrivial solutions. We quote for instance Mihailescu-Pucci-Rădulescu [89], Papageorgiou-Pudelko-Rădulescu [99], Papageorgiou-Rădulescu-Zhang [101] and the references therein."

The content is divided into the following five topics.

I. *Nonlinear elliptic equations* (singular problems, non-autonomous equations with unbalanced growth, nonhomogeneous elliptic operators, critical elliptic systems). We are interested in the qualitative analysis of solutions, including existence, nonexistence and parametric analysis for some classes of nonlinear elliptic systems with Dirichlet or Neumann boundary condition. The analysis is extended to a new class of non-homogeneous operators introduced recently by C. Stuart. The papers covering this field are [A5] and [A8].

II. *Regularity and gradient estimates of solutions.* The analysis covers both the isotropic case and the anisotropic setting associated with problems with variable exponent. In the first case, we discuss sharp gradient estimates for multi-phase problems in Campanato spaces. The anisotropic framework corresponds to degenerate fully nonlinear elliptic equations with variable exponent. The main results in this direction are included in the papers [A3] and [A4].

III. *Concentration and multiplicity properties of solutions.* The analysis covers both fractional double phase patterns described by unbalanced fractional Laplace operators, double phase problems with competing potentials, and nonlinear Schrödinger equations with magnetic potential. The arguments are at the interplay between the calculus of variations (critical point theory), topology (Ljusternik-Schnirelmann category), perturbation analysis, and asymptotic methods. The papers devoted to concentration phenomena are [A1], [A6] and [A11].

IV. *Qualitative analysis of double phase problems.* Even if the analysis of unbalanced problems is extended to other sections of this work, here we deal with two additional classes of double phase equations. We first discuss the case of non-autonomous (p,q)-equations and we establish the existence of steady-state solutions. The analysis is developed in Musielak-Orlicz-Sobolev spaces and it covers both the coercive resonant case and the noncoercive (asymptotic resonance or nonresonance) case. In the nonsmooth setting, we discuss a class of double phase *implicit* equations with *convection*. Problems of this type go back to the pioneering work by Stefan [119]. This field is covered by the papers [A9] and [A12].

V. *Nonlocal Kirchhoff, Choquard-Pekar and Stein-Weiss systems.* We are interested in the planar case associated with a reaction having critical exponential growth in the sense of Trudinger-Moser. The analysis is developed for trapping potentials of Rabinowitz-type. The Choquard-Pekar equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. We are interested in the existence of ground states, provided that the potential V is 1-periodic and the origin lies in a gap of the spectrum of the Schrödinger operator $-\Delta + V$. We also establish symmetry, regularity and asymptotic properties of solutions for a class of critical elliptic systems with Stein-Weiss convolution. The papers dealing with these classes of nonlocal equations are [A2], [A7] and [A10].

I. Nonlinear elliptic equations

The modern theory of elliptic equations started after the seminal contributions of H. Poincaré [108, 109, 110] who emphasized the close relationship between the mathematical analysis of elliptic equations and the wide range of applications of these problems. Poincaré established not only several basic results in the theory of linear elliptic equations but he also proved the existence of an infinite sequence of eigenvalues and corresponding eigenfunctions for the Laplace operator under the Dirichlet boundary condition. H. Poincaré is also a pioneer of the theory of *nonlinear* elliptic equations.

An excellent overview of the major contributions to the theory of elliptic equations in the 20th century can be found in the survey paper by H. Brezis and F. Browder [25]. During the preparation of my PhD and Habilitation at the Sorbonne University, I worked directly under the coordination of Professor Brezis, who introduced me to some modern directions of this field. After that I continued to be interested in the qualitative and quantitative treatment of elliptic equations and some contributions are described in what follows.

[A5] Stuart's operator: linear and sublinear cases

Background. C. Stuart and H.S. Zhou [123, 124] studied constrained minimization problems associated with variational integrals of the form

$$\int_\Omega \Gamma(\frac{1}{2}[u^2 + |\nabla u|^2])dx, \tag{1}$$

where $\Gamma \in C^1([0,\infty),\mathbb{R})$, $\Gamma(0) = 0$, $\gamma = \Gamma'$ is non-increasing on $[0,\infty)$ and $\gamma(\infty) = \lim_{t\to\infty}\gamma(t) > 0$. Problems of this type are associated with a model describing the guided traveling waves propagating through a self-focusing dielectric. Cf. C. Stuart and H.S. Zhou [123, 124], the mathematical analysis of such phenomena in a nonlinear dielectric medium is part of the study of special solutions to Maxwell's equations coupled with a nonlinear constitutive relation between the electric field and the electric displacement field.

The differential operator associated to the variational integral (1) is

$$-\text{div}\left(\gamma(\frac{1}{2}[u^2+|\nabla u|^2])\nabla u\right).$$

This operator is nonhomogeneous and it extends the standard elliptic differential operators. However, the structure of this operator is complex. In [122], C. Stuart considered a Dirichlet equation driven by this new differential operator and with a *linear reaction*. Problem (1.2) studied in [122] extends the classical Dirichlet linear problem studied in H. Brezis [24, pp. 291-293]. The proof of Theorem 1.1 in C. Stuart [122] develops a thorough analysis in order to establish fine multiplicity properties of solutions in various circumstances. However, the results are obtained under a strong uniform convexity hypothesis on the potential that characterizes the Stuart differential operator. In the case of small perturbations of the potential in the reaction term, the main result in [122] establishes the existence of two solutions with different level of energy.

Results. Paper [A5] is devoted to a related problem but in the case of a *nonlinear* reaction, which has either a sublinear behavior or a linear growth at infinity. In the case of a linear reaction studied in [122], one needs to understand precisely the interaction between the nonlinearity and the spectrum of a linear operator in order to prove the existence of *a priori* bounds on the Palais-Smale sequences. In [A5], this difficulty is reinforced by the nonlinear character of the quasilinear operator and also by the fact that its *nonhomogeneous* character does not permit to benefit from certain classical techniques as those developed by P. Takac, L. Tello and M. Ulm [126] in the homogeneous case. A feature of paper [A5] is that the strong uniform convexity assumption introduced in [122, p. 328] is replaced by the convexity of the potential, cf. hypothesis (q2) in [A5]. A basic example of potentials that fulfill all hypotheses in this paper is given by

$$\Gamma(t)=At+B[(1+t)^{p/2}-1],$$

where A, B are positive numbers and $1<p<2$. In this case, the energy functional associated to the problem is a double phase variational integral.

There are analyzed two cases. Firstly, it is assumed that the reaction has a sublinear growth and there are analyzed the effects of a perturbation by a nonnegative potential. The most interesting situation is described in Theorem 2.2(ii). In this framework, it is established that if the nonlinearity has a sublinear decay at $+\infty$ and at most a linear growth near the origin, then the positive parameter affecting the reaction must be large enough in order to guarantee the existence of solutions. This corresponds to *high perturbations* of the reaction. By contrast, if the right-hand side of the problem is affected by a nontrivial nonnegative perturbation, then the problem has solutions for all values of the positive parameter associated with the reaction term. This result implies some nice consequences in the case of some simple elliptic equations with nonstandard decay. For example, Theorem 2.2 in [A5] shows that the following Dirichlet problem with logarithmic nonlinearity

$$\begin{cases} -\Delta u+u=\nu\log(1+|u|) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

has only the trivial solution if $\nu > 0$ is small enough and that nontrivial solutions do exist as soon as $\nu > 0$ is sufficiently large. By contrast, if $h \gneqq 0$ (and $h \in L^2(\Omega)$), then the nonlinear problem

$$\begin{cases} -\Delta u + u = \nu \log(1+|u|) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has a non-negative solution for all $\nu > 0$.

Paper [A5] contains also an extension of the main results in [122], provided that the reaction has a linear decay at $+\infty$. This is done under an hypothesis concerning the strict convexity of the mapping $t \mapsto \Gamma(t^2)$, which is rather standard in the literature. The nonlinearity satisfies natural hypotheses, in strong relationship with the first eigenvalue. The main challenge is the assertion (ii) in Theorem 2.4, which establishes the existence of two non-negative solutions, provided that the reaction is affected by a nontrivial perturbation, which is small in the L^2-norm. A first solution corresponds to a local minima of an associated functional and the second one will lie at its mountain pass level. In this setting, to obtain a mountain pass critical point it suffices to show that the Palais-Smale sequence at the mountain pass level is, up to a subsequence, converging. In this direction a major difficulty is to show that the Palais-Smale sequence at the mountain pass level is bounded. This is usually achieved by applying the principle of *localizing the Palais-Smale sequence*. In [A5], in order to overcome this difficulty, we develop a different approach. On the one hand, we introduce a different choice of the energy functional in order to ensure, from the beginning, that a non-negative critical point will be obtained. On the other hand, we make use of the approach developed in [66] to obtain a bounded Palais-Smale sequence. With respect to [122], this method permits to work under weaker hypotheses.

The final version of this paper was analyzed by Professor H. Brezis who suggested several improvements of the initial version. I consider that the next step in the approach of problems driven by Stuart's differential operator is to develop a related analysis in the superlinear and critical cases.

[A8] Neumann problems with singular terms and convection

Background. Singular semilinear elliptic equations have been intensively studied in the last decades. Such problems arise in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, in the theory of heat conduction in electrically conducting materials. For instance, problems of this type characterize some reaction-diffusion processes where the solution $u \geq 0$ is viewed as the *density of a reactant* and the region where $u = 0$ is called the *dead core*, where no reaction takes place.

A major contribution to the field of singular elliptic equations is due to the pioneering papers by M. Crandall, P. Rabinowitz and L. Tartar [41] and J.I. Diaz, J.M. Morel and L. Oswald [50], who studied semilinear elliptic problems with Dirichlet boundary conditions and established several sufficient conditions for the existence of positive solutions, as well as some gradient estimates. Related important contributions are due to M. Coclite and G. Palmieri [37] and A. Lazer and P. McKenna [77].

Subsequently, combined effects of asymptotically linear and singular nonlinearities in bifurcation problems of Lane-Emden-Fowler type were studied by F. Cirstea, M. Ghergu and V. Rădulescu [36]. One of the first papers dealing with singular elliptic equations with convection is due to M. Ghergu and V. Rădulescu [59]. They studied multi-parameter bifurcation effects and asymptotics for the singular Lane-Emden-Fowler equation with convection. Another contribution to singular problems with gradient term is due to L. Dupaigne, M. Ghergu, and V. Rădulescu [52]. All these papers are devoted to singular elliptic equations with Dirichlet boundary condition. A treatment of semilinear parametric elliptic equations with both singular and convection terms and Dirichlet boundary condition can be found in the monograph by M. Ghergu and V. Rădulescu [60, Ch. 9].

Results. The case corresponding to Neumann boundary conditions is less studied in the framework of singular problems with convection. In the above mentioned singular equations with gradient term and Dirichlet boundary condition, the convection grows at most quadratically. As remarked by J. Serrin [118], Y. Choquet-Bruhat and J. Leray [33], and J. Kazdan and F. Warner [73], this requirement is natural in order to apply the maximum principle.

Paper [A8] deals with the following nonlinear Neumann problem with singular and convection terms

$$\begin{cases} -\Delta_p u(z) + \xi(z)u(z)^{p-1} = u(z)^{-\gamma} + f(z, u(z), Du(z)) \text{ in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega,\ u > 0,\ 1 < p < \infty,\ 0 < \gamma < 1. \end{cases} \tag{2}$$

A feature of this paper is that we do not impose any *global* growth conditions on the function $f(z, \cdot, y)$. Instead, we assume that $f(z, \cdot, y)$ exhibits a kind of *oscillatory* behaviour near zero. In this way we can employ truncation techniques and avoid any growth condition at $+\infty$. The presence of the gradient Du in the perturbation f, excludes from consideration a variational approach to dealing with problem (2). Instead, our main tool in the proof is topological and is based on fixed point theory, in particular, on the Leray-Schauder principle, cf. A. Granas and J. Dugundji [61, p. 124].

Essentially, the main problem in this paper is studied under three technical hypotheses on the Carathéodory function f. The first hypothesis is satisfied if, for example, there exists $\eta \in (0, +\infty)$ such that $\eta^{-\gamma} + f(z, \eta, y) \leq -c^* < 0$ for almost all $z \in \Omega$ and all $y \in \mathbb{R}^N$. The first two hypotheses together determine the oscillatory behavior of $f(z, \cdot, y)$ near 0^+. Finally, the third hypothesis is satisfied if we set $f(z, x, y) = 0$ for almost all $z \in \Omega$ and any $x \geq w(z)$, $y \in \mathbb{R}^N$ and require that the function $x \mapsto \frac{f(z,x,y)}{x^{p-1}}$ is nonincreasing on $(0, w(z)]$ for almost all $z \in \Omega$ and for all $y \in \mathbb{R}^N$. Due to the presence of the singular term, a truncated problem is an important auxiliary in the proof. In such a way, by an iteration process, one proves the existence of a positive solution in a suitable interval. The regularity is a delicate issue and it follows by applying the nonlinear regularity theory developed by G. Lieberman [81]. The existence of this solution enables us to define a convenient nonlinear operator (called *minimal solution map*), which has a fixed point, by the Leray-Schauder principle. The existence of a positive solution for problem (2) is straightforward and follows by combining these auxiliary results.

II. Regularity and gradient estimates of solutions

We are concerned with regularity theory for solutions of two classes of double phase problems. This theory was initiated by P. Marcellini who developed rigorous and sharp regularity estimates of minimizers for several classes of unbalanced variational integrals. In [85], P. Marcellini established regularity properties of minimizers for integrals in the calculus of variations with non standard growth conditions. He was mainly interested in the study of nonautonomous functionals, especially for the double phase problem

$$u \mapsto \int_\Omega (|Du|^p + a(x)|Du|^q)dx.$$

In [86] it is discussed the regularity and existence of solutions for elliptic equations with p, q-growth conditions. I also refer to the important developments of G. Mingione and his collaborators [20, 53], who established several regularity properties in the framework of double phase problems, including gradient estimates via nonlinear potentials and Lipschitz bounds for problems with nonuniform ellipticity. Regularity theory is at the center of the paper by G. Mingione and V. Rădulescu [91], which is concerned with some recent developments in problems with nonstandard growth and nonuniform ellipticity. This is the third most cited paper published in 2021 (cf. MathSciNet) and it is a Hot Paper and Highly Cited Paper (cf. Web of Science).

[A3] Regularity for degenerate fully nonlinear equations with variable exponent

Background. S. Byun and H. Lee in [27] considered inhomogeneous double phase equations with variable exponents and established the Calderón-Zygmund type estimates under some suitable conditions. On the other side, a fairly comprehensive investigation of regularity properties for fully nonlinear equations has been carried out by I. Birindelli and F. Demengel [23] who established the comparison principle and Liouville-type theorems in the singular setting. Alexandrov-Bakelman-Pucci estimates have also been performed for such class of equations in [64]. In particular, C. Imbert and L. Silvestre [65] proved the interior $C^{1,\alpha}$ estimates on the viscosity solutions. It is noteworthy to mention that C. De Filippis in [47] introduced the double phase type degeneracies to the fully nonlinear equation

$$\big[|Du|^p + a(x)|Du|^q\big]F(D^2u) = f(x), \quad 0 < p \le q,$$

and proved the $C^{1,\gamma}$ local regularity for viscosity solutions. I also point out the work by A. Bronzi, E. Pimentel, G. Rampasso and E. Teixeira [26] who proved that viscosity solutions to the following variable exponent fully nonlinear elliptic equations

$$|Du|^{p(x)}F(D^2u) = f$$

are locally of class $C^{1,\gamma}$ for a universal constant $\gamma \in (0, 1)$.

Results. Paper [A3] is motivated by the above mentioned results. In this paper it is considered the following fully nonlinear elliptic equation with variable exponent nonhomogeneous degeneracy

$$\big[|Du|^{p(x)} + a(x)|Du|^{q(x)}\big]F(D^2u) = f(x) \quad \text{in } \Omega, \tag{3}$$

where $a(x) \ge 0$ for all $x \in \Omega$.

It is assumed that the source term f belongs to $C(\Omega) \cap L^\infty(\Omega)$, while the fully nonlinear operator $F : Sym(n) \to \mathbb{R}$ is continuous and uniformly (λ, Λ)-elliptic in the sense that

$$\lambda\|N\| \le F(M + N) - F(M) \le \Lambda\|N\|$$

for some $0 < \lambda \le \Lambda$ and each $M, N \in Sym(n)$ with $N \ge 0$.

A function $u \in C(\Omega)$ is called a viscosity supersolution of problem (3), if for all $x_0 \in \Omega$ and $\varphi(x) \in C^2(\Omega)$ such that $u - \varphi$ attains a local minimum at x_0, one has

$$\big[|D\varphi(x_0)|^{p(x_0)} + a(x_0)|D\varphi(x_0)|^{q(x_0)}\big]F(D^2\varphi(x_0)) \le f(x_0).$$

A function $u \in C(\Omega)$ is a viscosity subsolution if for all $x_0 \in \Omega$ and $\varphi(x) \in C^2(\Omega)$ such that $u - \varphi$ attains a local maximum at x_0, one has

$$\big[|D\varphi(x_0)|^{p(x_0)} + a(x_0)|D\varphi(x_0)|^{q(x_0)}\big]F(D^2\varphi(x_0)) \ge f(x_0).$$

We say that u is a *viscosity solution* of problem (3) if it is viscosity super- and subsolution simultaneously.

The main result in [A3] establishes that if $u \in C(\Omega)$ is a viscosity solution of problem (3), then $u \in C^{1,\alpha}_{\text{loc}}(\Omega)$ for any α verifying

$$0 < \alpha < \min\left\{\overline{\alpha}, \frac{1}{1 + \sup_\Omega p(x)}\right\}.$$

Moreover, we have the following Hölder estimate on the gradient of solution, that is, for any subdomain $\Omega' \subset\subset \Omega$, there is a constant C depending on $n, \lambda, \Lambda, \alpha, \operatorname{dist}(\Omega', \Omega)$ and $\sup_\Omega p(x)$ such that

$$\sup_{\substack{x,y\in\Omega'\\ x\neq y}} \frac{|Du(x)-Du(y)|}{|x-y|^\alpha} \leq C\left(1+\|u\|_{L^\infty(\Omega)}+\|f\|_{L^\infty(\Omega)}^{\frac{1}{1+\inf_\Omega p(x)}}\right).$$

One of the ideas in the proof is to make use of the scaling properties of equation (3) to trace the problem back to a smallness regime. Namely, it is possible to assume that

$$\|u\|_{L^\infty(B_1)} \leq 1 \quad \text{and} \quad \|f\|_{L^\infty(B_1)} \leq \varepsilon \tag{4}$$

with $0 < \varepsilon \ll 1$. Next, it is provided the local Hölder continuity of viscosity solutions to the following problem

$$\left[|Du+\xi|^{p(x)} + a(x)|Du+\xi|^{q(x)}\right] F(D^2u) = f(x) \quad \text{in } B_1, \tag{5}$$

where ξ is an arbitrary vector in $\mathbb{R}^N$.

The Hölder estimate on viscosity solution provides the compactness with respect to the uniform convergence, which is the key ingredient of a basic approximation result. A nice regularity result on solutions to the homogeneous problem

$$F(D^2u) = 0 \quad \text{in } B_1$$

asserts that such solution is locally $C^{1,\overline{\alpha}}$ regular for $\overline{\alpha} \in (0,1)$ depending only on n, λ, Λ. Furthermore, cf. L. Caffarelli and X. Cabré [29, Ch. 5], there is a constant C depending on n, λ, Λ such that

$$\|u\|_{C^{1,\overline{\alpha}}(B_{1/2})} \leq C\|u\|_{L^\infty(B_1)}.$$

The proof is completed by combining the Ishii-Lions lemma [40, Theorem 3.2] with related estimates.

[A4] Gradient estimates for multi-phase problems in Campanato spaces

Background. In [49], E. DiBenedetto and J. Manfredi studied the higher integrability of the gradient of weak solutions to certain degenerate elliptic systems. They consider the system

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \operatorname{div}(|F|^{p-2}F) \text{ in } \Omega, \tag{6}$$

for a given $F = (F_1, F_2, \ldots, F_m) \in [L^p_{loc}(\Omega)]^m$, where $p > 1$. If $p = 2$, then the system is linear and it follows by the Calderón-Zygmund theory that if $F \in [L^q_{loc}(\Omega)]^m$ for some $q > p$, then $\nabla u \in [L^q_{loc}(\Omega)]^m$. This result was extended to the nonlinear degenerate case by E. DiBenedetto and J. Manfredi [49] who obtained gradient estimates in $[\mathrm{BMO}_{loc}(\Omega)]^{Nm}$.

Results. In paper [A4] it is established a new Campanato-type estimate for the weak solutions of a class of multi-phase problems. The problem under consideration is characterized by the fact that both ellipticity and growth switch between three different types of polynomial according to the position, which describes a feature of strongly anisotropic materials. The results obtained in [A4] are different from the BMO-type estimates for the usual p-Laplacian equation due to E. DiBenedetto and J. Manfredi.

Consider the following multi-phase problem in divergence form

$$\begin{aligned} &\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u + b(x)|\nabla u|^{s-2}\nabla u\right) = \\ &\operatorname{div}\left(|F|^{p-2}F + a(x)|F|^{q-2}F + b(x)|F|^{s-2}F\right) \quad \text{in } \Omega, \end{aligned} \tag{7}$$

which is described by the (p,q,s)-energy functional defined by

$$\mathcal{F}(u) = \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q + b(x)|\nabla u|^s)\, dx, \quad 1 < p < q \le s. \tag{8}$$

The double phase problem (if $b(x) \equiv 0$) is characterized by the fact that the ellipticity and growth rates of its integrand radically change with the position variable x. This problem was first studied by P. Marcellini [85, 86] and it provides a model for describing a feature of strongly anisotropic materials and new examples of Lavrentiev phenomenon.

If $a(\cdot) \in C^{0,\alpha}(\Omega)$ is nonnegative and $b(x) \equiv 0$ then M. Colombo and G. Mingione [38, 39] established the Hölder regularity of gradients of the minimizers for double phase energy functionals of the type

$$\mathcal{P}(u,\Omega) := \int_\Omega |\nabla u|^p + a(x)|\nabla u|^q\, dx. \tag{9}$$

They showed that if $q < p + \frac{\alpha p}{N}$, then minimizers of (9) are in $C^{1,\beta}$ for some $\beta \in (0,1)$, and if $q \le p + \alpha$, then bounded minimizers of (9) are in $C^{1,\beta}$ for some $\beta \in (0,1)$ as well.

The aim of paper [A4] is to establish a Campanato-type estimate of weak solutions to problem (7). If $s \ge 1$ and $\mu \ge 0$, recall that the Campanato space $\mathcal{L}^{s,\mu}(\Omega)$ is the class of all functions $u \in L^s(\Omega)$ such that

$$[u]_{s,\mu;\Omega} := \sup_{x\in\Omega,\, 0<\rho<\operatorname{diam}\Omega} \left(\rho^{-\mu} \int_{\Omega(x,\rho)} |u(z) - u_{x,\rho}|^s\, dz\right)^{\frac{1}{s}} < \infty,$$

where $\Omega(x,\rho) := \Omega \cap B(x,\rho)$,

$$u_{x,\rho} = \fint_{\Omega(x,\rho)} u(z)\, dz = \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} u(z)\, dz$$

and $|\Omega(x,\rho)|$ is the Lebesgue measure of $\Omega(x,\rho)$.

The Campanato space $(\mathcal{L}^{s,\mu}(\Omega), \|\cdot\|_{\mathcal{L}^{s,\mu}(\Omega)})$ is a Banach space that extends the notion of functions of bounded mean oscillation, which is due to F. John and L. Nirenberg [72]. The Campanato space describes situations where the oscillation of the function in a ball is proportional to some power of the radius other than the dimension.

For all $x \in \Omega$ and $z \in \mathbb{R}^N$, denote

$$A(x,z) = |z|^{p-2}z + a(x)|z|^{q-2}z + b(x)|z|^{s-2}z$$

and

$$H(x,z) = |z|^p + a(x)|z|^q + b(x)|z|^s.$$

Assume that the following hypotheses are fulfilled:

$$a(x), b(x) \ge 0, \quad a \in C^{0,\alpha}_{\mathrm{loc}}(\Omega), \quad b \in C^{0,\beta}_{\mathrm{loc}}(\Omega), \quad \alpha, \beta \in (0,1] \tag{10}$$

and

$$\frac{q}{p} \le 1 + \frac{\alpha}{N}, \quad \frac{s}{p} \le 1 + \frac{\beta}{N}. \tag{11}$$

The main result in [A4] establishes the following regularity property.

Theorem 1. *Let $u \in W^{1,1}_{\rm loc}(\Omega)$ be a local distributional solution to* (7) *with*

$$H(x, \nabla u) \in L^1_{\rm loc}(\Omega) \quad \text{and} \quad H(x, F) \in L^{1+\sigma}_{\rm loc}(\Omega)$$

for some $\sigma > 0$. Assume that (10), (11) *hold and*

$$1 < p < q \le s < \infty.$$

If $A(x, F) \in \mathcal{L}^{\frac{p}{p-1},\mu}_{\rm loc}(\Omega)$, where $0 < \mu < N$, then $\nabla u \in \mathcal{L}^{p,\widetilde{\mu}}_{\rm loc}(\Omega)$.

In the previous statement, the constant $\widetilde{\mu}$ is defined by

$$\widetilde{\mu} = \begin{cases} \mu & \text{if } 2 \le p < q \le s, \\ (p-1)\mu & \text{otherwise.} \end{cases}$$

The technical approach in the proof is based on different comparison estimates along with the good properties of homogeneous problems and after carefully controlling the interaction between the two phase transitions together with the appropriate *localization method.* Remark that if $a(x) = b(x) \equiv 0$, problem (7) becomes the usual p-Laplace type equation (6) considered by E. DiBenedetto and J. Manfredi in [49]. Notice that the result obtained in [A4] is different from the usual p-Laplace equation because of the presence of variable coefficients $a(\cdot)$ and $b(\cdot)$. It gives rise to the interesting new phenomenon that the BMO type estimate in [49] is no longer valid. In fact, we only have the Campanato-type estimates of weak solutions for problem (7). Note that paper [A4] does not touch Sobolev-Morrey and Besov-Morrey type spaces. I refer for details to the monograph [130] titled *Morrey and Campanato Meet Besov, Lizorkin and Triebel* by W. Yuan, W. Sickel and D. Yang.

In [A4] it is always assumed that $a(x), b(x) \ge 0$. Particularly, when $a(x) = b(x) \equiv 0$, problem (7) becomes the usual p-Laplace equation. For this case, if we choose $N < \mu \le N + \frac{p}{p-1}$, then we have the assumption

$$A(x, F) \in \mathcal{L}^{\frac{p}{p-1},\mu}_{\rm loc}(\Omega) \simeq C^{0,\delta_1}_{\rm loc}(\Omega) \text{ with } \delta_1 = \frac{(\mu - N)(p-1)}{p}.$$

Hence from Theorem 1 we know that $\nabla u \in C^{0,\delta_2}_{\rm loc}(\Omega)$, where $\delta_2 = \frac{\mu - N}{p}$. If $\mu = N$, it follows from $A(x, F) \in \mathcal{L}^{\frac{p}{p-1},\mu}_{\rm loc}(\Omega)$ that $A(x, F) \in BMO_{\rm loc}(\Omega)$, then $\nabla u \in BMO_{\rm loc}(\Omega)$, which recovers the classical BMO estimates established in [49].

III. Concentration and multiplicity properties of solutions

Concentration properties of solutions have been intensively studied in the last decades starting with the pioneering paper by M. del Pino and P. Felmer [48]. Consider the model example

$$\begin{cases} -\epsilon^p \Delta_p u + V(x)|u|^{p-2}u = K(x) f(x, u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{12}$$

where $1 < p < N$, $\Delta_p u = \text{div}(|\nabla u|^{-2} \nabla u)$, ϵ is a small positive parameter, V is a potential, and f is the reaction term.

Problems like (12) arise when one looks for the stationary solutions of reaction-diffusion systems of the form

$$u_t = \Delta_p u + c(x, u), \ x \in \mathbb{R}^N \text{ and } t > 0.$$

This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics, and chemical reaction design (see L. Cherfils and Y. Il'yasov [32]). In such applications, the function u is a state variable and describes density or concentration of multi-component substances, $\Delta_p u$ corresponds to the diffusion with a diffusion coefficient $|\nabla u|^{p-2}$, and $c(x,u)$ is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ has a polynomial form with respect to the unknown concentration denoted by u.

For the case when $\epsilon > 0$ is sufficiently small, the solutions of problem (12) are often referred to as *semiclassical states*, which possess many significant physical insights for ϵ small. More precisely, the concentration phenomenon of semiclassical states, as $\epsilon \to 0$, reflects the transformation process between quantum mechanics and classical mechanics. For such problem, some asymptotic behaviors of semiclassical states, such as concentration, convergence and exponential decay, are very interesting research topics in mathematics and physics. In this framework, from a mathematical viewpoint, it is worth to study not only the existence of semiclassical solutions but also their asymptotic behavior as $\epsilon \to 0$. Typically, solutions tend to concentrate around critical points of the potentials functions.

We extend this analysis to three classes of nonlinear stationary problems, namely fractional double phase equations, magnetic Schrödinger equations and quasilinear unbalanced equations with with competing potentials.

[A1] Fractional double-phase patterns: concentration and multiplicity of solutions

Background. For any $s \in (0,1)$ and $2 \le t < \frac{N}{s}$, let $(-\Delta)_t^s$ be the fractional t-Laplacian operator which, up to normalization factors, can be defined for every function $u \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ as

$$(-\Delta)_t^s u(x) = 2 \lim_{r\to 0} \int_{\mathbb{R}^N \setminus \mathcal{B}_r(x)} \frac{|u(x)-u(y)|^{t-2}(u(x)-u(y))}{|x-y|^{N+st}} dy \quad (x \in \mathbb{R}^N).$$

If $t = 2$, the equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u) \quad \text{ in } \mathbb{R}^N,$$

which has been extensively considered by several authors. In particular way, a great attention has been devoted to the study of solutions which concentrate around critical points of the potential V as $\varepsilon \to 0$, cf. J. Davila, M. del Pino and J. Wei [42] and G. Figueiredo, N. Ikoma and J. Santos Júnior [44].

If $t \neq 2$ and $\varepsilon = 1$, we obtain a class of fractional p-Laplacian equations:

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x,u) \text{ in } \mathbb{R}^N,$$

for which several existence and multiplicity results have been obtained in this last decade; see, for instance, P. Pucci, M. Xiang and B. Zhang [112]. From the mathematical point of view, the fractional p-Laplace operator has a great attractive since two phenomena are present: the nonlinearity of the operator and its nonlocal character. Indeed, some standard tools used to investigate the linear case $p = 2$ seem not to be trivially adaptable in the case $p \neq 2$ due to the lack of a Hilbertian structure for $W^{s,p}(\mathbb{R}^N)$ if $p \neq 2$.

Paper [A1] deals with the existence, multiplicity and concentration behavior of positive solutions for *fractional p&q-Laplacian problems*, which has a particular interest due to the unbalanced behaviour of the operator.

Results. Paper [A1] deals with the qualitative analysis of solutions and their concentration properties for the problem

$$\begin{cases} (-\Delta)_p^s u + (-\Delta)_q^s u + V(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N), \quad u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{13}$$

where $\varepsilon > 0$ is a small parameter, $s \in (0,1)$, $2 \leq p < q < \frac{N}{s}$, and $V : \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are continuous functions.

It is assumed that $V \in \mathcal{C}^0(\mathbb{R}^N, \mathbb{R})$ satisfies the following del Pino-Felmer conditions [48]:

(V_1) there exists $V_0 > 0$ such that $V_0 := \inf_{x\in\mathbb{R}^N} V(x)$,

(V_2) there exists an open bounded set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{\partial\Lambda} V \quad \text{and} \quad M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset,$$

M. del Pino and P. Felmer [48] assumed (V_1) and

$$\inf_{\Lambda} V < \min_{\partial\Lambda} V \tag{14}$$

instead of (V_2), and they showed that the following nonlinear Schrödinger equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \text{ in } \mathbb{R}^N, \tag{15}$$

admits a single-peak solution which concentrates around the minimum points of V in Λ. Their result can be seen as the localized version of the result of P. Rabinowitz [113] and X. Wang [131], who proved the existence of positive solutions to (15) for small $\varepsilon > 0$, by assuming the following global condition

$$\liminf_{|x|\to\infty} V(x) > V_0. \tag{16}$$

The relevance of (14) is that no restriction on the global behavior of V is required other that (V_1), and, in particular, V is not required to be bounded or to belong to a Kato class. Later, S. Cingolani and M. Lazzo [34], assuming (16), proved that the multiplicity of solutions to (15) is related to topology richness of the set $K := \{x \in \mathbb{R}^N : V(x) = V_0\}$. Subsequently, motivated by [34, 48], C. Alves and G. Figueiredo [7] assumed (V_1)-(V_2) and obtained multiple positive solutions for a quasilinear p-Laplacian problem. That is why, in order to get a multiplicity result for problem (13), we assume (V_1)-(V_2) as in [7]. Since we aim to relate the number of solutions of (13) with the topology of the set M of minima of the potential, it is worth recalling that if Y is a given closed set of a topological space X, we denote by $cat_X(Y)$ the Ljusternik-Schnirelmann category of Y in X, that is, the least number of closed and contractible sets in X which cover Y.

The continuous function f fulfills the following hypotheses:

(f_1) $\displaystyle\lim_{|t|\to 0} \frac{|f(t)|}{|t|^{p-1}} = 0$;

(f_2) there exists $\nu \in (q, q_s^*)$ such that $\displaystyle\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{\nu-1}} = 0$, where $q_s^* := \frac{Nq}{N-sq}$;

(f_3) there exists $\vartheta \in (q, q_s^*)$ such that $0 < \vartheta F(t) := \vartheta \displaystyle\int_0^t f(\tau)\,d\tau \leq t f(t)$ for all $t > 0$;

(f_4) the map $t \mapsto \dfrac{f(t)}{t^{q-1}}$ is increasing for $t > 0$.

The main result established in [A1] is the following.

Theorem 2. *Assume that* (V_1)-(V_2) *and* (f_1)-(f_4) *hold true. Then, for any* $\delta > 0$ *such that*

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leq \delta\} \subset M,$$

there exists $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *problem* (13) *has at least* $cat_{M_\delta}(M)$ *positive solutions. Moreover, if* u_ε *denotes one of these solutions and* $x_\varepsilon \in \mathbb{R}^N$ *is a global maximum point of* u_ε, *then*

$$\lim_{\varepsilon \to 0} V(\varepsilon x_\varepsilon) = V_0.$$

Since no information on the behavior of V at infinity are available, we adapt the penalization method in [48], which consists in making a suitable modification on f, solving an auxiliary problem and then check we that for $\varepsilon > 0$ small enough, all solutions of the new problem are also solutions of the original one. To obtain multiple solutions of the modified problem, we make use of some topological techniques proposed by V. Benci and G. Cerami [22], by means of accurate comparisons between the category of some sublevel sets of the modified functional and the category of the set M. Anyway, due to the fact that the nonlinearity is only continuous, one cannot apply standard $\mathcal{C}^1$-Nehari manifold arguments due to the lack of differentiability of the associated Nehari manifold. This difficulty will be overcome by using an abstract critical point method proposed by A. Szulkin and T. Weth [125]. Moreover, in order to show that the solutions of the modified problem are also solutions to (13), we cannot adapt in our setting the arguments used by C. Alves and G. Figueiredo [8], due to the nonlocal character of fractional $p\&q$-Laplacian operators. Also, it fails the strategy used by C. Alves and O. Miyagaki [10] based on some estimates coming from the properties of the Bessel kernel established by P. Felmer, A. Quaas and J. Tan [43]. In our situation, we develop the Moser iteration scheme introduced by J. Moser [95] to deduce L^∞-estimates and we establish a Hölder regularity result which extends in the fractional $p\&q$-case the interior regularity result proved by A. Iannizzotto, S. Mosconi and M. Squassina [63] for the fractional p-Laplacian. Indeed, the restriction $p \geq 2$ is related to the use of this regularity result because all variational and topological arguments used to obtain the existence and multiplicity of solutions for the modified problem hold for all $1 < p < q < \frac{N}{s}$.

[A6] Concentration of solutions to the nonlinear magnetic Schrödinger equation

Background. The Schrödinger equation is central in quantum mechanics and it plays the role of Newton's laws and conservation of energy in classical mechanics, that is, it predicts the future behaviour of a dynamical system. The *linear* Schrödinger equation is a central tool of quantum mechanics, which provides a thorough description of a particle in a non-relativistic setting. Schrödinger's linear equation is

$$\nabla^2\psi + \frac{8\pi^2 m}{\hbar^2}\,(E - V(x))\,\psi = 0\,,$$

where ψ is the Schrödinger wave function, m is the mass of the particle, $\hbar$ denotes Planck's renormalized constant, E is the energy, and V stands for the potential energy. The structure of the *nonlinear* Schrödinger equation is much more complicated.

E. Schrödinger also established the classical derivation of his equation, based upon the analogy between mechanics and optics, and closer to de Broglie's ideas. He developed a perturbation method, inspired by the work of Lord Rayleigh in acoustics, proved the equivalence between his wave mechanics and Heisenberg's matrix, and introduced the time dependent Schrödinger's equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\,\nabla^2\psi + V(x)\psi - \gamma|\psi|^{p-1}\psi, \qquad x \in \mathbb{R}^N\ (N \geq 2), \tag{17}$$

where $p < 2N/(N-2)$ if $N \geq 3$ and $p < +\infty$ if $N = 2$. In physical problems, a cubic nonlinearity corresponding to $p = 3$ in equation (17) is common. In this case, problem (17) is called the Gross-Pitaevskii equation.

The first result involving the magnetic field for the Schrödinger equation was obtained by M. Esteban and P.L. Lions [54]. Their arguments used the concentration-compactness principle and related minimization principles. Relevant results for equations with magnetic field are due to C. Alves, G. Figueiredo, M. Furtado [9], G. Arioli and A. Szulkin [14] and S. Cingolani and S. Secchi [35]. However, concentration phenomena associated with magnetic Schrödinger equations are less studied. I refer to C. Alves, G. Figueiredo and M. Furtado [9] (for multiple solutions of the nonlinear Schrödinger equation with magnetic fields) and C. Ji and V.D. Rădulescu [68, 69, 70, 71] (for magnetic Kirchhoff equations, magnetic Schrödinger equations with critical growth and magnetic Choquard equation with deepening potential well).

Results. Paper [A6] deals with the study of multiplicity and concentration properties for the following nonlinear magnetic Schrödinger equation

$$\left(\frac{\varepsilon}{i}\nabla - A(x)\right)^2 u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N \ (N \geq 2), \tag{18}$$

where $u \in H^1(\mathbb{R}^N, \mathbb{C})$, $\varepsilon > 0$ is a parameter, $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function, and the magnetic potential $A : \mathbb{R}^N \to \mathbb{R}^N$ is Hölder continuous with exponent $\alpha \in (0, 1]$.

A feature of paper [A6] is that f is only continuous, hence the arguments developed by C. Alves, G. Figueiredo and M. Furtado [9] fail. Moreover, due to the presence of the magnetic field $A(x)$, equation (18) cannot be changed into a pure real-valued problem, hence we must deal directly with a complex-valued problem, which causes several new difficulties.

The potential V satisfies the following hypotheses:

$(V1)$ there exists $V_0 > 0$ such that $V(x) \geq V_0$ for all $x \in \mathbb{R}^N$;

$(V2)$ there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 = \min_{x \in \Lambda} V(x) < \min_{x \in \partial\Lambda} V(x).$$

Set

$$M := \{x \in \Lambda : V(x) = V_0\} \neq \emptyset.$$

The subcritical nonlinearity f is increasing in $(0, \infty)$ and satisfies the Ambrosetti-Rabinowitz condition.

Under these hypotheses, the main result in this paper is the following.

Theorem 3. *For any $\delta > 0$ such that*

$$M_\delta := \{x \in \mathbb{R}^N : \mathrm{dist}\,(x, M) < \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $0 < \varepsilon < \varepsilon_\delta$, problem (18) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, for every sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0^+$ as $n \to +\infty$, if we denote by u_{ε_n} one of these solutions of problem (18) for $\varepsilon = \varepsilon_n$ and $\eta_{\varepsilon_n} \in \mathbb{R}^N$ is the global maximum point of $|u_{\varepsilon_n}|$, then

$$\lim_{\varepsilon_n \to 0^+} V(\eta_{\varepsilon_n}) = V_0.$$

The analysis is developed in the space

$$H^1_A(\mathbb{R}^N,\mathbb{C}) := \{u \in L^2(\mathbb{R}^N,\mathbb{C}) : |\nabla_A u| \in L^2(\mathbb{R}^N,\mathbb{R})\},$$

where

$$\nabla_A u := \Big(\frac{\nabla}{i} - A\Big)u,$$

The space $H^1_A(\mathbb{R}^N,\mathbb{C})$ is a Hilbert space endowed with the scalar product

$$\langle u,v\rangle := \mathrm{Re}\int_{\mathbb{R}^2}\Big(\nabla_A u\overline{\nabla_A v} + u\overline{v}\Big)dx, \quad \text{for any } u,v \in H^1_A(\mathbb{R}^N,\mathbb{C}).$$

On $H^1_A(\mathbb{R}^N,\mathbb{C})$ we will frequently use the following diamagnetic inequality (cf. E. Lieb and M. Loss [80, Theorem 7.21])

$$|\nabla_A u(x)| \geq |\nabla|u(x)||. \tag{19}$$

Thus, by a change of variables, problem (18) is equivalent to

$$\Big(\frac{1}{i}\nabla - A_\varepsilon(x)\Big)^2 u + V_\varepsilon(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{20}$$

where $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$. For this reason, we introduce the Hilbert space H_ε obtained as the closure of $C^\infty_c(\mathbb{R}^N,\mathbb{C})$ with respect to the scalar product

$$\langle u,v\rangle_\epsilon := \mathrm{Re}\int_{\mathbb{R}^N}\Big(\nabla_{A_\varepsilon} u\overline{\nabla_{A_\varepsilon} v} + V_\varepsilon(x)u\overline{v}\Big)dx.$$

The diamagnetic inequality (19) implies that if $u \in H^1_{A_\varepsilon}(\mathbb{R}^N,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N,\mathbb{R})$ and $\|u\| \leq C\|u\|_\varepsilon$. Therefore, the embedding $H_\varepsilon \hookrightarrow L^r(\mathbb{R}^N,\mathbb{C})$ is continuous for $2 \leq r \leq 2^*$ and the embedding $H_\varepsilon \hookrightarrow L^r_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C})$ is compact for $1 \leq r < 2^*$.

A key idea in the proof is to study the modified problem

$$\Big(\frac{1}{i}\nabla - A_\varepsilon(x)\Big)^2 u + V_\varepsilon(x)u = g(\varepsilon x,|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{21}$$

where the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g(x,t) := \chi_\Lambda(x)f(t) + (1-\chi_\Lambda(x))\tilde{f}(t). \tag{22}$$

Then the energy associated with problem (21) has a mountain pass geometry and satisfies the localized Palais-Smale condition at any positive level. By the Ambrosetti-Rabinowitz theorem [12], problem (21) has a ground state solution for any $\varepsilon > 0$.

The next step in the proof is to establish the existence of multiple solutions for the modified problem. This is achieved by applying the Ljusternik-Schnirelmann category theorem after observing a relation between the topology of M and the number of solutions of the modified problem. The final step is to complete the proof of Theorem 3. The idea is to observe that any solution u_ε of the modified problem satisfies

$$|u_\varepsilon(x)|^2 \leq a \text{ for } x \in \Lambda^c_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$$

for ε small. The proof of Theorem 3 follows by combining this estimate with the diamagnetic inequality via Nehari manifold analysis.

[A11] Concentration of ground states for double phase problems with competing potentials

Background. C. Alves and G. Figueiredo [8] studied the multiplicity and concentration of solutions for the following problem with linear potential

$$\begin{cases} -\Delta_p u - \Delta_q u + V(\epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N, \end{cases}$$

which is equivalent to the problem

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = f(u), & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), u > 0, & \text{in } \mathbb{R}^N. \end{cases}$$

Here, the authors assumed that the potential V satisfies the following global condition introduced by P. Rabinowitz [113]

$$0 < \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to \infty} V(x) < \infty,$$

and the nonlinear term f has C^1-smoothness with superlinear and subcritical growth. Using mountain pass arguments combined with the Ljusternik-Schnirelmann category theory, they proved the existence and multiplicity of positive solutions which concentrate at global minimum points of the potential V. Subsequently, the multiplicity result in [8] has been improved by V. Ambrosio and D. Repovš [13] by considering continuous nonlinearities.

The main purpose in paper [A11] is to develop a related analysis in the case of *two competing potentials*, namely an absorption potential V and a reaction potential K. A key problem in this new abstract setting is to establish the relationship between the number of positive solutions and the topology of the set where V attains its global minimum and K attains its global maximum.

Results. Paper [A11] deals with the study of the following perturbed double phase problem with competing potentials:

$$\begin{cases} -\epsilon^p \Delta_p u - \epsilon^q \Delta_q u + V(x)(|u|^{p-2}u + |u|^{q-2}u) = K(x)f(u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{23}$$

where $1 < p < q < N$, ϵ is a small positive parameter, V and K are potential functions and f is the reaction term with subcritical growth.

The features of paper [A11] are the following:

(1) the presence of several differential operators with different growth, which generate a double phase associated energy;

(2) the problem combines the multiple effects generated by *two* variable potentials;

(3) there exists a competition effect between the absorption potential and the reaction potential, which implies more complex phenomena to locate the concentration positions;

(4) the main concentration phenomenon creates a bridge between the global maximum point of the solution versus the global maximum of the reaction potential and the global minimum of the absorption potential;

(5) due to the unboundedness of the domain, the Palais-Smale sequences do not have the compactness property.

It seems that [A11] is the first work dealing with concentration properties for *double phase* problems in the presence of *two* competing potentials.

The subcritical nonlinearity f satisfies the Ambrosetti-Rabinowitz condition and it has a $(p-1)$-supercritical growth near the origin and a $(q-1)$-supercritical growth at $+\infty$.

Set

$$V_{\min} = \min V,\ \mathcal{V} = \{x \in \mathbb{R}^N : V(x) = V_{\min}\} \text{ and } V_\infty = \liminf_{|x|\to\infty} V(x)$$

$$K_{\max} = \max K,\ \mathcal{K} = \{x \in \mathbb{R}^N : K(x) = K_{\max}\} \text{ and } K_\infty = \limsup_{|x|\to\infty} K(x).$$

The potentials V and K satisfy the following conditions:

(A_0) $V, K \in C(\mathbb{R}^N, \mathbb{R})$ are bounded, $V_{\min} := \inf V > 0$ and $K_{\min} := \inf K > 0$;

(A_1) $V_{\min} < V_\infty$ and there is $x_v \in \mathcal{V}$ such that $K(x_v) \geq K(x)$ for all $|x| \geq R$ and some large $R > 0$;

(A_2) $K_{\max} > K_\infty$ and there is $x_k \in \mathcal{K}$ such that $V(x_k) \leq V(x)$ for all $|x| \geq R$ and some large $R > 0$;

(A_3) $V, K \in C(\mathbb{R}^N, \mathbb{R})$ are bounded functions such that $0 < V^\infty := \lim_{|x|\to\infty} V(x) \leq V(x)$ and $0 < K(x) \leq K^\infty := \lim_{|x|\to\infty} K(x)$, and $|\mathcal{V}| > 0$ or $|\mathcal{K}| > 0$, where

$$\mathcal{V} = \{x \in \mathbb{R}^N : V^\infty < V(x)\} \text{ and } \mathcal{K} = \{x \in \mathbb{R}^N : K^\infty > K(x)\}.$$

To describe the concentration of positive ground state solutions, we define sets that relate the competing potentials V and K:

$$\mathcal{A}_v := \{x \in \mathcal{V} : K(x) = K(x_v)\} \cup \{x \notin \mathcal{V} : K(x) > K(x_v)\},$$

and

$$\mathcal{A}_k := \{x \in \mathcal{K} : V(x) = V(x_k)\} \cup \{x \notin \mathcal{K} : V(x) < V(x_k)\}.$$

This kind of structure was introduced by Y. Ding and X. Liu [51] and it generalizes the global condition introduced by P. Rabinowitz [113].

Under the previous hypotheses, the existence of solutions for small $\varepsilon > 0$, their concentration properties, as well as their asymptotic behaviour is obtained in the first main result of [A11]. This property establishes an interesting relationship between the maximum points of the ground solutions, the minimum points of the absorption potential V and the maximum points of the reaction potential K. Roughly speaking, the ground state solutions concentrate at such points x_0 where $V(x_0)$ is small or $K(x_0)$ is large. As a special case, we can show that these ground state solutions concentrate around such points which are both the minima points of the potential V and the maximum points of the potential K.

Theorem 4. *For all small $\epsilon > 0$, the following properties hold:*

(i) *problem* (23) *has at least a positive ground state solution u_ϵ;*

(ii) *$\mathcal{L}_\epsilon$ is compact, where $\mathcal{L}_\epsilon$ denotes the set of all positive ground state solutions;*

(iii) *$u_\epsilon(x)$ possesses a maximum point x_ϵ such that, up to a subsequence, $x_\epsilon \to x_0$ as $\epsilon \to 0$, and $\lim_{\epsilon\to 0} dist(x_\epsilon, \mathcal{A}_v) = 0$, and $v_\epsilon(x) := u_\epsilon(\epsilon x + x_\epsilon)$ converges to a ground state solution of*

$$-\Delta_p u - \Delta_q u + V(x_0)(|u|^{p-2}u + |u|^{q-2}u) = K(x_0)f(u) \text{ in } \mathbb{R}^N.$$

In particular, if $\mathcal{V} \cap \mathcal{K} \neq \emptyset$, then $\lim_{\epsilon\to 0} dist(x_\epsilon, \mathcal{V} \cap \mathcal{K}) = 0$, and up to a subsequence, v_ϵ converges to a ground state solution of

$$-\Delta_p u - \Delta_q u + V_{\min}(|u|^{p-2}u + |u|^{q-2}u) = K_{\max}f(u) \text{ in } \mathbb{R}^N.$$

(iv) *We have $\lim_{|x|\to\infty} u_\epsilon(x) = 0$ and $u_\epsilon \in C^{1,\sigma}_{loc}(\mathbb{R}^N)$ with $\sigma \in (0,1)$. Furthermore, there exist positive constants c, C such that*

$$u_\epsilon(x) \le C \exp\left(-\frac{c}{\epsilon}|x - x_\epsilon|\right).$$

Another interesting problem is to investigate the relationship between the number of positive solutions and the topology of the set where V attains its global minimum and K attains its global maximum. For this purpose, let us assume that $\mathcal{V} \cap \mathcal{K} \neq \emptyset$ and denote

$$\Lambda := \mathcal{V} \cap \mathcal{V} \text{ and } \Lambda_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, \Lambda) \le \delta\} \text{ for } \delta > 0.$$

Assuming that $\Lambda \neq \emptyset$, then for any $\delta > 0$ there exists $\epsilon_\delta > 0$ such that, for all $\epsilon \in (0, \epsilon_\delta)$, problem (23) has at least $\text{cat}_{\Lambda_\delta}(\Lambda)$ positive solutions. Here, the Ljusternik-Schnirelmann category $\text{cat}_{\Lambda_\delta}(\Lambda)$ is the least number of closed and contractible sets in Λ_δ which cover Λ.

The proofs combine variational and topological methods with related asymptotic estimates. Due to the lack of compactness, an essential question is how to recover the Palais-Smale compactness condition. In this framework we apply the splitting lemma due to V. Ambrosio and D. Repovš [13] and an asymptotic integral estimate of C. Mercuri and M. Willem [88] for Lebesgue functions converging a.e. to 0. A key role is played by the analysis of a related limit problem, which implies an alternative property that involves the ground state energy value for this limit problem. In such a way it is obtained that all bounded Palais-Smale sequences are relatively compact with respect to levels lower than the ground state energy value of the limit problem. The multiplicity of solutions follows by topological arguments associated with the Ljusternik-Schnirelmann theory.

IV. Qualitative analysis of double phase problems

Problems with unbalanced growth have been studied for the first time by J. Ball [17] in relationship with patterns arising in nonlinear elasticity. The mathematical analysis of double-phase variational integrals has been initiated by P. Marcellini [85, 86]. These contributions are in relationship with the work of V. Zhikov [135], in order to describe the behavior of phenomena arising in nonlinear elasticity. In fact, Zhikov intended to provide models for strongly anisotropic materials in the context of homogenisation. These functionals revealed to be important also in the study of duality theory and in the context of the Lavrentiev phenomenon.

Double phase equations are also motivated by models arising in mathematical physics. Indeed, the Born-Infeld differential operator $\text{div}\left((1 - 2|\nabla u|^2)^{-1/2}\nabla u\right)$ that appears in electromagnetism can be approximated by the multi-phase differential operator

$$\Delta u + \Delta_4 u + \frac{3}{2}\Delta_6 u + \cdots + \frac{(2n-3)!!}{(n-1)!}\Delta_{2n} u.$$

Also, the fourth-order relativistic operator $\operatorname{div}(|\nabla u|^2(1-|\nabla u|^4)^{-3/4}\nabla u)$, which describes large classes of phenomena arising in relativistic quantum mechanics, can be approximated by the autonomous double phase operator $\Delta_4 u + \frac{3}{4}\Delta_8 u$.

[A9] Non-autonomous double phase equations

Background. V. Zhikov [135] considered the following unbalanced functional in relationship with the Lavrentiev phenomenon:

$$\mathcal{P}_{p,q}(u) := \int_\Omega (|\nabla u|^p + a(x)|\nabla u|^q)dx, \quad 0 \le a(x) \le L,\ 1 < p < q.$$

In the double phase functional $\mathcal{P}_{p,q}$, the modulating coefficient $a(x)$ dictates the geometry of the composite made by two differential materials, with hardening exponents p and q, respectively. The study of this non-autonomous functionals is characterized by the fact that its energy density changes the ellipticity and growth properties according to the point.

This research field has attracted a lot of interest. For instance, in the the recent the special issue [90] organized by the *Journal of Mathematical Analysis and Applications* and coordinated by G. Mingione and V. Rădulescu there are published 30 papers dealing with recent advances in *non-uniformly elliptic* and *nonstandard growth* problems. The survey paper [91] by G. Mingione and V. Rădulescu presents recent developments on existence and regularity problems involving variational integrals, provided that the integrand does not satisfy a standard p-polynomial growth condition. In the review of this paper for MathSciNet, the reviewer C. Mariconda points out that "the topic still contains many open questions and interactions with other fields like harmonic analysis and function space theory". Cf. MathSciNet, paper [91] is the third most cited paper among all the papers published in 2021.

Results. In [A9] it is studied the following double phase Dirichlet problem

$$\begin{cases} -\Delta_p^a u(z) - \Delta_q u(z) = f(z, u(z)) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{24}$$

where $1 < q < p < N$, $a \in L^\infty(\Omega) \setminus \{0\}$, $a(z) \ge 0$ for a.a. $z \in \Omega$, and Δ_p^a is the weighted r-Laplace differential operator defined by $\Delta_p^a u = \operatorname{div}\left(a(z)|Du|^{p-2}Du\right)$.

The features of paper [A9] are the following:

(i) The source term of problem (24) is driven by a differential operator with a power-type nonhomogeneous term.

(ii) The corresponding energy functional is a non-autonomous variational integral that satisfies nonstandard growth conditions of (p, q)-type.

(iii) The potential that describes the differential operator satisfies general regularity assumptions and it belongs to the p-Muckenhoupt class. Accordingly, the thorough spectral and the qualitative analysis contained in this paper are developed in Musielak-Orlicz-Sobolev spaces.

(iv) The paper covers both the coercive resonant case and the noncoercive (asymptotic resonance or nonresonance) case.

The weight function $a \in C^{0,1}(\overline{\Omega})$ satisfies $a(z) > 0$ for all $z \in \Omega$, while the exponents p and q satisfy $\frac{p}{q} < 1 + \frac{1}{N}$. This condition relating p, q, N is standard in Dirichlet double phase problems and it implies that $p < q^* = \frac{Nq}{N-q}$. This then leads to useful compact embeddings of some relevant function spaces. Moreover, this condition on p, q and N together with the Lipschitz continuity of $a(\cdot)$, implies that the Poincaré inequality is valid for the Musielak-Orlicz-Sobolev space corresponding to the function $\xi(z, t) := a(z)t^p + t^q$.

We denote by $L^\xi(\Omega)$ the Musielak-Orlicz space associated with ξ and let $W^{1,\xi}(\Omega)$ be the corresponding Musielak-Orlicz-Sobolev space.

The analysis of problem (24) depends on the spectral properties of the operator $\left(-\Delta_p^a, W_0^{1,\xi_0}(\Omega)\right)$, where $\xi_0(z,t) = a(z)t^p$, $z \in \overline{\Omega}$, $t \geq 0$. Let $m \in L^\infty(\Omega)$, $m(z) > 0$ for a.a. $z \in \Omega$ and consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p^a u(z) = \hat{\lambda} m(z) a(z) |u(z)|^{p-2} u(z) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < p < N. \end{cases} \tag{25}$$

By a careful analysis of the corresponding Rayleigh quotient, we deduce that problem (25) has a smallest eigenvalue $\hat{\lambda}_1^a(p,m) > 0$ which is simple and every corresponding eigenfunction $\hat{u} \in W_0^{1,\xi_0}(\Omega)$ satisfies $\hat{u} \in L^\infty(\Omega)$ and either $\hat{u}(z) > 0$ or $\hat{u}(z) < 0$ for a.a. $z \in \Omega$.

The first case studied in [A9] corresponds to *coercive resonant problems*. The analysis is developed if f has a subcritical growth and it fulfills an unbalanced behaviour near the origin and at infinity. The hypotheses imply that we have resonance with respect to $\hat{\lambda}_1 > 0$ as $x \to \pm\infty$. The resonance will occur from the left of $\hat{\lambda}_1$ and this implies that the energy functional of the problem is coercive. Also, the hypotheses imply that at zero we have nonuniform nonresonance with respect to $\hat{\lambda}_1(q) > 0$, that is, partial interaction with the spectrum of $\left(-\Delta_q, W_0^{1,q}(\Omega)\right)$.

The main result in the coercive case establishes that problem (24) has at least two nontrivial solutions $u_0, \hat{u} \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega)$. The basic idea in the proof is to observe that the associated non-autonomous energy functional has local linking at the origin with respect to the decomposition $W_0^{1,\xi}(\Omega) = \mathbb{R}\hat{u}_1(q) \oplus \hat{V}$, where $\hat{V} = V \cap W_0^{1,\xi}(\Omega)$ and

$$V = \left\{ u \in W_0^{1,q}(\Omega) : \int_\Omega \hat{u}_1(q)^{q-1} u dz = 0 \right\}, \ \hat{\lambda}_V = \inf \left\{ \frac{\|Du\|_q^q}{\|u\|_q^q} \ u \in V, u \neq 0 \right\}.$$

Since the energy functional is also coercive, then it satisfies the Palais-Smale condition. By the local linking theorem (cf. Theorem 5.4.17 in N. Papageorgiou, V. Rădulescu and D. Repovš [100, p. 410]), problem (24) has at least two nontrivial solutions $u_0, \hat{u} \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega)$.

The rest of the paper is concerned with the *nonresonant case* (nonuniform nonresonance) and then with the *resonant case* (resonance with respect to the principal eigenvalue $\hat{\lambda}_1 > 0$ from the right). The hypotheses imply that large classes of nonlinearities not satisfying the Ambrosetti-Rabinowitz condition are suitable for this analysis. In both cases it is established the existence of a nontrivial solution $u_0 \in W_0^{1,\xi}(\Omega) \cap L^\infty(\Omega)$. The arguments combine Cerami-type compactness, energy estimates, linking theory and Morse theory (critical groups).

[A12] Double phase implicit obstacle problems with convection

Background. Obstacle problems go back to the pioneering work of J. Stefan [119] who studied the temperature distribution in a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees centigrade submerged in water. Obstacle problems are also commonly used in physics, biology, and financial mathematics. Some relevant examples include the dam problem, the Hele-Shaw flow, pricing of American options, quadrature domains and random matrices. The role of obstacle problems in mathematical physics is described in the monograph by J.F. Rodrigues [117], while various relationships with elliptic equations and free boundaries are developed by G. Troianello [127] and A. Petrosyan, H. Shahgholian and N. Uraltseva [106].

In his Fermi Lectures given at the Scuole Normale Superiore di Pisa in 1998, L. Caffarelli [28] studied the classical obstacle problem for the Dirichlet energy $D(u) = \int_\Omega |\nabla u|^2 dx$. More precisely, given a smooth function f on $\partial\Omega$ and a smooth function φ on $\overline{\Omega}$ such that $\varphi|_{\partial\Omega} < f$, one looks for a D-minimizing function u in the class of all Sobolev functions $v \in H^1(\Omega)$ such that $v|_{\partial\Omega} = f$ and $v \geq \varphi$ a.e. on Ω.

Results. Paper [A12] deals with a mixed boundary value problem with a nonhomogeneous double phase differential operator, a nonlinear convection term (a reaction term depending on the gradient), three multivalued terms and an implicit obstacle constraint.

Let Ω be a bounded domain in $\mathbb{R}^N$ such that its boundary $\Gamma := \partial\Omega$ is Lipschitz continuous and it is divided into three mutually disjoint parts Γ_1, Γ_2 and Γ_3 with Γ_1 having positive Lebesgue measure. In our setting the parts Γ_2 and Γ_3 can be empty, that is, Γ_1 could be the whole boundary $\Gamma_1 = \Gamma$. Moreover, let $1 < p < q < N$ and let $\mu\colon \overline{\Omega} \to [0, \infty)$ be a given function, $U_1\colon \Omega \times \mathbb{R} \to 2^{\mathbb{R}}$, $U_2\colon \Gamma_2 \times \mathbb{R} \to 2^{\mathbb{R}}$ be two multivalued mappings, $\phi\colon \Gamma_3 \times \mathbb{R} \to \mathbb{R}$ be a convex function with respect to the second argument and $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a nonlinear convection function. We study the following problem

$$\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) &\in U_1(x,u) + f(x,u,\nabla u) && \text{in } \Omega,\\
u &= 0 && \text{on } \Gamma_1,\\
\frac{\partial u}{\partial \nu_a} &\in U_2(x,u) && \text{on } \Gamma_2,\\
-\frac{\partial u}{\partial \nu_a} &\in \partial_c \phi(x,u) && \text{on } \Gamma_3,\\
L(u) &\leq J(u),
\end{aligned} \tag{26}$$

where

$$\frac{\partial u}{\partial \nu_a} := \left(|\nabla u|^{p-2}\nabla u + \mu(x)|\nabla u|^{q-2}\nabla u\right) \cdot \nu,$$

with ν being the unit normal vector on Γ, $\partial_c\phi(x,u)$ is the convex subdifferential of $s \mapsto \phi(x,s)$, and $L, J\colon W^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ are given functions defined on the Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$.

This multivalued problem (which can be also regarded as a hemivariational inequality) is a nonsmooth (p,q)-problem (in the terminology of P. Marcellini). It includes as particular cases the double phase implicit obstacle system with Dirichlet boundary condition, the obstacle inclusion problem or the mixed boundary value problem without obstacle effect.

The features of paper [A12] are the following:

(i) the presence of a nonhomogeneous differential operator with different isotropic growth, which generates a double phase associated energy;

(ii) the analysis developed in this paper is concerned with the combined effects of a nonstandard operator with unbalanced growth, a convection nonlinearity, three multivalued terms, and an implicit obstacle constraint;

(iii) the proofs rely on fixed point methods for multivalued operators in combination with tools from nonsmooth analysis and theory of pseudomonotone operators.

Further interesting phenomena are the combination of an implicit obstacle effect along with mixed boundary conditions in a very general setting combined with the presence of multivalued mappings and convex subdifferentials. In several critical situations arising in engineering and economic models, such as Nash equilibrium problems with shared constraints, semipermeability problems with free boundary conditions and transport route optimization with feedback control, usually the constraint conditions depend explicitly on the unknown solution. Nowadays such problems with implicit obstacle effect and Clarke generalized gradient for homogeneous Dirichlet problems have been considered by many authors; see, e.g. S. Zeng, Y. Bai, L. Gasiński and P. Winkert [132].

The main contribution in paper [A12] establishes that the solution set of the implicit obstacle problem (26) is nonempty and weakly compact. The proof of the main result uses the Kakutani-Ky Fan fixed point theorem for multivalued operators (cf. N. Papageorgiou, S. Kyritsi-Yiallourou [98, Theorem 2.6.7]) along with the theory of nonsmooth analysis and variational methods for pseudomonotone operators. The main steps are described in what follows. We first introduce the following auxiliary problem

$$\begin{aligned} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u+\mu(x)|\nabla u|^{q-2}\nabla u\right) &= \eta(x)+f(x,u,\nabla u) && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Gamma_1,\\ \frac{\partial u(x)}{\partial \nu_a} &= \xi(x) && \text{on } \partial\Gamma_2,\\ -\frac{\partial u(x)}{\partial \nu_a} &\in \partial_c\phi(x,u) && \text{on } \partial\Gamma_3,\\ L(u) &\le J(w), \end{aligned} \tag{27}$$

where $X=L^p(\Omega)\times L^p(\Gamma_2)$ and $(\eta,\xi)\in X^*:=L^{p'}(\Omega)\times L^{p'}(\Gamma_2)$, $w\in V$ are arbitrary. Then, by applying an existence theorem for a class of mixed variational inequalities involving pseudomonotone operators in which the constraint set is a bounded, closed and convex set, we show that the solution map of the auxiliary problem (27) is well-defined and completely continuous. Moreover, by the Kakutani-Ky Fan fixed point theorem along with the theory of nonsmooth analysis we explore the nonemptiness and compactness of the solution set of problem (26).

V. Nonlocal Kirchhoff, Choquard-Pekar and Stein-Weiss systems

If the standard Laplace or p-Laplace operators have been a stimulating cornerstone of the theory of linear or quasilinear partial differential equations, the twentieth century has seen an increasing and outstanding activity in the study of various nonlocal operators. The analysis of these operators started with the pioneering contributions of G. Kirchhoff [74] in the study of transverse oscillations of a stretched string. More recently, the fractional Schrödinger equation was studied by N. Laskin [75, 76] as a manifestation of fractional quantum mechanics, namely as an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. The literature around the subject of nonlocal problems is huge. Variational and topological methods for the treatment of nonlocal problems are developed in the monograph by G. Molica Bisci, V.D. Rădulescu and R. Servadei [92] (the second most cited book published in 2016, cf. MathSciNet).

This part contains the analysis of three distinct classes of nonlocal equations, respectively two-dimensional critical Kirchhoff equations, ground states of the periodic Choquard-Pekar equation and critical Stein-Weiss systems.

[A2] Planar Kirchhoff equations with critical exponential growth and trapping potential

Background. After the pioneering contributions of J.-L. Lions [82] and S. Pohozaev [107], the following Kirchhoff problem

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(u), & \text{in } \mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N) \end{cases} \tag{28}$$

has been studied intensively by many researchers, where $a, b > 0$, $N \geq 2$, $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$ and $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. By variational methods, a number of important results of the existence and multiplicity of solutions for problem (28) were established under various conditions on V and f, especially when $N \geq 3$. In this framework, the nonlinearities are required to have polynomial growth, and the notion of criticality is associated to the sharp Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ with $2^* := 2N/(N-2)$. Coming to the case $N = 2$, much faster exponential growth is allowed for the nonlinearity and the Trudinger-Moser inequality replaces the sharp Sobolev inequality used for $N \geq 3$. Paper [A2] deals with the critical case if $N = 2$, when the nonlinearity exhibits the exponential growth, which is more complicated than the case $N \geq 3$.

The first result on planar Kirchhoff equation with critical exponential growth is due to G. Figueiredo and U. Severo [45]. Using the mountain pass theorem, they studied the following Kirchhoff equation on a bounded domain $\Omega \subset \mathbb{R}^2$

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{29}$$

and established sufficient conditions for the existence of a positive solution. The same result was obtained by D. Naimen and C. Tarsi [97] if the monotonicity assumption on the reaction used in [45] is replaced by the Ambrosetti-Rabinowitz condition.

Results. Paper [A2] is concerned with the following Kirchhoff equation

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = f(u), & x \in \mathbb{R}^2, \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{$\mathcal{K}$}$$

where $a, b > 0$ and $V \in \mathcal{C}(\mathbb{R}^2, (0, \infty))$ is the Rabinowitz-type trapping potential, namely it satisfies

(V1) $V_0 \leq V(x) \leq \liminf_{|y|\to\infty} V(y) = V_\infty$ for all $x \in \mathbb{R}^2$.

The nonlinearity $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ does not satisfy any monotonicity conditions but it fulfills the following hypotheses:

(F1) there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0, & \text{for all } \alpha > \alpha_0, \\ +\infty, & \text{for all } \alpha < \alpha_0; \end{cases} \tag{30}$$

(F2) $f(t) = o(t)$ as $t \to 0$ and $F(t) := \int_0^t f(s)\mathrm{d}s \geq 0$.

As in Adimurthi and S. Yadava [5], we say that f has critical exponential growth at $t = \pm\infty$ if condition (F1) holds. It was shown by N. Trudinger [128] and J. Moser [96] that this kind of nonlinearity has the maximal growth that can be treated variationally in $H^1(\mathbb{R}^2)$, which is motivated by the following Trudinger-Moser inequality.

Lemma 5. i) *If* $\alpha > 0$ *and* $u \in H^1(\mathbb{R}^2)$, *then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty;$$

ii) *if* $u \in H^1(\mathbb{R}^2), \|\nabla u\|_2^2 \leq 1, \|u\|_2 \leq M < \infty$*, and* $\alpha < 4\pi$*, then there exists a constant* $\mathcal{C}(M, \alpha)$*, which depends only on* M *and* α*, such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \leq \mathcal{C}(M, \alpha). \tag{31}$$

We first consider the following Kirchhoff equation with constant potential

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right) \Delta u + V_\infty u = f(u), \quad x \in \mathbb{R}^2; \\ u \in H^1(\mathbb{R}^2), \end{cases} \tag{$(\mathcal{K})_\infty$}$$

where f satisfies (F1), (F2) and

(F3) $\lim_{t\to+\infty} \frac{F(t)}{t^2} = +\infty$ and $f(t)t \geq 2F(t)$ for all $t \geq 0$,

Let $\Phi^\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ be the associated energy functional defined by

$$\Phi^\infty(u) := \frac{1}{2}\int_{\mathbb{R}^2} \left(a|\nabla u|^2 + V_\infty u^2\right) \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^2} F(u)\mathrm{d}x, \tag{32}$$

and denote by c^∞ the mountain pass level of Φ^∞, namely

$$c^\infty = \inf_{\gamma\in\Gamma^\infty} \max_{t\in[0,1]} \Phi^\infty(\gamma(t)), \tag{33}$$

where

$$\Gamma^\infty := \left\{\gamma \in \mathcal{C}([0,1], H^1(\mathbb{R}^2)) : \gamma(0) = 0, \Phi^\infty(\gamma(1)) < 0\right\}. \tag{34}$$

We say that a solution u of problem $(\mathcal{K})_\infty$ is a *least energy solution* if $\Phi^\infty(u) = m^\infty$ with

$$m^\infty := \inf\left\{\Phi^\infty(u) \;\middle|\; u \in H^1(\mathbb{R}^2) \setminus \{0\}, (\Phi^\infty)'(u) = 0\right\}. \tag{35}$$

Paper [A2] establishes the following qualitative properties.

Theorem 6. *Assume that* f *satisfies* (F1)-(F3)*. Then there exists* $V^* \in (0, +\infty]$ *such that for any* $V_\infty \in (0, V^*)$*, equation* $(\mathcal{K})_\infty$ *has a positive least energy solution. Moreover,* V^* *is equal to the Trudinger-Moser ratio:*

$$C^*_{TM}(F) := \sup\left\{\frac{2}{\|u\|_2^2}\int_{\mathbb{R}^2} F(u)\mathrm{d}x \;\middle|\; u \in H^1(\mathbb{R}^2) \setminus \{0\}, \|\nabla u\|_2^2 \leq \frac{4\pi}{\alpha_0}\right\}.$$

In particular, $V^* = +\infty$ *is equivalent to* $\lim_{t\to+\infty} \frac{t^2 F(t)}{e^{\alpha_0 t^2}} = +\infty$.

Theorem 7. *Under the assumptions of Theorem* 6*, the least energy level* m^∞ *is equal to the mountain pass value* c^∞*. Moreover, for any least energy solution* w *of* $(\mathcal{K})_\infty$*, there exists a path* $\tilde{\gamma} \in \Gamma^\infty$ *such that* $w \in \tilde{\gamma}([0,1])$ *and*

$$\max_{t\in[0,1]} \Phi^\infty(\tilde{\gamma}(t)) = \Phi^\infty(w).$$

The proofs of these results rely on the Pohozaev identity for problem $(\mathcal{K})_\infty$. We introduce the auxiliary functional $J^\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$J^\infty(u) = V_\infty \|u\|_2^2 - 2\int_{\mathbb{R}^2} F(u)\mathrm{d}x, \tag{36}$$

the set

$$\mathcal{P}_\infty := \left\{ u \in H^1(\mathbb{R}^2) \setminus \{0\} \;\middle|\; J^\infty(u) = 0 \right\} \tag{37}$$

and the constrained minimization problem

$$A^\infty := \inf_{u\in\mathcal{P}_\infty} \left(\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 \right) = \inf_{u\in\mathcal{P}_\infty} \Phi^\infty(u). \tag{38}$$

Based on a sufficient and necessary condition for compactness of general nonlinear functionals, it is established that A^∞ can be attained if V^∞ is less than the Trudinger-Moser ratio $C^*_{\mathrm{TM}}(F)$ depending on the Trudinger-Moser inequality. Next, under a suitable change of scale, the minimizer is a *least energy solution* of $(\mathcal{K})_\infty$. Different from the case $N \geq 3$, the minimum A^∞ has no saddle point structure with respect to the fibres $\{u(\cdot/t) : t > 0\} \subset H^1(\mathbb{R}^2)$, $u \in H^1(\mathbb{R}^2)$ since the Pohozaev functional $J^\infty(u)$ does not have a $\|\nabla u\|_2$-component. Thus, it is more complicated to establish the relation among the minimum A^∞, the least energy m^∞ and the mountain pass level c^∞ in the dimension $N = 2$. To address this issue, inspired by the idea of L. Jeanjean and K. Tanaka [67], we construct a new path belonging to Γ^∞ and we derive the mountain pass characterization of the *least energy solution* for $(\mathcal{K})_\infty$.

Paper [A2] is also concerned with the existence of *ground state solutions* for the critical exponential growth Kirchhoff equation $(\mathcal{K})$ with the trapping potential V satisfying (V1). Though this kind of potential has been studied in the literature, it seems that there is no paper associated with Kirchhoff equations dealing with the dimension $N = 2$ when the nonlinearity has critical exponential growth. Some effective methods, treating the dimension $N \geq 3$, do not work in this case due to the simultaneous appearance of the nonlocal term and the nonlinear term with critical exponential growth.

We say that a solution u of $(\mathcal{K})$ is a *ground state solution* (of Nehari type) if $\Phi(u) = c_N$ with

$$c_N := \inf_{u\in\mathcal{N}} \Phi(u) \tag{39}$$

and

$$\mathcal{N} := \{u \in H^1(\mathbb{R}^2) \setminus \{0\} \big| \langle \Phi'(u), u\rangle = 0\}. \tag{40}$$

It is established that if the potential V satisfies (V1) with $V_\infty \in (0, V^*)$, then problem $(\mathcal{K})$ has a ground state solution, where V^* is given by Theorem 6.

The proof is based on the mountain pass theorem. For this purpose, a standard procedure is to prove the boundeness of Cerami sequences, and verify that the weak limit of Cerami sequences is non-trivial and is also a weak solution. Nevertheless, to do that, compared with the previous works dealing with Kirchhoff-type equation (28) involving trapping potential V in $\mathbb{R}^N(N \geq 3)$, some new obstacles arise in the proofs, such as
(i) the lack of the monotonicity condition and the Ambrosetti-Rabinowitz type condition prevent us from using usual methods to prove the boundednss of Cerami sequences;
(ii) it is more difficult to rule out the concentration phenomena and the vanishing phenomena of Cerami sequences;
(iii) it does not work that the BL-splitting property for the energy functional along Cerami sequences

is caused by the appearance of the nonlinear term with critical growth, which is a powerful tool to restore the compactness of Cerami sequences.

[A7] Ground states for the periodic Choquard-Pekar equation

Background. The Choquard-Pekar equation

$$-\Delta u + u = \left(\frac{1}{|x|} * |u|^2\right) u \text{ in } \mathbb{R}^3. \tag{41}$$

was first introduced in the pioneering work of H. Fröhlich [56] and S. Pekar [102] for the modeling of quantum polaron. This model corresponds to the study of free electrons in a ionic lattice interact with photons associated to deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole). As pointed out by E. Lieb [78], Choquard used equation (41) to study steady states of the one component plasma approximation in the Hartree-Fock theory.

The Choquard-Pekar equation is also known as the Schrödinger-Newton equation in models coupling the Schrödinger equation of quantum physics together with nonrelativistic Newtonian gravity. The equation can also be derived from the Einstein-Klein-Gordon and Einstein-Dirac system. Such a model was proposed for boson stars phenomena. R. Penrose [104, 105] proposed equation (41) as a model of self-gravitating matter in which quantum state reduction was understood as a gravitational phenomenon.

Results. Paper [A7] is a Highly Cited paper (cf. Web of Science, October 2022) and it is concerned with the following non-autonomous Choquard-Pekar equation

$$\begin{cases} -\Delta u + V(x)u = (W * F(u))f(u), & x \in \mathbb{R}^N \ (N \geq 2), \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{42}$$

where V, W and f satisfy the following hypotheses:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, \mathbb{R})$, $V(x)$ is 1-periodic in x_i for $i = 1, 2, \ldots, N$, and

$$\sup[\sigma(-\Delta + V) \cap (-\infty, 0)] < 0 < \inf[\sigma(-\Delta + V) \cap (0, \infty)];$$

(W1) $W(x)$ is an even function, and there exist $1 \leq r_1 \leq r_2 < \infty$ such that $W \in L^{r_1}(\mathbb{R}^N) + L^{r_2}(\mathbb{R}^N)$;

(W2) $W(x) \geq 0$ and on a neighborhood of 0 we have $W(x) > 0$;

(W3) there exists $C_0 > 0$ such that for all nonnegative $\varphi, \psi \in L^1_{\text{loc}}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} (W * \varphi)\psi \mathrm{d}x \leq C_0 \sqrt{\int_{\mathbb{R}^N} (W * \varphi)\varphi \mathrm{d}x} \sqrt{\int_{\mathbb{R}^N} (W * \psi)\psi \mathrm{d}x};$$

(F1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and there exist $\mathcal{C}_0 > 0$ and $p_1, p_2 > 1$ with $(2r_2 - 1)/r_2 < p_1 \leq p_2 < (2r_1 - 1)2^*/2r_1$ such that for all $t \in \mathbb{R}$,

$$|f(t)| \leq \mathcal{C}_0(|t|^{p_1 - 1} + |t|^{p_2 - 1}); \tag{43}$$

(F2) $F(t) \geq 0$ and $\lim_{|t| \to \infty} \frac{F(t)}{|t|} = \infty$;

(F5) $f(-t) = -f(t)$ for all $t \in \mathbb{R}$.

According to J. Fröhlich, T.P. Tsai and H.T. Yau [57], if the particle interaction is attractive (that is, $W \geq 0$), problem (42) turns into (41) with $f(u) = u$, $N = 3$ and $V \equiv 1$, which also arises in the Hartree theory of bosonic systems.

In this paper we address the following

Problem. *Study problem* (42) *by getting rid of the Cauchy-Schwarz type inequality* (W3)*, which is a crucial hypothesis of the proof developed by N. Ackermann [4]. In this case, how to obtain the existence of ground state solutions to problem* (42) *without using the classical Ambrosetti-Rabinowitz condition.*

The analysis developed in [A7] brings some new difficulties both in the verification of the linking geometry and for proving the boundedness of Cerami sequences for the energy functional. Employing some new techniques and introducing some generic conditions on f, we obtain the existence of ground state solution for equation (42). We are also concerned with the existence of infinitely many geometrically distinct solutions of problem (42). Note that multiplicity results in [4, 19] were constructed by using pseudo-index and the $(PS)_I$-attractor or $(C)_I$-attractor in order to obtain the deformations. Moreover, instead of dealing directly with the different exponents r_1, r_2, p_1, p_2 in (W1) and (F1), it suffices to split W and F into a sum of functions, cf. [4]. One may ask whether is possible to consider the superquadratic part Ψ directly and to employ a direct approach to show the multiplicity results. In this paper we give an affirmative answer. First, we construct a profile decomposition of bounded sequences in $H^1(\mathbb{R}^N)$ in order to analyse Cerami sequences. Next, applying the Ljusternik-Schnirelmann theory and using deformation arguments, we obtain the existence of infinitely many geometrically distinct solutions of problem (42).

Consider the following additional hypotheses.

(F3) $\widehat{\mathcal{F}}(t) := f(t)t - F(t) \geq 0$, and there exist $c_1 > 0$, $\max\left\{1, \frac{(2r_1-1)N}{(N+2)r_1-N}\right\} < \kappa \leq \frac{p_2}{p_2-1}$ such that

$$|f(t)|^\kappa \leq c_1 \widehat{\mathcal{F}}(t), \qquad \forall\, t \in \mathbb{R};$$

(F3′) $\widehat{\mathcal{F}}(t) := f(t)t - F(t) \geq 0$, and there exist $c_1 > 0$, $\max\left\{1, \frac{(2r_1-1)N}{(N+2)r_1-N}\right\} < \kappa_i \leq \frac{2r_2-1}{r_2-1}$, $i = 1, 2$, such that

$$|f(t)\chi_{[0,1)}(|t|)|^{\kappa_1} + |f(t)\chi_{[1,+\infty)}(|t|)|^{\kappa_2} \leq c_1 \widehat{\mathcal{F}}(t), \qquad \forall\, t \in \mathbb{R};$$

(F4) $\lim_{t\to 0} \frac{f(t)}{t}$ exists.

We remark that condition (F3′) is weaker both than (F3) and than the Ambrosetti-Rabinowitz condition (AR).

The main results in paper [A7] are the following.

Theorem 8. *Assume that V, W and f satisfy* (V1), (W1), (W2), (F1), (F2) *and* (F3)*. Then problem* (42) *has a solution $\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{K}} \Phi > 0$, where $\mathcal{K} := \{u \in H^1(\mathbb{R}^N) \setminus \{0\} : \Phi'(u) = 0\}$. If, in addition, f satisfies* (F4) *and* (F5)*, then problem* (42) *possesses infinitely many pairs of geometrically distinct solutions $\pm u$.*

Theorem 9. *Assume that V, W and f satisfy* (V1), (W1), (W2), (W3), (F1), (F2) *and* (F3′)*. Then problem* (42) *has a solution $\bar{u} \in H^1(\mathbb{R}^N) \setminus \{0\}$ such that $\Phi(\bar{u}) = \inf_{\mathcal{K}} \Phi > 0$. If, in addition, f satisfies* (F4) *and* (F5)*, then problem* (42) *possesses infinitely many pairs of geometrically distinct solutions $\pm u$.*

I now sketch the proof of these results. In order to obtain the existence of ground state solutions for problem (42), we first construct the linking structure of the associated energy functional Φ. By the linking theorem, we find the Cerami sequences for Φ. The boundedness of these sequences is proved by virtue of the technical conditions (F3) and (F3′). Next, by applying a concentration compactness argument, we find nontrivial solutions of problem (42). Finally, we constrain the functional Φ on the critical point set $\mathcal{K}$ and we show that the corresponding infimum is positive, then ground states of (42) are obtained by a standard argument. The proof of multiplicity results is carried out via the Ljusternik-Schnirelmann theory and deformation arguments. In order to obtain the deformations, we use the decomposition of Φ along the Cerami sequences and verify the discreteness of such sequences. This requires a deep analysis of the profile decomposition of bounded sequences in $H^1(\mathbb{R}^N)$ due to different exponents r_1, r_2, p_1, p_2 that appear in (W1) and (F1). Next, arguing by contradiction, we succeed in establishing the existence of infinitely many geometrically distinct solutions for problem (42).

Paper [A7] also contains an existence theorem for the following N-dimensional Choquard equation with Riesz potential

$$-\Delta u + V(x)u = (I_\alpha * F(u))f(u), \quad u \in H^1(\mathbb{R}^N), \tag{44}$$

where $N \geq 3$, $\alpha \in (0, N)$ and $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{\frac{N}{2}} 2^\alpha |x|^{N-\alpha}}, \quad x \neq 0.$$

[A10] Critical Stein-Weiss elliptic systems

Background. A central role in the analysis of many singular phenomena is played by the fractional integral

$$(T_\mu \phi)(x) = \int_{\mathbb{R}^N} \frac{\phi(y)}{|x-y|^\mu}\, dy, \quad 0 < \mu < N.$$

Weighted L^p estimates for T_μ is a fundamental problem of harmonic analysis, with a wide range of applications. Starting from the classical one-dimensional case studied by G.H. Hardy and J. Littlewood, an exhaustive analysis has been made on the admissible classes of weights and ranges of indices. A detailed description is developed by E. Stein [120]. In the special case of power weights the optimal result is due to E. Stein and G. Weiss [121], which established a general weighted inequality, which is now called the *Stein-Weiss inequality.* This inequality extends the classical Hardy-Littlewood-Sobolev inequality. In this case, E. Lieb [79] applied the Riesz rearrangement inequalities to prove that the best constant for the classical Hardy-Littlewood-Sobolev inequality can be achieved by some extremals.

Nonlinear systems with Stein-Weiss convolutions have attracted a lot of interest in the last few years. I refer to the contributions of S. Peng [103] (Liouville theorems for coupled fractional elliptic system with Stein-Weiss convolution), M. Yang and X. Zhou [129] (coupled Schrödinger system with Stein-Weiss convolution) and Y. Zhang and X. Tang [134] (magnetic system with Stein-Weiss convolution).

Results. The weighted Stein-Weiss inequality is the following.

Proposition 10. *Let $1 < t, r < \infty$, $0 < \mu < N$, $\alpha + \beta \geq 0$ and $0 < \alpha + \beta + \mu \leq N$, $f \in L^t(\mathbb{R}^N)$, and $h \in L^r(\mathbb{R}^N)$. Then there exists a sharp constant $C_{t,r,\alpha,\beta,\mu,N}$ such that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x|^\alpha |x-y|^\mu |y|^\beta} dxdy \leq C(t, r, N, \mu, \alpha, \beta) \|f\|_t \|h\|_r,$$

where

$$\frac{1}{t}+\frac{1}{r}+\frac{\alpha+\beta+\mu}{N}=2$$

and

$$1-\frac{1}{t}-\frac{\mu}{N}<\frac{\alpha}{N}<1-\frac{1}{t},$$

where C is independent of f and h. Moreover, for any $h\in L^r(\mathbb{R}^N)$, we have

$$\|\int_{\mathbb{R}^N}\frac{h(y)}{|x|^\alpha|x-y|^\mu|y|^\beta}dy\|_s\le C(s,N,\mu,\alpha,\beta)\|h\|_r,$$

where s satisfies $1+\frac{1}{s}=\frac{1}{r}+\frac{\alpha+\beta+\mu}{N}$ and $\frac{\alpha}{N}<\frac{1}{s}<\frac{\alpha+\mu}{N}$.

E. Lieb [79] proved the existence of a sharp constant, provided that either one of r and t equals 2 or $r=t$. For $1<r,t<\infty$ with $\frac{1}{r}+\frac{1}{t}=1$, the sharp constant is given by W. Beckner [21]. The corresponding Euler-Lagrange equations for the Stein-Weiss inequality are the system of integral equations

$$\begin{cases}u(x)=\displaystyle\int_{\mathbb{R}^N}\frac{v^q(y)}{|x|^\alpha|x-y|^\mu|y|^\beta}dy,\\ v(x)=\displaystyle\int_{\mathbb{R}^N}\frac{u^p(y)}{|x|^\beta|x-y|^\mu|y|^\alpha}dy,\end{cases}\tag{45}$$

where $0<p,q<+\infty$, $0<\mu<N$, $\frac{\alpha}{N}<\frac{1}{p+1}<\frac{\mu+\alpha}{N}$ and $\frac{1}{p+1}+\frac{1}{q+1}=\frac{\mu+\alpha+\beta}{N}$.

Inspired by these results, paper [A10] deals with several classes of nonlocal elliptic systems with weighted Stein-Weiss convolution. It is first studied the following nonlocal Hartree system *without a variational structure*

$$\begin{cases}-\Delta u=\dfrac{1}{|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N}\frac{v^p(y)}{|x-y|^\mu|y|^\alpha}dy\right)u^q,\\ -\Delta v=\dfrac{1}{|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N}\frac{u^q(y)}{|x-y|^\mu|y|^\alpha}dy\right)v^p,\end{cases}\tag{46}$$

where $N\ge 3$, $\alpha\ge 0$, $0<\mu<N$, p, $q>1$ and $0<2\alpha+\mu\le N$.

This is related with the critical nonlocal Hartree equation

$$-\Delta u=\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^{2^*_{\alpha,\mu}}}{|x-y|^\mu|y|^\alpha}dy\right)|u|^{2^*_{\alpha,\mu}-2}u,\quad x\in\mathbb{R}^N,\tag{47}$$

which is a special case of the weighted Choquard equation

$$-\Delta u=\frac{1}{|x|^\alpha}\left(\int_{\mathbb{R}^N}\frac{|u(y)|^p(y)}{|x-y|^\mu|y|^\alpha}dy\right)|u|^{p-2}u,\quad x\in\mathbb{R}^N.\tag{48}$$

By translating the system (46) into an equivalent integral system with Riesz potential, we apply a regularity lifting lemma to obtain the regularity of the solutions and the moving plane methods in integral form to study the symmetry of positive solutions. More precisely, if $(u,v)\in L^{s_0}(\mathbb{R}^N)\times L^{s_0}(\mathbb{R}^N)$ is a pair of positive solutions of system (46) with $s_0=\frac{N(p+q-1)}{N+2-2\alpha-\mu}$, then u and v are radially symmetric and decreasing about the origin.

Next, we consider the *critical* setting $p+q=2\cdot 2^*_{\alpha,\mu}-1$. Thus, $s_0=\frac{2N}{N-2}$, and hence $(u,v)\in L^{2^*}(\mathbb{R}^N)\times L^{2^*}(\mathbb{R}^N)$. Applying the regularity lifting lemma of W. Chen, C. Li and C. Ma [84], we can improve the regularity properties of solutions. In such a way, if $3 \le N \le 6$ and $(u,v) \in L^{\frac{2N}{N-2}}(\mathbb{R}^N) \times L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ is a pair of positive solutions of system (46), where p, q satisfy

$$\frac{2(N-2\alpha-\mu)}{N-2} \le p, q \le \min\left\{\frac{4}{N-2}, \frac{N+6-2(2\alpha+\mu)}{N-2}\right\},$$

then $(u,v) \in L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ with $s \in \left(\frac{N}{N-2}, +\infty\right)$. At the same time, if $0 \le \alpha < 2$, then both solutions u and v are bounded and $u, v \in C^\infty(\mathbb{R}^N - \{0\})$. The growth of solutions at infinity is given by

$$u(x) \backsimeq \frac{C}{|x|^{N-2}} \text{ and } v(x) \backsimeq \frac{C}{|x|^{N-2}} \text{ for large } |x|.$$

Next, paper [A10] deals with the following nonlocal system with *variational structure*

$$\begin{cases} -\Delta u = \dfrac{1}{p|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N} \frac{v^p(y)}{|x-y|^\mu |y|^\alpha} dy\right) u^{q-1}, \\ -\Delta v = \dfrac{1}{q|x|^\alpha}\left(\displaystyle\int_{\mathbb{R}^N} \frac{u^q(y)}{|x-y|^\mu |y|^\alpha} dy\right) v^{p-1}, \end{cases} \tag{49}$$

where $N \ge 3$, $\alpha \ge 0$, $0 < \mu < N$, p, $q > 1$ and $0 < 2\alpha + \mu \le N$.

We are first concerned with the nonexistence of positive solutions to system (49) in the *critical* case, namely if $p + q = 2 \cdot 2^*_{\alpha,\mu}$. We first prove that system (49) has no positive solutions in the subcritical case. In the subcritical case if $p + q < 2 \cdot 2^*_{\alpha,\mu}$ we obtain a Pohozaev-type identity with Stein-Weiss potential. This result is essential to prove that if $(u,v) \in W^{2,2}_{loc}(\mathbb{R}^N) \times W^{2,2}_{loc}(\mathbb{R}^N)$ is a pair of solutions of (49) and $p + q < 2 \cdot 2^*_{\alpha,\mu}$, then $u \equiv v \equiv 0$.

Next, assuming that u, v are integrable positive solutions of system (49) belonging to $L^{s_0}(\mathbb{R}^N)$ with $s_0 = \frac{N(p+q-2)}{N+2-2\alpha-\mu}$, we establish that u and v are radially symmetric and decreasing about the origin. The proof follows by using moving plane arguments in integral form.

In the case $N \in \{3, 4, 5, 6\}$ and for the critical case $p + q = 2 \cdot 2^*_{\alpha,\mu}$, it is established a regularity property of positive solutions (u,v) to system (49). Namely, if

$$\frac{2(N-2\alpha-\mu)}{N-2} \le p-1, q-1 \le \min\left\{\frac{4}{N-2}, \frac{N+2+2(N+2-2\alpha-\mu)}{N-2}\right\},$$

then $(u,v) \in L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ with $s \in \left(\frac{N}{N-2}, +\infty\right)$. This follows by combining iterative arguments and singular integral analysis.

In the final part of paper [A10] it is studied the following Hamiltonian system with Stein-Weiss convolution

$$\begin{cases} -\Delta u = \dfrac{1}{|x|^{\alpha_1}}\left(\displaystyle\int_{\mathbb{R}^N} \frac{v^p(y)}{|x-y|^{\mu_1} |y|^{\alpha_1}} dy\right) v^{p-1}, \\ -\Delta v = \dfrac{1}{|x|^{\alpha_2}}\left(\displaystyle\int_{\mathbb{R}^N} \frac{u^q(y)}{|x-y|^{\mu_2} |y|^{\alpha_2}} dy\right) u^{q-1}, \end{cases} \tag{50}$$

where $N \ge 3$, $0 < \mu_1, \mu_2 < N$, α_1, $\alpha_2 \ge 0$, $0 < 2\alpha_1 + \mu_1 \le N$, $0 < 2\alpha_2 + \mu_2 \le N$ and p, $q > 1$. In the particular case $(p,q) = \left(\frac{2N-2\alpha_1-\mu_1}{N-2}, \frac{2N-2\alpha_2-\mu_2}{N-2}\right)$, if $(u,v) \in D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$ is a pair of positive solutions of (50), then u and v are radially symmetric and decreasing about the origin. The symmetry follows by applying the moving plane method in integral form, after reducing the system (50) to an equivalent integral system in $\mathbb{R}^N$.

Perspectives and open problems

(a) *Double phase Baouendi-Grushin operators.* The Baouendi-Grushin operator is a hypoelliptic operator extending the Tricomi operator and which is defined in $\mathbb{R}^{n+m}$ by $\Delta_G u := \Delta_x u + |x|^\alpha \Delta_y u$, where Δ_x and Δ_y stand for the standard Laplace operators on $\mathbb{R}^n$, respectively $\mathbb{R}^m$, α is a positive number and $u = u(x, y) \in \mathbb{R}^{n+m}$. The energy associated with this operator is of the type

$$u \mapsto \int_{\mathbb{R}^{n+m}} (|\nabla_x u|^2 + |x|^\alpha |\nabla_y u|^2) dx.$$

The study of this operator is motivated by the pioneering papers by M.S. Baouendi [18] and V.V. Grushin [62]. Relevant applications to transonic flow problems have been studied by C. Morawetz [93, 94]. The flow is supersonic in the elliptic region, while a shock wave is created at the boundary between the elliptic and hyperbolic regions.

A. Bahrouni, V.D. Rădulescu and D. Repovš [15, 16] extended the Baouendi-Grushin operator to the anisotropic setting and studied models described by the operator

$$\Delta_{G(x,y)} u = \operatorname{div}\left(\nabla_{G(x,y)} u\right) = \sum_{i=1}^{n} \left(|\nabla_x|^{G(x,y)-2} u_{x_i}\right)_{x_i} + |x|^\gamma \sum_{i=1}^{m} \left(|\nabla_y|^{G(x,y)-2} u_{y_i}\right)_{y_i},$$

where $G : \mathbb{R}^{n+m} \to (1, \infty)$ is a continuous function.

Here,

$$\nabla_{G(x,y)} u = \mathcal{A}(x) \begin{bmatrix} |\nabla_x|^{G(x,y)-2} & \nabla_x u \\ \\ |x|^\gamma |\nabla_y|^{G(x,y)-2} & \nabla_y u \end{bmatrix}$$

and

$$\mathcal{A}(x) = \begin{bmatrix} I_n & 0_{n,m} \\ \\ 0_{m,n} & |x|^\gamma I_m \end{bmatrix} \in \mathcal{M}_{N \times N}(\mathbb{R}).$$

This operator is degenerate along the m-dimensional subspace $M := \{0\} \times \mathbb{R}^m$ of $\mathbb{R}^N$. The analysis associated with this operator is of the type

$$u \mapsto \int_\Omega \frac{1}{G(x,y)} \left[|\nabla_x u|^{G(x,y)} + |x|^\gamma \, |\nabla_y u|^{G(x,y)}\right] dx\, dy, \tag{51}$$

which can be regarded as a degenerate anisotropic double phase energy.

The analysis developed in papers [A9] and [A11] corresponds to a double phase energy driven by the (p, q)-Laplace operator. I consider that a very interesting new research direction is to extend this analysis to the (isotropic or anisotropic) Baouendi-Grushin operator. At the same time, I do not have any information about double phase nonlocal problems driven by a fractional Baouendi-Grushin operator. Such analysis will extend in a substantial way the main contributions in [A1].

Since the energy functionals introduced in (51) has a degenerate action on the set where the gradient vanishes, it is a natural question to study what happens if the integrand is modified in such a way that, if $|\nabla u|$ is also small, there exists an imbalance between the two terms of every integrand.

(b) *Magnetic Baouendi-Grushin operators.* L. Aermark and A. Laptev [6] introduced the following Baouendi-Grushin operator with a magnetic field of Aharanov-Bohm type:

$$G_{\mathcal{A}} := -(\nabla_G + i\beta\mathcal{A}_0)^2 \quad \text{for } -\frac{1}{2} \le \beta \le \frac{1}{2},$$

where

$$\mathcal{A}_0 = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4) = \left(-\frac{\partial_y d}{d}, \frac{\partial_x d}{d}, -2y\frac{\partial_t d}{d}, 2x\frac{\partial_t d}{d}\right),$$

$$\nabla_G = (\partial_x, \partial_y, 2x\partial_t, 2y\partial_t),$$

with $z = (x, y)$, $|z| = \sqrt{x^2 + y^2}$, and $d(z, t) = (|z|^4 + t^2)^{1/4}$ is the Kaplan distance.

The analysis developed in [A6] is done for problems driven by the nonlinear magnetic Schrödinger operators. I appreciate that an interesting new research direction corresponds to a related analysis in the framework of the magnetic operator $G_{\mathcal{A}}$.

(c) *Problems with mixed subcritical-critical-supercritical regime.* Consider the Lane-Emden equation with Dirichlet boundary condition

$$\begin{cases} -\Delta_p u = |u|^{q-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega, \end{cases} \tag{52}$$

where $p, q \in (1, \infty)$.

Usually, the analysis of this problem is developed in relationship with the values of q with respect to the Sobolev critical exponent p^* of p and we distinguish the following situations:

(i) $q < p^*$ (*subcritical case*);

(ii) $q = p^*$, provided that $1 < p < N$ (*critical case*);

(iii) $q > p^*$, provided that $1 < p < N$ (*supercritical case*).

In the case of variable exponents, the Lane-Emden problem (52) becomes

$$\begin{cases} -\Delta_{p(x)} u = |u|^{q(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \not\equiv 0 & \text{in } \Omega. \end{cases} \tag{53}$$

Problem (53) can fulfill a mixed "subcritical-critical-supercritical" triple regime, in the sense that $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ and

$$\begin{aligned} q(x) &< p^*(x) \quad \text{if } x \in \Omega_1, \\ q(x) &= p^*(x) \quad \text{if } x \in \Omega_2, \\ q(x) &> p^*(x) \quad \text{if } x \in \Omega_3. \end{aligned}$$

The study of these problems was initiated by C. Alves and V.D. Rădulescu [11] but the analysis is far to be complete. A feature of this new abstract setting is that, under general hypotheses, the associated energy functional is not coercive or it does not have a mountain pass geometry. Also, the presence of critical and supercritical regions implies serious problems due to the lack of compactness of

Sobolev embeddings. I propose to develop this research both in the local case (to extend results from my joint monograph [116] with D.D. Repovš) but also in the nonlocal setting.

(d) *Anisotropic nonlocal problems.* It seems that the results obtained in the papers [A1] and [A11] have not been extended to anisotropic nonlocal problems. They should be the local analog of those that in the local case are given by functionals of the type

$$w \mapsto \int_\Omega \varphi_1(|Dw|) + \varphi_2(|Dw|)\, dx,$$

where the conditions satisfied by $\varphi_i(t)$ are typically given by

$$1 < i \leq \frac{\varphi_i'(t)t}{\varphi_i(t)} \leq s. \tag{54}$$

For instance, in the setting corresponding to paper [A1] we have $\varphi_1(t) = t^p$ and $\varphi_2(t) = t^q$.

(e) *Superlinear and critical Stuart-type problems.* Problems [A5] is concerned with a Dirichlet problem driven by the Stuart differential operator and there are studied the cases where the reaction is either sublinear or it has a linear growth at infinity. The cases where the nonlinearity is either superlinear or critical are still open.

(f) *Anisotropic double phase integrands.* Paper [A9] addresses the study of unbalanced variational integrals of the type

$$u \mapsto \int_\Omega (a(x)|\nabla u|^p + |\nabla u|^q)\, dx. \tag{55}$$

As pointed out in my joint paper [133, p. 201] with Q. Zhang, a very interesting research direction concerns the study of energy functionals like (55) for anisotropic integrands of the type

$$\Phi(x, |\xi|) = \begin{cases} a(x)|\xi|^p + |\xi|^q & \text{if } |\xi| \leq 1, \\ a(x)|\xi|^{p_1} + |\xi|^{q_1} & \text{if } |\xi| \geq 1. \end{cases}$$

4 Scientific activity carried out at more than one university

Presentation of significant scientific or artistic activity carried out at more than one university, scientific or cultural institution, especially at foreign institutions

4.1 Extended research visits

1. December 15, 1997–February 15, 1998: Universities of Sussex and Oxford, with a Royal Society Research Fellowship

2. 1998-2001 (4 months every year): PAST Visiting Professor at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (now, Paris Sorbonne University)

3. September 1–November 30, 2002: Université de Savoie–Chambéry with a CNRS research visiting position (Poste Rouge)

4. January 6–July 4, 2014: Isaac Newton Institute, Cambridge, Programme *Free Boundary Problems and Related Topics* (G.-Q. Chen, H. Shahgholian, J.-L. Vázquez, organizers)

Prior to the award of the Ph.D.:

1. January 1992 – July 1994 (6 months every year): PhD Scholarship at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (now Paris Sorbonne University), funded by the European Community
2. January 1995 – July 1995: PhD Scholarship at the Laboratoire d'Analyse Numérique, Université Pierre et Marie Curie–Paris 6 (now, Paris Sorbonne University), funded by the French Government

4.2 Short term research visits

1. Politecnico di Milano (March 1996, with a CNR research grant)
2. Freie Universität in Berlin (two weeks in May 1996)
3. Aristotle University in Thessaloniki (June 1996)
4. Leiden University (October and November 1996)
5. Università Cattolica di Brescia (March 1997, with a CNR research grant)
6. Aristotle University in Thessaloniki (May 15 - June 15, 1997)
7. Université Catholique de Louvain (Belgium) in November 1998
8. University of Perugia (Nov. 15 - Dec. 15, 1999, with a CNR research grant)
9. Université Catholique de Louvain (Belgium) in October 2001
10. Université de Picardie "Jules Verne", Amiens (February 2002)
11. Politecnico di Milano (June–July 2002, with a GNAMPA–INdAM Visiting Professor position)
12. Central-European University, Budapest (10 days in September 2002)
13. Université de Picardie "Jules Verne", Amiens (February 2003)
14. Université de Tunis El Manar (two weeks in April 2003)
15. Institut Elie Cartan, Université Henri Poincaré (Nancy I) (May 2003)
16. Mathematisches Institut, Basel Universität (two weeks in June 2003)
17. Université de Perpignan (July 2003)
18. Université de Picardie "Jules Verne", Amiens (February 2004)
19. Université de Savoie–Chambéry (two weeks in March 2004)
20. Université de Tunis El Manar (two weeks in April 2004)
21. Université Catholique de Louvain (Belgium) in November 2004
22. Université de Picardie "Jules Verne", Amiens (February 2005)

23. Universidad Complutense de Madrid (one week in March 2005)
24. City University of Hong Kong (two weeks in April 2005)
25. Université de Tunis El Manar (two weeks in May 2005)
26. Université de Franche Comté and Université de Limoges (two weeks in November 2005)
27. Université de Picardie “Jules Verne”, Amiens (February 2006)
28. Université de Tunis El Manar (one week in May 2006)
29. Université de Poitiers (June 2006)
30. Université de Savoie (two weeks in August 2006)
31. Central European University in Budapest (one week in September 2006)
32. Université de Picardie “Jules Verne”, Amiens (one week in October 2006)
33. University of Perugia (November 2006, with a GNAMPA–INdAM Visiting Professor position)
34. Université de Picardie “Jules Verne”, Amiens (February 2007)
35. Université de Tunis El Manar (one week in March 2007)
36. Université de Haute Alsace (May 2007)
37. Université de La Rochelle (one week in July 2007)
38. Approximation and Wavelets, Bilateral Workshop Romania-Germany, October 1-4, 2007, Königswinter, Germany
39. Université Catholique de Louvain (December 2007)
40. Université de Picardie “Jules Verne”, Amiens (February 2008)
41. Université de Tunis El Manar (two weeks in March 2008)
42. Université de Limoges (May 2008)
43. Université de Tours (June 2008)
44. University of Perugia (two weeks in July 2008) with a GNAMPA–INdAM Visiting Professor position
45. Visiting Professor, Institute of Mathematics, Physics and Mechanics, University of Ljubljana (July–September 2008)
46. University of Cagliari (two weeks in October 2008)
47. Scuola Normale Superiore di Pisa (one week in October 2008)
48. City University of Hong Kong (one week in December 2008)

49. Université de Picardie "Jules Verne", Amiens (February 2009)
50. Université de Tunis El Manar (one week in April 2009)
51. University of Rzeszów (one week in May 2009)
52. University of Ljubljana (one week in May 2009)
53. Université Pierre et Marie Curie Paris VI (one week in August 2009)
54. Université de La Rochelle (one week in September 2009)
55. Université de Picardie "Jules Verne", Amiens (February 2010)
56. University of Messina, Italy (one week in April 2010)
57. Université de Tunis El Manar (one week in January 2011)
58. Université de Picardie "Jules Verne", Amiens (May 2011)
59. University of Oxford (one week in November 2011)
60. University of Monastir (one week in March 2012)
61. Université de Poitiers (one week in March 2012)
62. University of Perugia (15 May–15 June 2012) with a GNAMPA–INdAM Visiting Professor position
63. Université de Picardie "Jules Verne", Amiens (November 2012)
64. Universities of Catania and Reggio Calabria (two weeks in January 2013)
65. Université de Besançon (March 2013)
66. Université de Poitiers (April 2013)
67. ICTP Trieste (one week in May 2013)
68. King Abdulaziz University, Jeddah, Saudi Arabia (one week in September 2013)
69. Universities of Reggio Calabria and Messina (one week in October 2013)
70. Université de Picardie "Jules Verne", Amiens (November 2013)
71. University of Ljubljana (one week in January 2014)
72. Université Cadi Ayyad, Marrakech (one week in March 2014)
73. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in April 2014)
74. University of Pisa (one week in May 2014)
75. Recent Trends in Nonlinear Partial Differential Equations and Applications Celebrating Enzo Mitidieri's 60th Birthday, University of Trieste, 28–30 May 2014

76. Université de Picardie “Jules Verne”, Amiens (November 2014)
77. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in December 2014)
78. University of Perugia (one week in January 2015)
79. Senior Research Fellow, City University of Hong Kong (February 2015)
80. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in April 2015)
81. King Saud University, Riyadh, Saudi Arabia (one week in May 2015)
82. Université de Pau (two weeks in October 2015)
83. King Saud University, Riyadh, Saudi Arabia (one week in November 2015)
84. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in December 2015)
85. University of Stockholm (one week in January 2016)
86. Université de Tunis (one week in March 2016)
87. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in April 2016)
88. University of Perugia (one week in September 2016)
89. King Saud University, Riyadh, Saudi Arabia (one week in October 2016)
90. Université de Picardie “Jules Verne”, Amiens (November 2016)
91. King Abdulaziz University, Jeddah, Saudi Arabia (two weeks in December 2016)
92. University of Perugia (one week in January 2017)
93. University of Perugia (one week in January 2018)
94. University of Stockholm (one week in February 2018)
95. King Saud University, Riyadh, Saudi Arabia (one week in April 2018)
96. Harbin Engineering University, China (three weeks in November 2018)
97. Université de Picardie “Jules Verne”, Amiens (one week in December 2018)
98. University of Trieste (three weeks in March 2019)
99. King Saud University, Riyadh, Saudi Arabia (one week in April 2019)
100. University of Pisa, Italy (one week in April 2019)
101. Università di Urbino Carlo Bo, Urbino, Italy (one week in May 2019)
102. Université de Picardie “Jules Verne”, Amiens (one week in May 2019)
103. Central South University, Changsha, China (one month in November 2019)
104. Brno University of Technology (one week in September 2021 and September 2022)

5 Teaching and organizational achievements

Presentation of teaching and organizational achievements as well as achievements in popularization of science and art

1. AGH University of Science and Technology (since 2018): *Elliptic Equations* (master)

2. University of Craiova (since 1990): *Analysis I* (bachelor), *Partial Differential Equations* (bachelor), *Functional Analysis* (bachelor), *Numerical Analysis* (bachelor), *Nonlinear Partial Differential Equations* (master), *Nonlinear Analysis* (master), Calculus of Variations (master)

3. Central European University, Budapest, September 2002: *Functional Analysis* (master)

4. École Normale Supérieure, Bucharest, academic year 2005-2006: *Nonlinear Analysis and Mathematical Physics* (Ph.D.)

5. École Normale Supérieure, Bucharest, academic year 2010-2011: *Applied Functional Analysis and Partial Differential Equations* (Ph.D.)

6. University of Padova, June 18-22, 2012: *Comparison Principles and Critical Point Methods in Nonlinear Analysis* (Ph.D.), Mini-courses in Mathematical Analysis

7. ICTP Trieste, May 27–June 1, 2013: *Singular Phenomena in Nonlinear Elliptic Equations* (Ph.D.), Mini-courses in Partial Differential Equations, Women in Mathematics Summer School

8. Between 2002 and 2016, I organized the *Ateliers d'Écriture Scientifique* (Scientific Writing Laboratory) at the Doctoral School of the Université de Picardie "Jules Verne", Amiens, France

9. Lectures notes "Partial Differential Equations" and "Treatment Methods of the Elliptic Problems", published by Craiova University Press.

10. H. Brezis, *Analyse fonctionnelle: théorie, méthodes et applications*, Masson, Paris, 1992. Translation from French. Romanian title: *Analiză funcţională: teorie, metode şi aplicaţii*, Editura Academiei Române, Bucureşti, 2002, 275 pp.

11. Notes of the Master lectures
    H. Brezis, *Équations de Ginzburg-Landau et singularités*, Université Pierre et Marie Curie (Paris 6), 2001 (written by Vicenţiu Rădulescu)

6 Other information about professional career

Apart from information set out in previous sections, the applicant may include other information about his/her professional career, which he/she deems important.

6.1 Editor of special issues

1. H. Le Dret, V.D. Rădulescu, R. Wong, Special Issue of *Communications in Pure and Applied Analysis* dedicated to the 70th anniversary of Professor Philippe G. Ciarlet, Vol. 8, Issue 1, 491 pp., 2009.

2. V.D. Rădulescu, Special Issue *Degenerate and Singular Partial Differential Equations and Phenomena*, *Journal of Mathematical Analysis and Applications*, Vol. 352, Issue 1, 572 pp., 2009.

3. C. Alves, V.D. Rădulescu, Special Issue *Degenerate and Singular Differential Operators with Applications to Boundary Value Problems*, *Boundary Value Problems*, Volume 2010 (2010).

4. A. Pankov, R. P. Gilbert, V.D. Rădulescu, S. Antontsev, Special Issue *Sobolev Spaces with Variable Exponent and Related Elliptic Problems: Theory and Applications*, *Complex Variables and Elliptic Equations* **56**, Issue 7–9, 2011.

5. G. Da Prato, V.D. Rădulescu, Special Issue *Stochastic PDEs in Fluid Dynamics, Particle Physics and Statistical Mechanics*, *Journal of Mathematical Analysis and Applications* **384** (2011), Issue 1.

6. V.D. Rădulescu, Special Issue *Singular and Degenerate Phenomena in Nonlinear Analysis*, *Nonlinear Analysis: Theory, Methods & Applications* **119** (2015), 1-500.

7. V.D. Rădulescu, Special Issue dedicated to Acad. Marius Iosifescu on the occasion of his 80th anniversary, *Annals Univ. Craiova Ser. Mat. Inform.* **43**, No. 1, 2016.

8. V.D. Rădulescu, Special Issue *Pure and Applied Nonlinear Analysis*, *Opuscula Mathematica* **39**, No. 2, 2019.

9. P. Pucci, V.D. Rădulescu, Special Issue *Progress in Nonlinear Kirchhoff Problems*, *Nonlinear Analysis: Theory, Methods and Applications* **186** (2019), 1-258.

10. V.D. Rădulescu, D. Repovš, Special Issue *Elliptic Equations and Their Synergies*, *Complex Variables and Elliptic Equations*, vol. 65, no. 7, 2020.

11. S. Krantz, V.D. Rădulescu, Special Issue *Perspectives of Geometric Analysis in PDEs*, *Journal of Geometric Analysis*, vol. 30, no. 2, 2020.

12. G. Mingione, V.D. Rădulescu, Special Issue *Non-uniformly elliptic problems and nonstandard elliptic equations*, *Journal of Mathematical Analysis and Applications*, vol. 501, issue 1, 2021.

13. V.D. Rădulescu, Special Issue *Nonlinear Analysis & its Synergies*, *Rendiconti del Circolo Matematico di Palermo Series 2*, in press.

6.2 Editorial activities

1. Member of the Editorial Board of the new Academic Press *Mathematics in Science and Engineering* Book Series (Elsevier)

2. Editor of the *De Gruyter Series in Nonlinear Analysis and Applications*

3. Editor-in-Chief and founder of *Advances in Nonlinear Analysis* (Walter de Gruyter); the journal is ranked 3/332 in Mathematics by Clarivate Analytics 2022 according with the Impact Factor and 3/474 according with the Journal Citation Indicator

4. Editor-in-Chief of *Boundary Value Problems* (Springer Open)

5. Associate Editor of the *Journal of Geometric Analysis* (Springer)

6. Member of the Editorial Board of *Bulletin of Mathematical Sciences* (World Scientific)

7. Advisory Editor of *Mathematical Methods in the Applied Sciences* (Wiley)

8. Associate Editor of *Asymptotic Analysis* (IOS Press)

9. Member of the Editorial Board of *Complex Variables and Elliptic Equations* (Taylor & Francis)

10. Member of the Advisory Board of *Rendiconti del Circolo Matematico di Palermo* (Springer)

11. Member of the Editorial Advisory Board of *Demonstratio Mathematica* (Walter de Gruyter)

12. Associate Editor of *Discrete and Continuous Dynamical Systems, Series S* (American Institute of Mathematical Sciences)

13. Member of the Editorial Committee of *Opuscula Mathematica* (AGH University of Science and Technology)

14. Member of the Editorial Board of *Journal of Numerical Analysis and Approximation Theory* (Romanian Academy)

15. Member of the Editorial Board of *Ann. St. Univ. Ovidius Constanta*

16. *Nonlinear Analysis* (Associate Editor: 2009-2018, Editor-in-Chief: 2018-2020)

17. *Journal of Mathematical Analysis and Applications* (Associate Editor: 2004-2021)

6.3 Organizer of international conferences

1. 7ème Colloque Franco–Roumain de Mathématiques Appliquées, Craiova (Romania), September 2004

2. 8th International Conference of Mathematical Analysis and Applications, Craiova, September 23–24, 2005

3. Conférence Francophone sur la Modélisation Mathématique en Biologie et en Médecine, Craiova (Romania), 12–14 juillet 2006

4. 6th Congress of Romanian Mathematicians, Bucharest, June 28–July 4, 2007

5. Bilateral Workshop Romania–Germany Approximation and Wavelets, Königswinter, Germany, October 1–4, 2007

6. Current and Prospective Trends in Mathematical Research, Institute of Mathematics Simion Stoilow of the Romanian Academy, Bucharest, September 17–18, 2008

7. International Conference on Partial Differential Equations and Applications - in Honor of Professor Philippe G. Ciarlet's 70th Birthday, City University, Hong Kong, December 5-8, 2008

8. Romania-Germany Workshop Nonlinear Analysis and Mathematical Physics, University Lucian Blaga of Sibiu, May 14–16, 2009

9. 7th Congress of Romanian Mathematicians, Brasov, June 29–July 5, 2011

10. New Trends in Modern Analysis: Probabilistic and Analytic Methods in PDEs and Spectral Theory, Hammamet (Tunisia), October 24-28, 2011
11. Lectures on Partial Differential Equations, International Conference in Honor of Professor Patrizia Pucci's 60th birthday, University of Perugia, May 28–June 1, 2012
12. Special Session "Analyse et Analyse des Équations aux Dérivées Partielles" (with L. Rifford), XIème Colloque Franco-Roumain de Mathématiques Appliquées, Bucharest, August 24-30, 2012
13. Workshop "New Trends in Pure and Applied Nonlinear Analysis", Sibiu, March 2013
14. International Conference "Recent Advances in PDEs and Applications" (on occasion of Professor Hugo Beirao da Veiga's 70th birthday), Levico Terme (Trento), Italy, February 17-21, 2014
15. Special Session "Discrete and Continuous Boundary Value Problems and Applications", 10th AIMS Conference in Dynamical Systems, Differential Equations and Applications, Madrid, July 7-11, 2014
16. International Workshop on Nonlinear Analysis and Applications to Economics dedicated to Professor Dučan Repovš on his 60th birthday, University of Craiova, 25 September 2014
17. Section "Ordinary and Partial Differential Equations, Variational Methods", 8th Congress of Romanian Mathematicians, Iasi, June 26–July 1, 2015
18. Equilibrium and Optimization Methodology in Finance and Economics, King Saud University, Riyadh, Saudi Arabia, 9-11 November 2015
19. 9th Congress of Romanian Mathematicians, Galaţi, June 28–July 3, 2019
20. Elsevier–JMAA Conference on Nonlinear Analysis at AGH-UST, Kraków, 11–12 October 2019
21. Member of the Scientific Committee of the 5th Conference on Mathematical Science and Applications, KAUST, Saudi Arabia (April 7-9, 2020), organized by the Saudi Association for Mathematical Science and the King Abdullah University of Science and Technology
22. Methods of Nonlinear Analysis in Differential and Integral Equations, Rzeszów University of Technology, 15-16 May and 22-23 May 2021

6.4 Scientific and honorary awards

1. Simion Stoilow Prize of the Romanian Academy, 1999
2. Prize for Excellence in Research of the Romanian Research Council, 2007
3. Distinguished Foreign Professor, University of Ljubljana (July–September 2008)
4. Best Associate Editor of the *Journal of Mathematical Analysis and Applications*, 2009
5. Member of the *Accademia Peloritana dei Pericolanti*, Messina (since January 2014)
6. *Highly Cited Researcher* (2014, 2019, 2020, 2021)

7. Honorary Director, Institute of Mathematics of the Heilongjiang Institute of Technology, Harbin, China (2014-2019)
8. Senior Research Fellow, City University of Hong Kong, 2015
9. Member of the *Accademia delle Scienze dell'Umbria*, Perugia (since 2017)
10. Guest Professor, Harbin Engineering University, 2018
11. Senior Visiting Scholar, Central South University, Changsha, 2019
12. First Prize at the Gala of the Best Researchers, University of Craiova (2017–2021)
13. First Prize of the Rector of the AGH University of Science and Technology for scientific achievements, 2020
14. World's Top 2 Scientists List of Stanford University (2019–2020)
15. Honorary Citizen, municipality of Caracal (Romania), 2021
16. AGH University of Science and Technology Rector's Award, 2021

During 2008 and 2015, I was a member of the Scientific Board of the *Laboratoire Européen Associé CNRS Franco–Roumain Mathématiques & Modélisation* between the *Laboratoire de Mathématiques de l'Université Paris-Sud (Orsay)* and the "Simion Stoilow" Mathematics Institute of the Romanian Academy.

6.5 Coordination of PhD theses and postdoctoral researchers

Starting with October 2000, I have been entitled to coordinate PhD theses in Romania. In February 2003 I received the Habilitation at the University "Pierre et Marie Curie"–Paris 6 (Paris Sorbonne University). According with Mathematics Genealogy Project, I have coordinated 15 PhD theses at the University of Craiova. An additional 16th PhD thesis was defended on 22 June 2022. At this moment, I coordinate 2 PhD theses at the AGH University of Science and Technology. I also coordinated 2 Master theses at the AGH University of Science and Technology and I gave 7 talks in the Seminars of Functional Analysis and Differential Equations of the Faculty of Applied Mathematics, AGH University of Science and Technology.

During the past years, I have coordinated at the University of Craiova the following doctoral and postdoctoral researchers from Central South University in Changsha, Harbin Engineering University and Southeast University in Nanjing: Youpei Zhang, Wen Zhang, Jian Zhang, Lixi Wen, Shuai Yuan, Qiang Lin, Siyan Guo, Chunxia Ma, Tiantian Pang, Yitian Wang, Yue Pang, Xueying Sun, and Li Cai. They visited the University of Craiova for one or two years with the financial support of China Scholarship Council.

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