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Research Article

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Global well-posedness for a class of fourth-order nonlinear strongly damped wave equations

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Abstract: In this paper, we consider the initial boundary value problem for a class of fourth-order wave equations with strong damping term, nonlinear weak damping term, strain term and nonlinear source term in polynomial form. First, the local solution is obtained by using fix point theory. Then, by constructing the potential well structure frame, we get the global existence, asymptotic behavior and blowup of solutions for the subcritical initial energy and critical initial energy respectively. Ultimately, we prove the blowup in finite time of solutions for the arbitrarily positive initial energy case.

Keywords: Fourth-order wave equations, strain term, asymptotic behavior, blowup, arbitrarily positive initial energy

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1 Introduction

In this paper, we investigate the following fourth-order nonlinear strongly damped wave equations with strain term and nonlinear source term:

$$u_{tt} + \Delta^2 u - \Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \sigma_i(u_{x_i}) - \Delta u_t + |u_t|^{r-1} u_t = f(u), \qquad (x,t) \in \Omega \times (0,\infty), \tag{1.1}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad x \in \Omega, \tag{1.2}$$

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{or} \quad u = \Delta u = 0, \qquad x \in \partial\Omega,\ t \geq 0, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$ and $r > 1$ is a constant.

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(H_1) The nonlinear function $f(u)$ satisfies
- (i) $f \in C^1$ and $f(0) = f'(0) = 0$,
- (ii) $f(u)$ is monotone and is convex for $u > 0$, concave for $u < 0$,
- (iii) $|f(u)| \leq a_1|u|^p$ and $(p+1)F(u) \leq uf(u)$ for some $a_1 > 0$ ($1 < p < \infty$ if $n \leq 4$ and $1 < p < \frac{n}{n-4}$ if $n \geq 5$),
- (iv) $F(u) = \int_0^u f(s)\,ds$.

(H_2) The function $\sigma_i (i = 1, \dots, n)$ satisfies
- (i) $\sigma_i(s) \in C^1$ and $\sigma_i(0) = \sigma_i'(0) = 0$,
- (ii) $\sigma_i(s)$ are monotone, and are convex for $s > 0$, concave for $s < 0$,
- (iii) $|\sigma_i(s)| \leq a_2|s|^q$ and $(q+1)G_i(s) \leq s\sigma_i(s)$ for some $a_2 > 0$ ($1 < q < \infty$ if $n = 1, 2$ and $1 < q < \frac{n}{n-2}$ if $n \geq 3$),
- (iv) $G_i(s) = \int_0^s \sigma_i(\tau)\,d\tau$, $1 \leq i \leq n$.

In view of its structure, equation (1.1) is a very complex model. By the strong damping term Δu_t, nonlinear weak damping term $|u_t|^{r-1}u_t$, fourth-order term $\alpha\Delta^2 u$, strain term $\sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})$ and generalized source term $f(u)$ in the equation, it takes into account not only many physical factors, but also a lot of complex physical processes. Apparently, such model equation does not exist in the first place, so we will introduce its development and evolution to show its background by listing a lot of related model equations. We shall tell the stories of the following model equations not only to illustrate the corresponding physical background, but also to describe the mathematical achievements in order to summarize the corresponding unsolved problems. Hence the main goal of this paper is to try to solve some of those unsolved problems systematically by considering a more general model equation, that is, equation (1.1). First we shall begin with fourth-order wave equations.

Usually, the fourth-order wave equations can be regarded as the formal extension of the classical second-order Klein–Gordon equation in the way $\partial_{tt} - \Delta \to \partial_{tt} + \Delta^2$ and also the model of Schrödinger equation by $(\partial_t + i\Delta)(\partial_t - i\Delta) = \partial_{tt} + \Delta^2$. This model is not only of mathematical interest, and its physical background can be traced back to 1964, by recalling the work [8] by Bretherton as the first one, where a one-dimensional nonlinear fourth-order wave equation describing dispersive waves interacting weakly with a quadratic source term was introduced in order to investigate the resonant interactions between waves. Meanwhile, for the linear fourth-order wave equation as a model for a suspension bridge, the so-called beam equation describing the one-dimensional beam suspended by cables, some interesting phenomena like traveling waves and nonlinear oscillations were studied in [27, 28]. We also refer to the pioneering contributions of P. Pucci and J. Serrin [34, 36, 37] who investigated qualitative properties of dissipative wave systems, including asymptotic stability, local asymptotic stability and related blow-up phenomena. Some very recent results contained in [21, 23] paralleled the conclusions of second-order Klein–Gordon equations established in [17, 19–26, 41, 42, 51] by proving the local existence, stability and instability of solitary waves, decay and the optical decay rate of the solution to the Cauchy problem of fourth-order wave equation ($\lambda = 1$)

$$u_{tt} + \Delta^2 u + mu = \lambda|u|^{p-1}u.$$

Besides the above research for $\lambda > 0$ as the focusing case, the defocusing case $\lambda < 0$ was considered in [31] by proving the Levandosky–Strauss conjecture, that is, the scattering theory in the energy space. Although the classical nonlinear second-order wave equations attract much more attention than the fourth-order ones do, especially in the frame of variational methods to investigate the conditions of initial data for global existence in time [17, 24–40, 51] and non-global existence [32], the abstract model considered by Levine in [24] can include the fourth-order case. This observation that a lot of effects were devoted to extend the conclusions of the second-order wave equation to the fourth-order wave equations, especially the classification of the initial data leading to global existence in time or finite time blowup of solutions, motivates us to establish the corresponding theories for the fourth-order case, which is also part of our motivations.

Given a model with strain term like $u_{tt} - u_{xxt} = \sigma(u_x)_x$ introduced to describe the longitudinal motion of a homogeneous bar, in the absence of monotonicity of σ and the strict monotonicity of τ, the existence of a Holder solution and time-asymptotic properties dependent on boundary conditions were presented by Dafermos in [11]. Furthermore, the local and global existence of solutions to the initial boundary value problem of such a one-dimensional physical model equation were proved in [5]; then improvements were made in [6] by considering more general monotone σ and obtaining the asymptotic behavior. The model equation also attracts a lot of attentions in various directions like well-posedness of weak solution, asymptotic analysis and stability [15, 33, 38], and it was also extended to be considered in the form of nonlinear systems of viscoelasticity [12].

The one-dimensional fourth-order wave equation

$$u_{tt} + u_{xxxx} = a(u_x^2)_x + f(x,t), \quad x \in (0,1),\ t > 0$$

was proposed in [4] to describe the elastoplastic microstructure models for longitudinal motion of an elastoplastic bar [2, 3]. And its two-dimensional case is related to some Kirchhoff–Boussinesq models [9, 10]. An abstract model equation including strain and dissipative terms, and linear source term was treated by the Galerkin method in [7] in order to get the global existence and asymptotic behavior of the solution. A class of fourth-order wave equations with weak damping term and nonlinear strain term in the form

$$u_{tt} + \Delta^2 u + \lambda u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) \quad \text{in } \Omega \times (0,+\infty)$$

was considered in [53]. The global existence and large time decay of the solution were obtained for the initial datum in the potential well with the restriction on the initial energy, that is, $E(0) < \frac{m-1}{4(m+1)}(\frac{1}{C_* b})^{\frac{2}{m-1}} < d$. The finite time blowup of the solution was also proved for the negative initial energy, that is, $E(0) < 0$. Later, the above results were generalized and extended to both the cases $E(0) < d$ and $E(0) = d$ in [26]; especially, the finite time blowup for positive initial energy was derived with the restriction on the dissipative parameter γ in [26], which will be removed in this work.

Further, these two research groups of authors in [53] and [26] turned to consider the model wave equation with linear strong damping term, nonlinear strain term, nonlinear weak damping term and nonlinear source term in unbounded domain as follows:

$$u_{tt} - \Delta u_t - \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + f(u_t) = g(u) \quad \text{on } \mathbb{R}^N \times (0,+\infty).$$

Under some restrictions on these terms, the global existence and finite time blowup for negative initial energy, that is, $E(0) < 0$, were derived in [52]. Then these results were improved in [51] by relaxing these restrictions and considering the positive initial energy case in the potential well, that is, $E(0) < d$, for both global existence and finite time blowup of solutions.

Jorge A. Esquivel-Avila [13] studied the initial boundary value problem for a fourth-order wave equation with nonlinear strain and source terms

$$u_{tt} + \Delta^2 u - \alpha\Delta u \pm \beta \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(|u_{x_i}|^{m-2}u_{x_i}) = \mu|u|^{r-2}u, \quad x \in \Omega,\ t > 0,$$

and derived the sufficient conditions on the initial data for global solution and finite time blow-up solution when $E(0) \le d$. Later Liu and Xu [25] extended these results by considering more general strain terms and nonlinear source terms, giving vacuum isolating phenomena of the solution, obtaining some new invariant sets and global existence theorems, and also generalizing some exponents. Then Shen et al. [18] pushed the above restrictions on the initial energy to the arbitrarily positive initial energy level, that is, they proved the finite time blowup for $E(0) > 0$ by taking advantage of the concavity method.

The dissipative model with linear weak damping $u_{tt} + u_t + \Delta^2 u - \alpha\Delta u + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) = f(u)$ was discussed by Wang and Wang in [44]. They proved the finite time blowup for both negative initial energy, that is, $E(0) < 0$, and arbitrary positive initial energy, that is, $E(0) > 0$, and global existence and exponential decay of solutions starting from the initial data in the so-called potential well, that is, $E(0) < \frac{p-2}{2p} k_p^{-\frac{2p}{p-2}}$ and $\|u_0\|_B < k_p^{-\frac{2p}{p-2}}$. The corresponding model with strong damping term was considered in [14], and the global existence and exponential attractors were established; here we specially remind that the term $\Delta_p u$ in their model plays the role of the strain term in the present paper. Not only that, Jorge Silva and Ma [19, 20] studied a viscoelastic version of the problem with a memory term; they got the existence of global attractor and asymptotic stability, respectively.

Terms included in the equation							
u_{tt}	$\Delta^2 u$	Δu	$\sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})$	Δu_t	$\lvert u_t\rvert^{r-1}u_t$	$f(u)$	**Results**
•			•	•			Global existence for $E(0) < d$ (see [5])
•	•		•			•	Describe the elastoplastic microstructure models for longitudinal motion of an elastoplastic bar [4]
•			•	•	•		Global existence and asymptotic behavior for $E(0) < d$ (see [7])
•	•		•		•		Global existence and asymptotic behavior for $E(0) < \frac{m-1}{4(m+1)}(\frac{1}{C_* b})^{\frac{2}{m-1}} < d$ and finite time blowup for $E(0) < 0$ (see [53])
•	•		•		•		Global existence and finite time blowup for $E(0) < d$, global existence for $E(0) = d$ (see [26])
•			•		•	•	Global existence and finite time blowup for $E(0) < 0$ (see [52])
•	•		•			•	Global solution and finite time blowup for $E(0) \le d$ (see [13])
•	•	•	•			•	Global existence and finite time blowup for $E(0) < d$, global existence for $E(0) = d$ (see [25])
•	•	•	•			•	Finite time blowup for $E(0) > 0$ (see [18])
•	•	•	•		•	•	Finite time blowup for $E(0) < 0$ and $E(0) > 0$, global existence and exponential decay for $E(0) < \frac{p-2}{2p} k_p^{-\frac{2p}{p-2}}$ and $\lVert u_0\rVert_B < k_p^{-\frac{2p}{p-2}}$ (see [44])

Table 1: The mathematical physics models and compositions of the results.

The potential well theory [32, 40, 43] and its improved version [26] allow us to consider the case without positive definite energy, and also a lot of efforts devoted to extend the negative initial energy blowup for $E(0) < 0$ to the positive initial energy blowup for $E(0) < d$ (see [35]), to study the critical initial energy $E(0) = d$ (see [45]) and the arbitrary positive initial energy $E(0) > 0$ (see [18, 48]). All of above efforts push such studies to a new stage, so we expect to conduct such research in a comprehensive and systematical way, that is, to solve the problem related to subcritical energy level $E(0) < d$, critical energy level $E(0) = d$ and arbitrary high initial level $E(0) > 0$ in a uniform frame [46, 47, 49, 50]. Then we can easily figure out the corresponding unsolved problems related to different equations. For example, the case $E(0) < 0$ was considered in [52], and the case $E(0) < d$ is not solved; the case $E(0) < d$ was considered in [53], and the case $E(0) = d$ is not solved; the case $E(0) \le d$ was considered in [13], and the case $E(0) > 0$ is not solved. It seems necessary to deal with the unsolved problems by the above observations for different model equations, but we realize that this kind of studies may produce a lot of paper fragments, that is to say, maybe one paper deals with an equation with nonlinear strain and source terms when $E(0) < d$ and another paper tackles an equation with more general strain term and nonlinear source terms when $E(0) > 0$. In order to avoid such problems, we consider a more general model, although there is no actual physical background for it, but this allows us to gather all the terms apparently in different model equations together in order to better solve these unsolved problems comprehensively and systematically. Although readers can easily find these unresolved problems for corresponding model equations from the above discussion, we still feel that it is necessary to present these problems in a more understandable way, so we give the following Table 1. In this table, each black dot represents a specific term in the model, and the far right description shows the results that have been obtained. By comparison, it is easy to see which problem is not solved for this model. So, next, we will follow the following train of thought to solve the problem. In Section 2, some preparing knowledge are introduced, such as stable sets and the depth of the potential well. In Section 3, local existence and uniqueness of the solution is proved. Section 4 is devoted to obtain the global existence, asymptotic behavior and nonexistence of solutions with initial energy $E(0) < d$. In Section 5, the global existence, asymptotic behavior and finite time blowup for the critical initial energy $E(0) = d$ are studied. In Section 6, we prove that the solution to problem (1.1)–(1.3) blows up in finite time for $E(0) > 0$ when $r = 1$. We refer to the recent monograph by Papageorgiou, Rădulescu and Repovš [30] for various analytic tools used in this paper.

2 Preliminaries

Throughout the present paper, the following notations are used for precise statements: $L^p(\Omega)(2 \le p < +\infty)$ denotes the usual space of all L^p-functions on Ω with norm $\|u\|_{L^p(\Omega)} = \|u\|_p$, $\|u\|_{L^2(\Omega)} = \|u\|$ and the inner product $(u, v) = \int_\Omega uv\,dx$. Let

$$H := \begin{cases} H_0^2(\Omega) & \text{for } u = 0 \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \\ H^2(\Omega) \cap H_0^1(\Omega) & \text{for } u = 0 \text{ and } \Delta u = 0 \text{ on } \partial\Omega, \end{cases}$$

and $\|u\|_H^2 := \|\Delta u\|^2 + \|\nabla u\|^2$. In addition, we denote the duality paring between H^{-1} and H by $\langle \cdot, \cdot \rangle$.

Lemma 2.1 ([1]). *For any $u \in H^2(\Omega) \cap H_0^1(\Omega)$, the norm $\|\Delta u\|$ is equivalent to $\|u\|_{2,2}$.*

Corollary 2.2. *For any $u \in H$, the norm $\|u\|_H$ is equivalent to $\|u\|_{2,2}$. Let C be the optimal constant such that $\|u\|_H^2 \ge C\|u\|$.*

Corollary 2.3 ([1]). *Let p and q be defined by* (H_1) *and* (H_2). *Then*
(i) *$H \hookrightarrow L^{p+1}(\Omega)$ compactly and $\|u\|_{p+1} \le C_1\|u\|_H$,*
(ii) *$H \hookrightarrow W^{1,q+1}(\Omega)$ compactly and $\|u\|_{1,q+1} \le C_2\|u\|_H$,*
where C_1 and C_2 are constants independent of u.

Definition 2.4. Function $u(x, t)$ is called a weak solution to problem (1.1)–(1.3) on $\Omega \times [0, T)$ if

$$u(x, t) \in L^\infty(0, T; H), \quad u_t(x, t) \in L^2(0, \infty; H_0^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$$

satisfying

$$\begin{aligned} \langle u_{tt}, w\rangle + \int_\Omega \nabla u \nabla w\,dx + \int_\Omega \Delta u \Delta w\,dx + \int_\Omega \nabla u_t \nabla w\,dx + \int_\Omega |u_t|^{r-1} u_t w\,dx \\ = (f(u), w) + \sum_{i=1}^n (\sigma_i(u_{x_i}), w_{x_i}) \quad \text{for all } w \in H,\ 0 < t < T. \end{aligned} \tag{2.1}$$

Next, for problem (1.1)–(1.3), we introduce the energy functional

$$E(t) := E(u(t)) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_H^2 - \sum_{i=1}^n \int_\Omega G_i(u_{x_i})\,dx - \int_\Omega F(u)\,dx, \tag{2.2}$$

the potential energy functional

$$J(u) := J(u(t)) = \frac{1}{2}\|u\|_H^2 - \sum_{i=1}^n \int_\Omega G_i(u_{x_i})\,dx - \int_\Omega F(u)\,dx, \tag{2.3}$$

the Nehari function

$$I(u) := I(u(t)) = \|u\|_H^2 - \sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx - \int_\Omega uf(u)\,dx, \tag{2.4}$$

the potential well (stable set) $W = \{u \in H \mid I(u) > 0\} \cup \{0\}$ and the outer space of the potential well (unstable set) $V = \{u \in H \mid I(u) < 0\}$.

Lemma 2.5. *Let $u(x, t)$ be a solution to problem* (1.1)–(1.3) *with $u_0 \in H$ and $u_1 \in L^2(\Omega)$. Then the energy functional $E(t)$ is non-increasing about t.*

Proof. Multiplying equation (1.1) by u_t and integrating it over $\Omega \times [0, t)$, we obtain

$$E(t) + \int_0^t \|\nabla u_\tau(\tau)\|^2\,d\tau + \int_0^t \|u_\tau(\tau)\|_{r+1}^{r+1}\,d\tau = E(0) \tag{2.5}$$

and $E'(t) = -\|\nabla u_t\|^2 - \|u_t\|_{r+1}^{r+1} \le 0$. □

Lemma 2.6. *From* (H_1) *and* (H_2), *we can easily derive*

$$E(0) \geq E(t) = \frac{1}{2}\|u_t\|^2 + J(u) \geq \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|u\|_H^2 + \frac{1}{p+1}I(u). \tag{2.6}$$

Now we define the depth of the potential well as

$$d := \inf_{u \in \mathcal{N}} J(u), \tag{2.7}$$

where $\mathcal{N} := \{u \in H \setminus \{0\} \mid I(u) = 0\}$.

Lemma 2.7. *Let* (H_1) *and* (H_2) *hold and* $q \geq p$. *Then*

$$d = \Big(\frac{1}{2} - \frac{1}{p+1}\Big)r^2, \tag{2.8}$$

where r *is the unique real root of the equation* $h(r) := a_1 C_1^{p+1} r^{p-1} + a_2 C_2^{q+1} r^{q-1} = 1$.

Proof. From (2.7), this implies $I(u) = 0$, which, along with (H_1) and (H_2), gives

$$\|u\|_H^2 = \sum_{i=1}^{n} \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx. \tag{2.9}$$

Recalling (H_1) and (H_2) and Corollary 2.3, we get

$$\sum_{i=1}^{n} \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx \leq a_1\|u\|_{p+1}^{p+1} + a_2\sum_{i=1}^{n}\|u_{x_i}\|_{q+1}^{q+1} \leq a_1 C_1^{p+1}\|u\|_H^{p+1} + a_2 C_2^{q+1}\|u\|_H^{q+1}. \tag{2.10}$$

Incorporating (2.9), (2.10) and the fact that $h(r)$ is increasing and $\|u\|_H \geq r$ mean $h(\|u\|_H) \geq 1$. Then, by (2.6) and (2.7), we gain

$$J(u) \geq \frac{p-1}{2(p+1)}\|u\|_H^2 + \frac{1}{p+1}I(u) = \frac{p-1}{2(p+1)}\|u\|_H^2 \geq \frac{p-1}{2(p+1)}r^2,$$

which implies (2.8). □

3 Local solution

This section considers the existence and uniqueness of a local solution by employing the contraction mapping principle. For any fixed time T, we consider the space $\mathscr{H} := C([0,T]; H) \cap C^1([0,T]; L^2(\Omega))$ with the norm $\|u\|_{\mathscr{H}}^2 := \max_{t\in[0,T]}(\|u_t\|^2 + \|u\|_H^2)$.

First, in order to obtain the local existence and uniqueness of solution, we introduce the following lemma.

Lemma 3.1. *For any* $T > 0$, *if* $u \in \mathscr{H}$ *solves problem* (1.1)–(1.3) *with the initial data* (u_0, u_1), *then there exists a unique* $v \in \mathscr{H} \cap C^2([0,T], H^{-1})$ *with* $v_t \in L^2([0,T], H_0^1(\Omega))$ *satisfying the linear problem*

$$\begin{cases} v_{tt} + \Delta^2 v - \Delta v - \Delta v_t + |v_t|^{r-1}v_t = f(u) - \displaystyle\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\sigma_i(u_{x_i}), & x \in [0,T) \times \Omega, \\ v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,t) = 0, & x \in \partial\Omega,\ t \geq 0. \end{cases} \tag{3.1}$$

Proof. The proof of this lemma is based on the Galerkin method. We set $W_h := \mathrm{Span}\{\omega_1, \dots, \omega_h\}$ for every $h \geq 1$, where ω_h is the orthogonal complete system of the eigenfunction of the equation $\Delta\omega + \lambda\omega = 0$, $\omega|_{\partial\omega} = 0$, such that $\|\omega_j\| = 1$ for all j. Let

$$u_0^h := \sum_{j=1}^{h}\Big(\int_\Omega \Delta u_0 \Delta\omega_j\,dx\Big)\omega_j, \quad u_1^h := \sum_{j=1}^{h}\Big(\int_\Omega u_1\omega_j\,dx\Big)\omega_j.$$

Then $u_0^h \in W_h$, $u_1^h \in W_h$, and

$$u_0^h \to u_0 \in H, \qquad h \to +\infty,$$
$$u_1^h \to u_1 \in L^2(\Omega), \quad h \to +\infty.$$

For any $h \geq 1$, we seek functions $y_1^h, \dots, y_h^h \in C^2[0, T]$ such that

$$v_h(t) = \sum_{j=1}^{h} y_j^h(t)\omega_j \tag{3.2}$$

solves the problem

$$\begin{cases} \int\limits_\Omega (\ddot{v}_h(t) + \Delta^2 v_h(t) - \Delta v_h(t) - \Delta \dot{v}_h(t) + |v_h(t)|^{r-1} v_h(t))\eta\, dx = f(u) - \sum\limits_{i=1}^{n} \dfrac{\partial}{\partial x_i}\sigma_i(u_{x_i}), \\ v^h(0) = u_0^h, \quad \dot{v}^h(0) = u_1^h, \end{cases} \tag{3.3}$$

where $\eta \in W_h$ and $t \geq 0$. Taking $\eta = \omega_j$ for $j = 1, \dots, h$ in (3.3), we can derive the following Cauchy problem for an ordinary differential equation with the unknown y_j^h:

$$\begin{cases} \ddot{y}_j^h(t) + \lambda_j^2 y_i^h(t) + \lambda_j y_j^h(t) + \lambda_j \dot{y}_j^h(t) + |y_j^h(t)|^{r+1} y_j^h(t) = \psi_j(t), \\ y_j^h(0) = \int\limits_\Omega \Delta u_0 \Delta \omega_j\, dx, \\ \dot{y}_j^h(0) = \int\limits_\Omega u_1 \omega_j\, dx, \end{cases} \tag{3.4}$$

where

$$\psi_j(t) = \int\limits_\Omega f(u(t))\omega_j\, dx - \int\limits_\Omega \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})\omega_j\, dx, \quad t \in C[0, T].$$

Then, for all j, problem (3.4) yields a unique global solution $y_j^h \in C^2[0, T]$, which gives a unique v_h defined by (3.2) satisfying (3.3). In particular, (3.2) implies that $\dot{v}_h(t) \in H_0^1(\Omega)$ for $t \in [0, T]$. Then, from Poincaré's inequality, we can see that

$$\|\nabla \dot{v}_h(t)\|^2 \geq c\|\dot{v}_h(t)\|^2, \quad t \in [0, T]. \tag{3.5}$$

Taking $\eta = \dot{v}_h(t)$ into (3.3) and integrating the obtained results over $[0, t] \subset [0, T]$, we have

$$\begin{aligned} \|\dot{v}_h(t)\|^2 + \|v_h(t)\|_H^2 + 2\int_0^t \|\nabla \dot{v}_h(\tau)\|^2\, d\tau + 2\int_0^t \|\dot{v}_h(\tau)\|_{r+1}^{r+1}\, d\tau \\ = \|u_1^h(t)\|^2 + \|u^h(0)\|_H^2 + 2\int_0^t \Big(\int_\Omega f(u(\tau))\dot{v}_h(\tau)\, dx\Big)\, d\tau \\ - 2\int_0^t \Big(\int_\Omega \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})\dot{v}_h(\tau)\, dx\Big)\, d\tau, \quad h \geq 1. \end{aligned} \tag{3.6}$$

Next we estimate the last two terms on the right-hand side of (3.6) as follows. For the first term, from (H_1), Hölder, Sobolev and Young inequalities, and for all $u \in \mathscr{H}$, it follows that

$$\begin{aligned} 2\int_0^t \int_\Omega f(u(\tau))\dot{v}_h(\tau)\, d\tau &\leq 2\int_0^t \int_\Omega a_1|u|^p \dot{v}_h(\tau)\, dx\, d\tau \leq 2a_1 \int_0^t \|u\|_{2p}^p \|\dot{v}_h(\tau)\|\, d\tau \\ &\leq 2a_1 2c^{\frac{1}{2}} \widetilde{C}^p \int_0^t \|u\|_H^p (2c)^{-\frac{1}{2}} \|\dot{v}_h(\tau)\|\, d\tau \\ &\leq C\int_0^t \|u\|_H^{2p}\, d\tau + \frac{1}{2c}\int_0^t \|\dot{v}_h(\tau)\|^2\, d\tau \leq C_1(T) + \frac{1}{2}\int_0^t \|\nabla \dot{v}_h(\tau)\|^2\, d\tau, \end{aligned} \tag{3.7}$$

where c is defined in (3.5).

For the second term, from (H_2), this implies $-\sigma_i(s) \le a_2|s|^q$, which helps us to derive

$$\begin{aligned}-2\int_0^t\int_\Omega\sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})\dot v_h(\tau)\,dx\,d\tau &= 2\int_0^t\int_\Omega\sum_{i=1}^n \sigma_i(u_{x_i})\nabla\dot v_h(\tau)\,dx\,d\tau\\ &\le 2\int_0^t\int_\Omega\sum_{i=1}^n a_2|u_{x_i}|^q\nabla\dot v_h(\tau)\,dx\,d\tau\\ &\le 2a_2\int_0^t\sum_{i=1}^n\|u_{x_i}\|_{2q}^q\|\nabla\dot v_h(\tau)\|\,d\tau\\ &\le 2a_2C_2^q\int_0^t\|u\|_H^q\|\nabla\dot v_h(\tau)\|\,d\tau\\ &\le \widetilde{C_2}\int_0^t\|u\|_H^{2q}\,d\tau+\frac12\int_0^t\|\nabla\dot v_h(\tau)\|^2\,d\tau\\ &\le C_2(T)+\frac12\int_0^t\|\nabla\dot v_h(\tau)\|^2\,d\tau. \end{aligned} \tag{3.8}$$

Substituting (3.7) and (3.8) into (3.6) yields

$$\|\dot v_h(t)\|^2+\|v_h(t)\|_H^2+2\int_0^t\|\nabla\dot v_h(\tau)\|^2\,d\tau+2\int_0^t\|\dot v_h(\tau)\|_{r+1}^{r+1}\,d\tau\le C_T, \tag{3.9}$$

where $C_T=\|u_1^h\|^2+\|u_0^h\|_H^2+C_1(T)+C_2(T)$ is independent of h recalling that u_0^h and u_1^h converge. Hence (3.9) implies that

- $\{v_h\}$ is bounded in $L^\infty([0,T],H)$,
- $\{\dot v_h\}$ is bounded in $L^{r+1}([0,T],L^{r+1}(\Omega))\cap L^\infty([0,T],L^2(\Omega))\cap L^2([0,T],H_0^1(\Omega))$,
- $\{\ddot v_h\}$ is bounded in $L^2([0,T],H^{-1})$.

Therefore, up to a subsequence, we may pass to the limit in (3.3) and get a weak solution $v(t)$ to a problem satisfying the above regularity. So the existence of v to problem (3.1) is obtained.

Uniqueness follows by contradiction: if v and $\overline{v}$ were two solutions to problem (3.1) with the same initial data, by subtracting the equations and testing with $v_t-\overline{v}_t$, instead of (3.6), we get

$$\|v_t(t)-\overline{v}_t(t)\|_2^2+\|v(t)-\overline{v}(t)\|_H^2+2\int_0^t\|\nabla v_\tau(\tau)-\nabla\overline{v}_\tau(\tau)\|^2\,d\tau+2\int_0^t\|v_\tau(\tau)-\overline{v}_\tau(\tau)\|_{r+1}^{r+1}\,d\tau=0,$$

which gives $v\equiv\overline{v}$. The proof of the lemma is now completed. □

Now we show the existence and uniqueness of a local solution to problem (1.1)–(1.3).

Theorem 3.2 (Local existence and uniqueness). *Let $u_0\in H$, $u_1\in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold. Then there exists a unique local solution of problem* (1.1)–(1.3)

$$\begin{aligned}&u\in C([0,T_m),H),\\ &u_t\in L^2([0,T_m),H_0^1(\Omega))\cap L^\infty([0,T_m),L^{r+1}(\Omega))\quad\text{for some } T_m>0.\end{aligned}$$

Proof. Take (u_0,u_1) satisfying (1.2). Let $R_0^2:=\frac12(\|u_0\|_H^2+\|u_1\|^2)$, and for any $T>0$, we consider

$$\mathcal{M}_{\mathcal{T}}=\{u\mid u\in\mathcal{H}: u(x,0)=u_0,\ u_t(x,0)=u_1,\ \|u\|_{\mathcal{H}}^2\le R_0^2\}\quad\text{with } R\ge R_0.$$

Then, from Lemma 3.1, we can define a map satisfying $\Phi(\mathcal{M}_{\mathcal{T}})\subseteq\mathcal{M}_{\mathcal{T}}$, and $v=\Phi(u)$ is the unique solution for any $u\in\mathcal{M}_{\mathcal{T}}$.

Next we will state that Φ is a contractive map from $\mathcal{M}_{\mathcal{T}}$ into itself for sufficiently large R and small enough T, and then, by the contraction mapping principle, we can obtain a unique local solution to problem (1.1)–(1.3). Now we prove it by two steps.

Step (I). We claim that Φ maps $\mathscr{M}_{\mathscr{T}}$ into itself for sufficiently large R and small enough T. In other words, we will show $\|\Phi(u)\|_{\mathscr{H}} \le R$ provided that $\|u\|_{\mathscr{H}} \le R$. If $\|u\|_{\mathscr{H}} \le R$, taking $\Phi(u) = v$ into the energy identity (see (3.6)) yields

$$\begin{aligned}\|v_t\|^2 + \|v\|_H^2 + 2\int_0^t \|\nabla v_\tau\|^2\,d\tau + 2\int_0^t \|v_\tau\|_{r+1}^{r+1}\,d\tau \\ = \|u_1\|^2 + \|u_0\|_H^2 + 2\int_0^t\int_\Omega f(u(\tau))v_\tau(\tau)\,d\tau - 2\int_0^t\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})v_\tau(\tau)\,d\tau. \end{aligned} \tag{3.10}$$

In the following, we will employ the Hölder inequality, the Young inequality, the Cauchy–Schwarz inequality, the Sobolev embedding inequality and (H_1) and (H_2) to estimate the third term on the right side of (3.10) as

$$\begin{aligned} 2\int_0^t\int_\Omega f(u(\tau))v_\tau(\tau)\,d\tau &\le 2\int_0^t\int_\Omega a_1|u(\tau)|^p v_\tau(\tau)\,dx\,d\tau = 2a_1\int_0^t \|u\|_{2p}^p\|v_\tau(\tau)\|\,d\tau \le 2a_1\widetilde{C}^p\int_0^t \|u\|_H^p\|v_\tau(\tau)\|\,d\tau \\ &\le C_1\int_0^t \|u\|_H^{2p}\,d\tau + \int_0^t \|v_\tau(\tau)\|^2\,d\tau \le \widetilde{C_1}(T) + \int_0^t \|v_\tau(\tau)\|^2\,d\tau, \end{aligned} \tag{3.11}$$

and the last term is estimated as

$$\begin{aligned} -2\int_0^t\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})v_t(\tau)\,d\tau &\le 2\int_0^t\int_\Omega \sum_{i=1}^n a_2|u_{x_i}|^q\nabla v_\tau(\tau)\,d\tau \le 2a_2C_2^q\int_0^t \|u\|_H^q\|\nabla v_\tau(\tau)\|\,d\tau \\ &\le C_2\int_0^t \|u\|_H^{2q}\,d\tau + \int_0^t \|\nabla v_\tau(\tau)\|^2\,d\tau \le \widetilde{C_2}(T) + \int_0^t \|\nabla v_\tau(\tau)\|^2\,d\tau. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), equation (3.10) becomes

$$\begin{aligned}\|v_t\|^2 + \|v\|_H^2 + 2\int_0^t \|\nabla v_\tau\|^2\,d\tau + 2\int_0^t \|v_\tau\|_{r+1}^{r+1}\,d\tau \\ = \|u_1\|^2 + \|u_0\|_H^2 + 2\int_0^t\int_\Omega f(u(\tau))v_\tau(\tau)\,d\tau - 2\int_0^t\int_\Omega \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(u_{x_i})v_\tau(\tau)\,d\tau \\ \le \|u_1\|^2 + \|u_0\|_H^2 + C_m + \int_0^t \|v_\tau(\tau)\|^2\,d\tau + \int_0^t \|\nabla v_\tau(\tau)\|^2\,d\tau, \end{aligned}$$

that is,

$$\|v_t\|^2 + \|v\|_H^2 \le 2R_0^2 + C_m + \int_0^t \|v_\tau(\tau)\|^2\,d\tau \le 2R_0^2 + C_m + \int_0^t (\|v_\tau\|^2 + \|v\|_H^2)\,d\tau,$$

where $C_m = \widetilde{C_1}(T) + \widetilde{C_2}(T)$. By the Gronwall inequality, we can get $\|v_t\|^2 + \|v\|_H^2 \le (2R_0^2 + C_m)e^T$. At this point, we can choose a small enough T such that $(2R_0^2 + C_m)e^T < R$. Then we have $\|v\|_{\mathscr{H}} \le R$, which shows that $\Phi(\mathscr{M}_{\mathscr{T}}) \subseteq \mathscr{M}_{\mathscr{T}}$.

Step (II). We claim that Φ is contractive in $\mathscr{M}_{\mathscr{T}}$, that is, there exists a positive constant α with $0 < \alpha < 1$ such that $\|\Phi(w_1) - \Phi(w_2)\|_{\mathscr{H}} \le \alpha\|w_1 - w_2\|_{\mathscr{H}}$ for $w_1, w_2 \in \mathscr{M}_{\mathscr{T}}$.

For $w_1, w_2 \in \mathscr{M}_{\mathscr{T}}$, set $v_1 = \Phi(w_1)$, $v_2 = \Phi(w_2)$, $z = w_1 - w_2$. Then we have

$$\begin{cases} z_{tt} - \Delta^2 z - \Delta z + \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(w_{1_{x_i}}) - \sum_{i=1}^n \frac{\partial}{\partial x_i}\sigma_i(w_{2_{x_i}}) - \Delta z_t \\ \qquad - |w_{1t}|^{r-1}w_{1t} - |w_{2t}|^{r-1}w_{2t} = f(w_1) - f(w_2), \\ z(x,0) = 0, \quad z_t(x,0) = 0 \quad \text{if } x \in \Omega, \\ z(x,t) = 0 \quad \text{if } x \in \partial\Omega,\ t \ge 0. \end{cases} \tag{3.13}$$

Multiplying the first equation of problem (3.13) by z_t and integrating the obtained results over $(0,T)\times\Omega$ yields

$$\frac{1}{2}\|z_t(t)\|^2+\frac{1}{2}\|z(t)\|_H^2+\int_0^t\|\nabla z_\tau\|^2\,d\tau=\int_0^t\int_\Omega(f(w_1(\tau))-f(w_2(\tau)))z_\tau(\tau)\,dx\,d\tau$$
$$-\int_\Omega\int_0^t(|w_{1\tau}|^{r-1}w_{1\tau}-|w_{2\tau}|^{r-1}w_{2\tau})z_\tau\,dx\,d\tau$$
$$-\int_0^t\int_\Omega\Big(\sum_{i=1}^n\frac{\partial}{\partial x_i}\sigma_i(w_{1_{x_i}})-\sum_{i=1}^n\frac{\partial}{\partial x_i}\sigma_i(w_{2_{x_i}})\Big)z_\tau\,dx\,d\tau. \tag{3.14}$$

According to (H_1), we can get $|f(w_1)-f(w_2)|\le a_1(|w_1|^p-|w_2|^p)$. Since $w_1,w_2\in\mathscr{M}_{\mathscr{T}}$, we can estimate the first term on the right side of (3.14) as

$$\Big|\int_0^t\int_\Omega(f(w_1(\tau))-f(w_2(\tau)))z_\tau(\tau)\,d\tau\Big|\le a_1\int_0^t\int_\Omega(|w_1|^p-|w_2|^p)z_\tau(\tau)\,dx\,d\tau$$
$$\le a_1\int_0^t\int_\Omega\xi(t)(w_1(\tau)-w_2(\tau))z_\tau(\tau)\,dx\,d\tau,$$

where $\xi(t)\ge0$ is given by the Lagrange theorem such that $\xi(t)\ge p(|w_1(t)|+|w_2(t)|)^{p-1}$. Then we get

$$a_1\int_0^t\int_\Omega\xi(t)(w_1(\tau)-w_2(\tau))z_\tau(\tau)\,dx\,d\tau\le CR^{2(p-1)}T\|w_1(\tau)-w_2(\tau)\|_{\mathscr{H}}^2+\frac{1}{2}\int_0^t\|\nabla z_\tau\|^2\,d\tau. \tag{3.15}$$

For the second term, we notice that the nonlinear function $h(s)=|s|^{r-1}s$ is increasing. For the third term on the right-hand side of (3.14), according to condition (H_2) and the mean value theorem, it shows

$$-\int_0^t\int_\Omega\Big(\sum_{i=1}^n\frac{\partial}{\partial x_i}\sigma_i(w_{1_{x_i}})-\sum_{i=1}^n\frac{\partial}{\partial x_i}\sigma_i(w_{2_{x_i}})\Big)z_\tau\,dx\,d\tau\le a_2\int_0^t\int_\Omega\sum_{i=1}^n\frac{\partial}{\partial x_i}\big(|w_{1_{x_i}}|^q-|w_{2_{x_i}}|^q\big)z_\tau\,dx\,d\tau$$
$$\le a_2\int_0^t\int_\Omega\sum_{i=1}^n\frac{\partial}{\partial x_i}\vartheta(t)(w_{1_{x_i}}-w_{2_{x_i}})z_\tau\,dx\,d\tau,$$

where $\vartheta(t)\ge0$ is given by the Lagrange theorem such that $\vartheta(t)\le q(|w_{1_{x_i}}(t)|+w_{2_{x_i}}(t))^{q-1}$. Then we get

$$a_2\int_0^t\int_\Omega\sum_{i=1}^n\frac{\partial}{\partial x_i}\vartheta(t)(w_{1_{x_i}}-w_{2_{x_i}})z_\tau\,dx\,d\tau\le CR^{2(q-1)}T\|w_1(\tau)-w_2(\tau)\|_{\mathscr{H}}^2+\frac{1}{2}\int_0^t\|\nabla z_\tau\|^2\,d\tau. \tag{3.16}$$

Therefore, by arguing as in (3.15) and (3.16), we obtain

$$\frac{1}{2}\|z_t(t)\|^2+\frac{1}{2}\|z(t)\|_H^2+\int_0^t\|\nabla z_\tau\|^2\,d\tau$$
$$\le CR^{2(p-1)}T\|w_1(\tau)-w_2(\tau)\|_{\mathscr{H}}^2+\frac{1}{2}\int_0^t\|\nabla z_\tau\|^2\,d\tau+CR^{2(q-1)}T\|w_1(\tau)-w_2(\tau)\|_{\mathscr{H}}^2+\frac{1}{2}\int_0^t\|\nabla z_\tau\|^2\,d\tau$$
$$=CT(R^{2(p-1)}+R^{2(q-1)})\|w_1(\tau)-w_2(\tau)\|_{\mathscr{H}}^2+\int_0^t\|\nabla z_\tau\|^2\,d\tau,$$

that is,

$$\|z_t(t)\|^2+\|z(t)\|_H^2\le 2CT(R^{2(p-1)}+R^{2(q-1)})\|z\|_{\mathscr{H}}^2.$$

By taking a small enough T, we can obtain the contractiveness of Φ. Hence the proof of this theorem is completed. □

4 Subcritical initial energy case $E(0) < d$

4.1 Global existence for $E(0) < d$

Lemma 4.1 (Invariant set W for $E(0) < d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold. Then the solution to problem* (1.1)–(1.3) *with $E(0) < d$ belongs to W, provided that $u_0 \in W$.*

Proof. Let $u(t)$ be a solution to problem (1.1)–(1.3) with $E(0) < d$ and $u_0 \in W$, and let T be the maximum existence time of $u(t)$. Then it follows from Lemma 2.5 that $E(u(t)) \le E(0) < d$. Thus it suffices to show that $I(u(t)) > 0$ for $0 < t < T$. Arguing by contradiction, we suppose that there exists $t_1 \in (0, T)$ such that $I(u(t_1)) \le 0$. From the continuity of the solution in time, there exists $t_* \in (0, T)$ such that $I(u(t_*)) = 0$. Then, from the definition of d, we have $d \le J(u(t_*)) \le E(u(t_*)) \le E(0) < d$, which is a contradiction. □

Theorem 4.2 (Global existence for $E(0) < d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold and $q \ge p$. Assume that $E(0) < d$ and $u_0 \in W$. Then problem* (1.1)–(1.3) *admits a global weak solution $u(t) \in L^\infty(0, \infty; H)$, $u_t \in L^\infty(0, \infty; L^2(\Omega))$.*

Proof. Let $\{w_j(x)\}$ be a system of base functions in H. Construct the approximate solution to problem (1.1)–(1.3)

$$u_m(x,t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x), \quad m = 1, 2, \dots$$

satisfying

$$\begin{aligned} &\langle u_{mtt}, w_s\rangle + (\Delta u_m, \Delta w_s) + (\nabla u_m, \nabla w_s) + (\nabla u_{mt}, \nabla w_s) - (f(u_m), w_s) \\ &\quad = -(|u_{mt}|^{r-1} u_{mt}, w_s) + \sum_{i=1}^{n} (\sigma_i(u_{mx_i}), w_{sx_i}), \quad s = 1, 2, \dots, m, \end{aligned} \tag{4.1}$$

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } H, \tag{4.2}$$

$$u_{mt}(x,0) = \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1(x) \quad \text{in } L^2(\Omega). \tag{4.3}$$

Multiplying (4.1) by $g'_{sm}(t)$ and summing for s, we get

$$\frac{d}{dt}\left(\frac{1}{2}\|u_{mt}\|^2 + \|u_m\|_H^2 - \sum_{i=1}^{n} \int_\Omega G_i(u_{mx_i})\,dx - \int_\Omega F(u_m)\,dx\right) = -\|u_{mt}\|_{r+1}^{r+1} - \|\nabla u_{mt}\|^2. \tag{4.4}$$

Integrating (4.4) over $(0, t)$, we can obtain

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \int_0^t (\|u_{m\tau}\|_{r+1}^{r+1} + \|\nabla u_{m\tau}\|^2)\,d\tau = E_m(0).$$

From (4.2) and (4.3), we get $E_m(0) \to E(0)$, $m \to +\infty$. Hence, for sufficiently large m, we have

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) + \int_0^t (\|u_{m\tau}\|_{r+1}^{r+1} + \|\nabla u_{m\tau}\|^2)\,d\tau < d. \tag{4.5}$$

Recalling Lemma 2.6, we note that

$$J(u_m) \ge \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_m\|_H^2 + \frac{1}{p+1} I(u_m). \tag{4.6}$$

Hence, from (4.5) and (4.6), we get

$$\frac{1}{2}\|u_{mt}\|^2 + \frac{p-1}{2(p+1)}\|u_m\|_H^2 + \frac{1}{p+1} I(u_m) + \int_0^t (\|u_{m\tau}\|_{r+1}^{r+1} + \|\nabla u_{m\tau}\|^2)\,d\tau < d. \tag{4.7}$$

By $u_0 \in W$ and $\frac{1}{2}\|u_{mt}(0)\|^2 + J(u_m(0)) = E(0)$, (4.2) and (4.3), we can get $u_m(0) \in W$ for sufficiently large m. From (4.5) and an argument similar to the proof of Lemma 4.1, we can prove that $u_m(t) \in W$ for $0 \le t < +\infty$ and sufficiently large m. Thus (4.7) gives

$$\frac{1}{2}\|u_{mt}\|^2 + \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u_m\|_H^2 + \int_0^t (\|u_{m\tau}\|_{r+1}^{r+1} + \|\nabla u_{m\tau}\|^2)\,d\tau < d \tag{4.8}$$

for sufficiently large m and $0 \le t < +\infty$.

Inequality (4.8) gives

$$u_m \text{ is bounded in } L^\infty(0,\infty;H), \tag{4.9}$$

$$u_{mt} \text{ is bounded in } L^2(0,\infty;H_0^1(\Omega)) \cap L^\infty(0,\infty;L^2(\Omega)), \tag{4.10}$$

$$|u_{mt}|^{r-1}u_{mt} \text{ is bounded in } L^\infty(0,\infty;L^{q_1}(\Omega)), \quad \text{where } q_1 = \frac{r+1}{r}, \tag{4.11}$$

$$|u_m|^{p-1}u_m \text{ is bounded in } L^\infty(0,\infty;L^{q_2}(\Omega)), \quad \text{where } q_2 = \frac{p+1}{p}. \tag{4.12}$$

Hence, integrating (4.1) with respect to t, for $0 \le t < +\infty$, we get

$$\begin{aligned}(u_{mt}, w_s) &+ \int_0^t (\Delta u_m, \Delta w_s)\,d\tau + \int_0^t (\nabla u_m, \nabla w_s)\,d\tau + (\nabla u_m, \nabla w_s) + \int_0^t (|u_{m\tau}|^{r-1}u_{m\tau}, w_s)\,d\tau \\ &= \int_0^t (f(u_m), w_s)\,d\tau + (u_{m1}, w_s) + (\nabla u_{m0}, \nabla w_s) \\ &\quad + \sum_{i=1}^n \int_0^t (\sigma_i(u_{mx_i}), w_{sx_i})\,d\tau \quad \text{for all } s \in H,\ 0 < t < T.\end{aligned} \tag{4.13}$$

Therefore, up to a subsequence, by (4.9)–(4.12), we may pass to the limit in (4.13) and obtain a weak solution $u(x,t)$ to problem (1.1)–(1.3) with the above regularity and (2.1). On the other hand, from (4.2) and (4.3), we have $u(x,0) = u_0(x)$ in H and $u_t(x,0) = u_1(x)$ in $L^2(\Omega)$. □

4.2 Exponential decay for $E(0) < d$

This section considers the special case of (H_1) and (H_2), namely,

$(\widetilde{\mathrm{H}_1})$ (i) $f \in C^1$ and $f(0) = f'(0) = 0$,
(ii) $f(u)$ is monotone and is convex for $u > 0$, concave for $u < 0$,
(iii) $|f(u)| \le a_1|u|^p$ and $(p+1)F(u) = uf(u)$ for some $a_1 > 0$ ($1 < p < \infty$ if $n \le 4$ and $1 < p < \frac{n}{n-4}$ if $n \ge 5$),
(iv) $F(u) = \int_0^u f(s)\,ds$;

$(\widetilde{\mathrm{H}_2})$ (i) $\sigma_i(s) \in C^1$ and $\sigma_i(0) = \sigma_i'(0) = 0$.
(ii) $\sigma_i(s)$ are monotone, and are convex for $s > 0$, concave for $s < 0$.
(iii) $|\sigma_i(s)| \le a_2|s|^q$ and $(q+1)G_i(s) = s\sigma_i(s)$ for some $a_2 > 0$ ($1 < q < \infty$ if $n = 1, 2$ and $1 < q < \frac{n}{n-2}$ if $n \ge 3$).
(iv) $G_i(s) = \int_0^s \sigma_i(\tau)\,d\tau$ for $1 \le i \le n$.

Lemma 4.3. *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let (H_1) and (H_2) hold and $q \ge p$. Assume that $u_0 \in W$ and $E(0) < d$. Then, for all $t \in [0,T)$, we have*

$$J(t) \equiv J(u(t)) \ge \frac{p-1}{2(p+1)}\|u(t)\|_H^2. \tag{4.14}$$

Proof. The assertion (4.14) follows directly from Lemma 4.1 and Lemma 2.6. □

Next we present the asymptotic behavior of the solution to problem (1.1)–(1.3) for $E(0) < d$.

Theorem 4.4 (Exponential decay for $E(0) < d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let $(\widetilde{\mathrm{H}_1})$ and $(\widetilde{\mathrm{H}_2})$ hold. Assume that $E(0) < d$ and $u_0 \in W$. Then there exist two positive constants K and k such that $E(t) \le Ke^{-kt}$ for $t \ge 0$.*

Remark 4.5. It is easy to see that, from (2.8), the condition $E(0) < d$ is equivalent to the inequality

$$\beta = a_1 C_1^{p+1}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{p-1}{2}} + a_2 C_2^{q+1}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{q-1}{2}} < 1,$$

which will be used in the proof of Theorem 4.4.

Proof. Inspired by the idea in [16, 29], we define the auxiliary function

$$G(t) := E(t) + \varepsilon(u, u_t) + \frac{\varepsilon}{2}\|\nabla u\|^2. \tag{4.15}$$

We can choose ε so small that

$$\alpha_1 E(t) \le G(t) \le \alpha_2 E(t) \quad \text{for some } \alpha_1, \alpha_2 > 0. \tag{4.16}$$

Testing equation (1.1) by $u(x, t)$, we have

$$\langle u, u_{tt}\rangle + (\nabla u, \nabla u_t) + \|u\|_H^2 + \int_\Omega |u_t|^{r-1}u_t u\,dx = \sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx. \tag{4.17}$$

Then, from (4.15), Lemma 2.5 and (4.17), we can obtain

$$\begin{aligned} G'(t) &= E'(t) + \varepsilon(\|u_t\|^2 + \langle u, u_{tt}\rangle + (\nabla u, \nabla u_t)) \\ &= -\|u_t\|_{r+1}^{r+1} - \|\nabla u_t\|^2 + \varepsilon\|u_t\|^2 - \varepsilon\int_\Omega |u_t|^{r-1}u_t u\,dx + \varepsilon\Big(\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx - \|u\|_H^2\Big). \end{aligned} \tag{4.18}$$

From (2.2), (2.3), Lemma 2.6 and (2.4), we have

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + J(u) \ge \frac{1}{2}\|u_t\|^2 + \Big(\frac{1}{2} - \frac{1}{p+1}\Big)\|u\|_H^2 + \frac{1}{p+1}I(u) \\ &= \frac{1}{2}(\|u_t\|^2 + \|u\|_H^2) + \frac{1}{p+1}(I(u) - \|u\|_H^2) \\ &= \frac{1}{2}(\|u_t\|^2 + \|u\|_H^2) - \frac{1}{p+1}\Big(\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx\Big). \end{aligned}$$

Now we use Lemma 2.6, Lemma 2.7 and Remark 4.5 to estimate

$$\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx,$$

which yields

$$\begin{aligned} \sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx &= a\Big(\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx\Big) \\ &\quad + (1-a)\Big(\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx\Big) \\ &\le \frac{(q+1)a}{2}(\|u_t\|^2 + \|u\|_H^2) - a(q+1)E(t) + \beta(1-a)\|u\|_H^2, \end{aligned} \tag{4.19}$$

where $0 < a < 1$. By Young's inequality to estimate $\int_\Omega |u_t|^{r-1}u_t u\,dx$, we have

$$\Big|\int_\Omega |u_t|^{r-1}u_t u\,dx\Big| \le \delta\|u_t(t)\|_{r+1}^{r+1} + C(\delta)\|u(t)\|_{r+1}^{r+1} \quad \text{for } \delta > 0. \tag{4.20}$$

By (4.19), (4.20) and a simple computation, (4.18) becomes

$$G'(t) \le \varepsilon b\|u_t\|^2 + \varepsilon\Big(\frac{(q-1)a}{2} - (1-\beta)(1-a)\Big)\|u\|_H^2 - \varepsilon(q+1)aE(t) + (\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} + \varepsilon C(\delta)\|u(t)\|_{r+1}^{r+1} - \|\nabla u_t\|^2, \tag{4.21}$$

where $b = \frac{(q+1)a}{2} + 1$. By (4.14) and choosing a close to 1 so that $\frac{(q-1)a}{2} - (1-\beta)(1-a) \ge 0$, then (4.21) reaches to

$$G'(t) \le \varepsilon b\|u_t\|^2 + \varepsilon\Big(\frac{(q-1)a}{2} - (1-\beta)(1-a)\Big)\frac{2(p+1)}{p-1}E(t) - \|\nabla u_t\|^2 - \varepsilon(q+1)aE(t) + (\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} + \varepsilon C(\delta)\|u(t)\|_{r+1}^{r+1}.$$

By the Poincaré inequality $\|u_t\| \le C\|\nabla u_t\|$, we can obtain

$$G'(t) \le (\varepsilon bC^2 - 1)\|\nabla u_t\|^2 + \varepsilon\Big(\frac{(q-1)a}{2} - (1-\beta)(1-a)\Big)\frac{2(p+1)}{p-1}E(t) - \varepsilon(q+1)aE(t) + (\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} + \varepsilon C(\delta)\|u(t)\|_{r+1}^{r+1}. \tag{4.22}$$

Next we exploit the Sobolev embedding inequality and Lemma 4.3 to estimate $\|u(t)\|_{r+1}^{r+1}$ as

$$\|u(t)\|_{r+1}^{r+1} \le C_{r+1}^{r+1}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{r-1}{2}}\|u\|_H^2 \le C_{r+1}^{r+1}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{r-1}{2}}\frac{2(p+1)}{p-1}E(t).$$

Therefore, (4.22) turns into

$$G'(t) \le -\varepsilon\Big((1-\beta)(1-a) - C(\delta)C_{r+1}^{r+1}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{r-1}{2}}\Big)\frac{2(p+1)}{p-1}E(t) + (\varepsilon\delta - 1)\|u_t\|_{r+1}^{r+1} + (\varepsilon bC^2 - 1)\|\nabla u_t\|^2 + \varepsilon a\Big(q - 1 - \frac{(q+1)(p-1)}{p+1}\Big)\frac{p+1}{p-1}E(t). \tag{4.23}$$

At this point, we first choose δ so small that

$$\gamma_1 = (1-\beta)(1-a) - C(\delta)C_{r+1}^{r+1}\Big(\frac{2(p+1)}{p-1}E(0)\Big)^{\frac{r-1}{2}} > 0,$$

and then, once δ is fixed (hence γ_1 is fixed also), we can pick $q < \frac{(p+1)\gamma_1}{a} + p$ so that

$$\gamma_2 = \gamma_1 - a\Big(q - 1 - \frac{(q+1)(p-1)}{p+1}\Big) > 0.$$

Moreover, once δ is chosen, we can take ε small so that $\gamma_3 = \varepsilon bC^2 - 1 \le 0$ and $\gamma_4 = \varepsilon\delta - 1 \le 0$, and (4.16) remains valid. Consequently, (4.23) becomes

$$G'(t) \le \gamma_3\|u_t\|^2 + \gamma_4\|u_t\|_{r+1}^{r+1} - \varepsilon\gamma_2\frac{2(p+1)}{p-1}E(t). \tag{4.24}$$

A simple integration of (4.24) then leads to $G(t) \le G(0)e^{-kt}$, which, together with (4.16), gives $E(t) \le Ke^{-kt}$, where $k = \frac{2\varepsilon\gamma_2(p+1)}{\alpha_2(p-1)}$. □

4.3 Finite time blowup for $E(0) < d$

In what follows, we state the finite time blowup of the solution to problem (1.1)–(1.3). By the same argument as Lemma 4.3, we can get the following lemma.

Lemma 4.6 (Invariant set for $E(0) < d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold. Then the solution to problem* (1.1)–(1.3) *with* $E(0) < d$ *belongs to* V, *provided that* $u_0 \in V$.

Lemma 4.7. *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold and $q \geq p$. Assume that $u_0 \in V$ and $E(0) < d$. Then we have*

$$\frac{p-1}{2(p+1)}\|u\|_H^2 > d. \tag{4.25}$$

Proof. By Lemma 4.6, we get $u(t) \in V$, that is, $I(u) < 0$, which, together with (H_1), (H_2) and Corollary 2.3, gives

$$\begin{aligned}\|u\|_H^2 < \sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx &\leq a_1\|u\|_{p+1}^{p+1} + a_2\sum_{i=1}^n \|u_{x_i}\|_{q+1}^{q+1}\\ &\leq a_1 C_1^{p+1}\|u\|_H^{p+1} + a_2 C_2^{q+1}\|u\|_H^{q+1} = h(\|u\|_H)\|u\|_H^2\end{aligned}$$

or, equivalently, $h(\|u\|_H) > 1$ and $\|u\|_H > r$. Hence, by Lemma 2.6, we have

$$J(u) \geq \frac{p-1}{2(p+1)}\|u\|_H^2 + \frac{1}{p+1}I(u) = \frac{p-1}{2(p+1)}\|u\|_H^2 > \frac{p-1}{2(p+1)}r^2,$$

which, together with (2.8), gives (4.25). □

Theorem 4.8 (Finite time blowup for $E(0) < d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, $q \geq p > r > 1$, and let* (H_1) *and* (H_2) *hold. Assume that $E(0) < d$ and $u_0 \in V$. Then the solution to problem* (1.1)–(1.3) *blows up in finite time.*

Proof. Let $u(t)$ be the solution to problem (1.1)–(1.3) with $E(0) < d$ and $I(u_0) < 0$, and let T be the maximum existence time of $u(t)$. Then we prove $T < +\infty$. Arguing by contradiction, we suppose $T = +\infty$. For any $T > 0$, we define $\theta(t) := \|u\|^2$ and

$$\vartheta(t) := a_1\|u\|_{p+1}^{p+1} + a_2\sum_{i=1}^n \|u_{x_i}\|_{q+1}^{q+1}.$$

Then, for $t \in [0, T]$, we get $\theta'(t) = 2(u, u_t)$ and

$$\begin{aligned}\theta''(t) &= 2\langle u, u_{tt}\rangle + 2\|u_t\|^2\\ &= 2\|u_t\|^2 - 2\|u\|_H^2 - 2(|u_t|^{r-1}u_t, u) - 2(\nabla u, \nabla u_t)\\ &\quad + 2\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + 2\int_\Omega uf(u)\,dx.\end{aligned} \tag{4.26}$$

We now exploit the Hölder inequality, the so-called interpolation inequality, Lemma 4.6 and (H_2) to estimate the third term of (4.26). From Lemma 4.6 and (H_2), we have

$$\|u\|^2 < c\|u\|_H^2 < c\bigg(\sum_{i=1}^n \int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \int_\Omega uf(u)\,dx\bigg) < c\vartheta(t)$$

and

$$c_0\|u\|_{p+1}^{p+1} \leq a_1\|u\|_{p+1}^{p+1} + a_2\sum_{i=1}^n \|u_{x_i}\|_{q+1}^{q+1} = \vartheta(t),$$

which, together with the Hölder inequality and the so-called interpolation inequality, give

$$\big|(|u_t|^{r-1}u_t, u)\big| \leq \|u\|_{r+1}\|u_t\|_{r+1}^r \leq \|u\|^\delta\|u\|_{p+1}^{1-\delta}\|u_t\|_{r+1}^r < C\vartheta(t)^{\frac{1}{r+1}}\vartheta(t)^{\frac{1-\delta}{p+1}-\frac{1}{r+1}+\frac{\delta}{2}}\|u_t\|_{r+1}^r, \tag{4.27}$$

where $\delta = (\frac{1}{r+1} - \frac{1}{p+1})/(\frac{1}{2} - \frac{1}{p+1})$. It is easy to see that $\frac{1-\delta}{p+1} - \frac{1}{r+1} + \frac{\delta}{2} = 0$. By applying Young's inequality to estimate the right side of (4.27), we have

$$|(|u_t|^{r-1}u_t, u)| < \eta_1\vartheta(t) + c_{\eta_1}\|u_t\|_{r+1}^{r+1} \quad \text{for any } \eta_1 > 0. \tag{4.28}$$

Therefore, from (2.4) and (4.28), equation (4.26) becomes

$$\theta''(t) > 2\|u_t\|^2 - 2(\nabla u_t, \nabla u) - 2I(u) - 2\eta_1\vartheta(t) - 2c_{\eta_1}\|u_t\|_{r+1}^{r+1}. \tag{4.29}$$

Recalling (2.2) and (2.4), we get

$$\begin{aligned} I(t) &\le I(t) + \sigma(E(0) - E(t)) \\ &\le I(t) + \sigma E(0) - \frac{\sigma}{2}\|u_t\|^2 - \frac{\sigma}{2}\|u\|_H^2 + \frac{\sigma}{p+1}\sum_{i=1}^{n}\int_\Omega u_{x_i}\sigma_i(u_{x_i})\,dx + \frac{\sigma}{p+1}\int_\Omega uf(u)\,dx \\ &\le \Big(\frac{\sigma}{p+1} - 1\Big)\vartheta(t) + \sigma E(0) - \frac{\sigma}{2}\|u_t\|^2 + \Big(1 - \frac{\sigma}{2}\Big)\|u\|_H^2, \end{aligned} \tag{4.30}$$

where the constant $\sigma > 2$ will be chosen later. By (4.29) and (4.30), we have

$$\begin{aligned} \theta''(t) &> (2+\sigma)\|u_t\|^2 + (\sigma - 2)\|u\|_H^2 - 2\sigma E(0) - 2(\nabla u, \nabla u_t) \\ &\quad + 2\Big(1 - \frac{\sigma}{p+1} - \eta_1\Big)\vartheta(t) - 2c_{\eta_1}\|u_t\|_{r+1}^{r+1}, \end{aligned} \tag{4.31}$$

which, along with Young inequality, gives

$$\begin{aligned} \theta''(t) &> (2+\sigma)\|u_t\|^2 + (\sigma-2)\|u\|_H^2 - 2\sigma E(0) - 2\eta_2\|\nabla u\|^2 \\ &\quad - 2c_{\eta_2}\|\nabla u_t\|^2 + 2\Big(1 - \frac{\sigma}{p+1} - \eta_1\Big)\vartheta(t) - 2c_{\eta_1}\|u_t\|_{r+1}^{r+1} \\ &> (2+\sigma)\|u_t\|^2 + (\sigma-2)\|u\|_H^2 - 2\sigma E(0) - 2\eta_2 C\|u\|^2 \\ &\quad - 2c_{\eta_2}\|\nabla u_t\|^2 + 2\Big(1 - \frac{\sigma}{p+1} - \eta_1\Big)\vartheta(t) - 2c_{\eta_1}\|u_t\|_{r+1}^{r+1} \\ &> (2+\sigma)\|u_t\|^2 + (\sigma-2)\|u\|_H^2 - 2\sigma E(0) - 2c_{\eta_2}\|\nabla u_t\|^2 \\ &\quad + 2\Big(1 - \frac{\sigma}{p+1} - \eta_1 - cC\eta_2\Big)\vartheta(t) - 2c_{\eta_1}\|u_t\|_{r+1}^{r+1} \end{aligned} \tag{4.32}$$

for any $\eta_2 > 0$, where C is the best constant of Poincaré's inequality $C\|u\| \le \|\nabla u\|$. We choose the constant σ so that

$$\frac{2(p+1)d}{(p+1)d - (p-1)E(0)} < \sigma < p+1,$$

which guarantees that $\sigma > 2$ since $E(0) < d$. Then, by this choice and (4.25), we get

$$(\sigma-2)\|u\|_H^2 - 2\sigma E(0) > \frac{2(p+1)}{p-1}(\sigma-2)d - 2\sigma E(0) = 2\Big(\frac{p+1}{p-1}d - E(0)\Big)\sigma - \frac{4(p+1)}{p-1}d > 0. \tag{4.33}$$

Once the constant σ is fixed, we choose the constant $\eta = \max\{\eta_1, cC\eta_2\}$ so that

$$C = 1 - \frac{\sigma}{p+1} - \eta > 0. \tag{4.34}$$

Hence, by (4.32), (4.34), (4.33) and Lemma 4.7, inequality (4.31) becomes

$$\theta''(t) + c_\eta(2\|u_t\|_{r+1}^{r+1} + 2\|\nabla u_t\|^2) \ge 2C\vartheta(t) \ge 2C\|u\|_H^2 \ge \frac{4C(q+1)}{q-1}d > 0, \tag{4.35}$$

where $c_\eta = \max\{c_{\eta_1}, c_{\eta_2}\}$. Integrating the last inequality of (4.35) about t yields

$$\theta'(t) + c_\eta\int_0^t (2\|u_\tau\|_{r+1}^{r+1} + 2\|\nabla u_\tau\|^2)\,d\tau > \frac{4C(q+1)}{q-1}td + \theta'(0). \tag{4.36}$$

Note that

$$\int_0^t \|\nabla u_\tau\|^2\,d\tau + \int_0^t \|u_\tau\|_{r+1}^{r+1}\,d\tau = E(0) - E(t) < D. \tag{4.37}$$

By (4.37), inequality (4.36) becomes

$$\theta'(t) > \frac{4C(q+1)}{q-1}td + \theta'(0) - c_\eta D. \tag{4.38}$$

Integrating (4.38) from 0 to t, we can obtain

$$\theta(t) > \frac{4C(q+1)}{q-1} t^2 d + (\theta'(0) - c_\eta D)t + \theta(0) \quad \text{for all } t > 0. \tag{4.39}$$

That is, $\theta(t)$ has quadratic growth as $t \to +\infty$.

On the other hand, we estimate $\|u(t)\|^2$ as follows. By the regularity of $u(t)$ in $L^2(\Omega)$, the Hölder inequality and (4.38), for all $t \geq 0$, we have

$$\begin{aligned}
\int_\Omega |u(t)|^2\,dx &= \int_\Omega \Big(u_0 + \int_0^t u_t(t)\,dt\Big)^2 dx \\
&\leq 2\|u_0\|^2 + 2t \int_\Omega \int_0^t |u_t|^2\,dt\,dx \\
&\leq 2\|u_0\|^2 + Ct^{1+\frac{r-1}{r+1}} \Big(\int_0^t \int_\Omega |u_t|^{r+1}\,dx\,dt\Big)^{\frac{2}{r+1}} \\
&\leq 2\|u_0\|^2 + Ct^{\frac{2r}{r+1}} d^{\frac{2}{r+1}}.
\end{aligned} \tag{4.40}$$

Hence, by (4.40), for all $t \geq 0$, we have

$$\theta(t) \leq 2\|u_0\|^2 + Ct^{\frac{2r}{r+1}} d^{\frac{2}{r+1}}. \tag{4.41}$$

Since the power $\frac{2r}{r+1}$ is smaller than 2, then (4.41) contradicts (4.39), which shows that $\theta(t)$ has at least quadratic growth for $t > 0$. Therefore, the solution cannot be extended to the whole interval $[0, +\infty)$. This completes the proof of Theorem 4.8. □

5 Critical initial energy case $E(0) = d$

In this section, we study the global existence, exponential decay and finite time blowup for the critical initial energy case $E(0) = d$.

Lemma 5.1. *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let (H_1) and (H_2) hold. If $u(x, t)$ is an unsteady solution to problem (1.1)–(1.3), then there exists a $\bar{t} \in (0, T)$ such that*

$$\int_0^{\bar{t}} \|\nabla u_\tau\|^2\,d\tau + \int_0^{\bar{t}} \|u_\tau\|_{r+1}^{r+1}\,d\tau > 0. \tag{5.1}$$

Proof. Let $u(t)$ be an unsteady solution to problem (1.1)–(1.3) with $E(0) = d$, and let T be the maximum existence time of $u(t)$. We prove that there exists a $\bar{t} \in (0, T)$ such that (5.1) holds. Arguing by contradiction, we suppose that, for $t \in [0, T)$,

$$\int_0^t \|\nabla u_\tau\|^2\,d\tau + \int_0^t \|u_\tau\|_{r+1}^{r+1}\,d\tau \equiv 0,$$

which implies $\|\nabla u_t\|^2\,d\tau + \|u_t\|_{r+1}^{r+1} = 0$ for all $x \in \Omega$, $t \in [0, T)$. Thus we can conclude $u(x, t) \equiv u_0$ for all $x \in \Omega$, $t \in [0, T)$, that is, $u(x, t)$ is a steady solution to problem (1.1)–(1.3), which is a contradiction. □

5.1 Global existence for $E(0) = d$

Next we give the invariance of the stable set W under problem (1.1)–(1.3) at the critical initial energy level $E(0) = d$.

Lemma 5.2 (Invariant set W for $E(0) = d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let (H_1) and (H_2) hold. If $E(0) = d$, then the stable set W is invariant for problem (1.1)–(1.3).*

Proof. Let $u_0 \in W$ be any solution to problem (1.1)–(1.3) with $E(0) = d$, and let T be the existence time of $u(t)$. Arguing by contradiction, we suppose that there exists a $t_0 \in (0, T)$ such that $u(t_0) \in \partial W$, that is, $I(u(t_0)) = 0$, $\|u(t_0)\|_H \neq 0$. Combining with the definition of the depth of potential well, we have $J(u(t_0)) \geq d$. According to Lemma 2.5, we have

$$\frac{1}{2}\|u_t(t_0)\|^2 + J(u(t_0)) + \int_0^{t_0} \|\nabla u_\tau(\tau)\|^2 \, d\tau + \int_0^{t_0} \|u_\tau(\tau)\|_{r+1}^{r+1} \, d\tau = E(0) = d.$$

Therefore, we can get

$$\frac{1}{2}\|u_t(t_0)\|^2 + \int_0^{t_0} \|\nabla u_\tau(\tau)\|^2 \, d\tau + \int_0^{t_0} \|u_\tau(\tau)\|_{r+1}^{r+1} \, d\tau = 0,$$

which implies that $\frac{du}{dt} = 0$ for all $0 \leq t \leq t_0$ and $x \in \Omega$. In other words, we have $u(x, t) = u_0(x)$ for all $0 \leq t \leq t_0$ and $x \in \Omega$. Then $I(u(t_0)) = I(u_0) > 0$, obviously, which contradicts assumption $I(u(t_0)) = 0$. This completes the proof. □

The global existence for problem (1.1)–(1.3) with $E(0) = d$ is given as follows.

Theorem 5.3 (Global existence for $E(0) = d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let (H_1) and (H_2) hold. If $E(0) = d$ and $u_0 \in W$, then problem (1.1)–(1.3) admits a global solution.*

Proof. It suffices to prove the maximal time $T = +\infty$. Obviously, if $u(x, t)$ is a steady solution to problem (1.1)–(1.3), then we have $T = +\infty$. If $u(x, t)$ is a solution but not a steady solution to problem (1.1)–(1.3), then there exists a $\bar{t} \in (0, T)$ from Lemma 5.1 such that

$$\int_0^{\bar{t}} \|\nabla u_\tau(\tau)\|^2 \, d\tau + \int_0^{\bar{t}} \|u_\tau(\tau)\|_{r+1}^{r+1} \, d\tau > 0.$$

Combining Lemma 2.5 and $E(0) = d$, we have $E(\bar{t}) < d$ and $u(\bar{t}) \in W$ due to Lemma 5.2, that is, $I(u(\bar{t})) > 0$ or $\|u(\bar{t})\|_H = 0$. Let $v(t) = u(t + \bar{t})$ for $t \geq 0$. Then $v(t)$ is a solution to problem (1.1)–(1.3). Therefore, the maximum time of $v(t)$ is infinite due to Theorem 4.2, which implies $T = +\infty$. This completes the proof. □

5.2 Exponential decay for $E(0) = d$

Theorem 5.4 (Exponential decay for $E(0) = d$). *Under the conditions of Theorem 5.3, for some positive constants K_1 and k, we have*

$$0 < E(t) \leq K_1 e^{-kt}, \quad 0 \leq t < +\infty. \tag{5.2}$$

Proof. Immediately, (5.2) follows from Lemma 5.1, Theorem 5.3 and Theorem 4.4. □

5.3 Finite time blowup for $E(0) = d$

Similar to Lemma 5.2, we infer the following.

Lemma 5.5 (Invariant set V for $E(0) = d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let (H_1) and (H_2) hold. If $E(0) = d$, then the unstable set V is invariant for problem (1.1)–(1.3).*

Theorem 5.6 (Finite time blowup for $E(0) = d$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let (H_1) and (H_2) hold. If $E(0) = d$ and $u_0 \in V$, then the solution to problem (1.1)–(1.3) blows up in finite time.*

Proof. Similar to the proof of Theorem 5.3, this conclusion follows from Lemma 5.1, Lemma 5.5 and Theorem 4.8. □

6 Arbitrarily positive initial energy case $E(0) > 0$ when $r = 1$

Lemma 6.1 (Invariant set V for $E(0) > 0$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold. Then the solution to problem* (1.1)–(1.3) *belongs to V, provided that $u_0 \in V$ and*

$$\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) > AE(0) > 0, \tag{6.1}$$

where $A = \frac{2(p+1)(C+2)}{(p-1)C}$, C is the best constant of Poincaré's inequality

$$\|\nabla u\|^2 \ge C\|u\|^2. \tag{6.2}$$

Proof. We prove $u(t) \in V$ for $t \in [0, T_0)$. Arguing by contradiction, we suppose that $t_0 \in (0, T)$ is the first time such that

$$I(u(t_0)) = 0 \tag{6.3}$$

and $I(u(t)) < 0$ for $t \in [0, t_0)$. First we introduce the auxiliary function $F(t) := \|\nabla u(t)\|^2 + \|u(t)\|^2 + 2(u, u_t)$. Then, from equation (1.1), it follows

$$F'(t) = 2(\nabla u, \nabla u_t) + 2\langle u, u_{tt}\rangle + 2\|u_t\|^2 + 2(u, u_t) = 2\|u_t\|^2 - 2I(u).$$

Hence, by $u(t) \in V$, we have $F'(t) > 0$ for $t \in [0, +\infty)$. Moreover, from (6.1) and $E(0) > 0$, this implies that

$$F(0) = \|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) > AE(0) > 0, \quad \text{where} \quad A = \frac{2(p+1)(C+2)}{(p-1)C}.$$

Therefore, we can see that $F(t) > F(0) > 0$, which shows that $\{t \mapsto \|\nabla u(t)\|^2 + \|u(t)\|^2 + 2(u, u_t)\}$ is strictly increasing, namely,

$$\|\nabla u(t)\|^2 + \|u(t)\|^2 + 2(u, u_t) \ge \|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0, u_1) \ge AE(0), \quad t \in [0, t_0).$$

Besides, it gives

$$\|\nabla u(t_0)\|^2 + \|u(t_0)\|^2 + 2(u(t_0), u_t(t_0)) > \frac{2(p+1)(C+2)}{(p-1)C}E(0), \tag{6.4}$$

according to the continuity of $u(t)$ and $u_t(t)$ in t. Recalling (2.2) and (2.5), we gain

$$\begin{aligned} E(0) &= E(t) + \int_0^t \|\nabla u_\tau\|^2\, d\tau + \int_0^t \|u_\tau\|^2\, d\tau \\ &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_H^2 - \sum_{i=1}^n \int_\Omega G_i(u_{x_i})\, dx - \int_\Omega F(u)\, dx + \int_0^t \|\nabla u_\tau\|^2\, d\tau + \int_0^t \|u_\tau\|^2\, d\tau \\ &= \frac{1}{2}\|u_t\|^2 + \frac{p-1}{2(p+1)}\|u\|_H^2 + \frac{1}{p+1}I(u) + \int_0^t \|\nabla u_\tau\|^2\, d\tau + \int_0^t \|u_\tau\|^2\, d\tau, \end{aligned} \tag{6.5}$$

which, together with (6.3), (6.2) and the Cauchy–Schwarz inequality, shows that

$$\begin{aligned} E(0) = E(t_0) &\ge \frac{1}{2}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|u(t_0)\|_H^2 \\ &\ge \frac{(p-1)C}{2(p+1)(C+2)}\|u_t(t_0)\|^2 + \frac{p-1}{2(p+1)}\|\nabla u(t_0)\|^2 \\ &= \frac{(p-1)C}{2(p+1)(C+2)}\|u_t(t_0)\|^2 + \frac{(p-1)C}{2(p+1)(C+2)}\|\nabla u(t_0)\|^2 + \frac{p-1}{2(p+1)(C+2)}\|\nabla u(t_0)\|^2 \\ &\ge \frac{(p-1)C}{2(p+1)(C+2)}\|u_t(t_0)\|^2 + \frac{(p-1)C}{2(p+1)(C+2)}\|\nabla u(t_0)\|^2 + \frac{(p-1)C}{2(p+1)(C+2)}\|\nabla u(t_0)\|^2 \\ &\ge \frac{(p-1)C}{2(p+1)(C+2)}(\|\nabla u(t_0)\|^2 + \|u(t_0)\|^2 + 2(u(t_0), u_t(t_0))). \end{aligned} \tag{6.6}$$

Obviously, (6.4) contradicts (6.6). This completes the proof. □

In the end, we present the finite time blow-up result of the solution to problem (1.1)–(1.3) with arbitrarily high initial energy.

Theorem 6.2 (Finite time blowup for $E(0) > 0$). *Let $u_0 \in H$, $u_1 \in L^2(\Omega)$, and let* (H_1) *and* (H_2) *hold. Assume that $u_0 \in V$ and* (6.1) *holds. Then the solution to problem* (1.1)–(1.3) *blows up in finite time.*

Proof. Let $u(t)$ be any solution to problem (1.1)–(1.3), (6.1) and $u_0 \in V$. Then, from Lemma 6.1, it follows that $u \in V$. Next we prove that the solution to problem (1.1)–(1.3) blows up in finite time. Arguing by contradiction, we suppose that the solution $u(x,t)$ is global. Then, for any $T_0 > 0$, we introduce the auxiliary function

$$B(t) := \|u\|^2 + \int_0^t (\|\nabla u(\tau)\|^2 + \|u(\tau)\|^2)\, d\tau + (T_0 - t)(\|\nabla u_0\|^2 + \|u_0\|^2).$$

It is obvious that $B(t) > 0$ for all $t \in [0, T_0]$. From the continuity of $B(t)$ in t, it is easy to see that there exists $\rho > 0$ (independent of the choice of T_0) such that

$$B(t) \geq \rho \quad \text{for all } t \in [0, T_0]. \tag{6.7}$$

Moreover, for $t \in [0, T_0]$, we can get

$$B'(t) = 2(u, u_t) + 2(\|\nabla u\|^2 + \|u\|^2) - 2(\|\nabla u_0\|^2 + \|u_0\|^2) = 2(u, u_t) + 2\int_0^t \int_\Omega (\nabla u \nabla u_\tau + u u_\tau)\, dx\, d\tau, \tag{6.8}$$

$$B''(t) = 2\|u_t\|^2 + 2\langle u, u_{tt}\rangle + 2\int_\Omega (\nabla u \nabla u_\tau + u u_\tau)\, dx = 2\|u_t\|^2 - 2I(u). \tag{6.9}$$

From (6.8) this implies that

$$(B'(t))^2 = 4\bigg((u, u_t)^2 + 2(u, u_t)\int_0^t \int_\Omega (\nabla u \nabla u_\tau + u u_\tau)\, dx\, d\tau\bigg) + 4\bigg(\int_0^t \int_\Omega (\nabla u \nabla u_\tau + u u_\tau)\, dx\, d\tau\bigg)^2. \tag{6.10}$$

Then, from the Cauchy–Schwarz inequality, it follows

$$(u, u_t)^2 \leq \|u\|^2 \|u_t\|^2,$$

$$\bigg(\int_0^t \int_\Omega (\nabla u \nabla u_\tau + u u_\tau)\, dx\, d\tau\bigg)^2 \leq \int_0^t (\|\nabla u\|^2 + \|u\|^2)\, d\tau \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau$$

and

$$\begin{aligned} 2(u, u_t)\int_0^t \int_\Omega (\nabla u \nabla u_\tau + u u_\tau)\, dx\, d\tau &\leq 2\|u\|\|u_t\|\bigg(\int_0^t (\|\nabla u\|^2 + \|u\|^2)\, d\tau\bigg)^{\frac{1}{2}} \bigg(\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau\bigg)^{\frac{1}{2}} \\ &\leq \|u\|^2 \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau + \|u_t\|^2 \int_0^t (\|\nabla u\|^2 + \|u\|^2)\, d\tau. \end{aligned}$$

Therefore, (6.10) becomes

$$\begin{aligned} (B'(t))^2 &\leq 4\bigg(\|u\|^2 + \int_0^t (\|\nabla u\|^2 + \|u\|^2)\, d\tau\bigg)\bigg(\|u_t\|^2 + \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau\bigg) \\ &\leq 4B(t)\bigg(\|u_t\|^2 + \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau\bigg). \end{aligned} \tag{6.11}$$

Hence, from (6.9) and (6.11), we have

$$\begin{aligned} B''(t)B(t) - \frac{\lambda + 3}{4}(B'(t))^2 &\geq B(t)\bigg(B''(t) - (\lambda + 3)\bigg(\|u_t\|^2 + \int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau\bigg)\bigg) \\ &\geq B(t)\bigg(-(\lambda + 1)\|u_t\|^2 - 2I(u) - (\lambda + 3)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\, d\tau\bigg). \end{aligned} \tag{6.12}$$

Now we define

$$\xi(t) := -(\lambda+1)\|u_t\|^2 - 2I(u) - (\lambda+3)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\,d\tau. \tag{6.13}$$

From (6.5) and the Cauchy–Schwarz inequality, we gain

$$\begin{aligned}
\xi(t) &= (p-\lambda)\|u_t\|^2 + (p-1)(\|\Delta u\|^2 + \|\nabla u\|^2) \\
&\quad - 2(p+1)E(0) - (2p-\lambda-1)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\,d\tau \\
&\geq (p-\lambda)\|u_t\|^2 + \frac{2(p-\lambda)}{C}\|\nabla u\|^2 + \Big(p-1-\frac{2(p-\lambda)}{C}\Big)\|\nabla u\|^2 \\
&\quad - 2(p+1)E(0) - (2p-\lambda-1)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\,d\tau \\
&\geq (p-\lambda)\|u_t\|^2 + 2(p-\lambda)\|u\|^2 + \Big(p-1-\frac{2(p-\lambda)}{C}\Big)\|\nabla u\|^2 \\
&\quad - 2(p+1)E(0) - (2p-\lambda-1)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\,d\tau \\
&\geq (p-\lambda)(\|u_t\|^2 + 2\|u\|^2) + \Big(p-1-\frac{2(p-\lambda)}{C}\Big)\|\nabla u\|^2 \\
&\quad - 2(p+1)E(0) - (2p-\lambda-1)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\,d\tau \\
&\geq (p-\lambda)(2(u,u_t) + \|u\|^2) + \Big(p-1-\frac{2(p-\lambda)}{C}\Big)\|\nabla u\|^2 \\
&\quad - 2(p+1)E(0) - (2p-\lambda-1)\int_0^t (\|\nabla u_\tau\|^2 + \|u_\tau\|^2)\,d\tau.
\end{aligned} \tag{6.14}$$

At this point, we choose $\lambda := p - \frac{C(p-1)}{C+2}$, which guarantees that $\lambda \in (1,p)$ since $p > 1$. Then, by a simple computation and $\lambda < 1 + 2p$, we can get

$$\begin{aligned}
\xi(t) &\geq \frac{C(p-1)}{C+2}(\|\nabla u\|^2 + \|u\|^2 + 2(u,u_t)) - 2(p+1)E(0) \\
&> \frac{C(p-1)}{C+2}(\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0,u_1)) - 2(p+1)E(0).
\end{aligned} \tag{6.15}$$

Let

$$\sigma := \frac{C(p-1)}{C+2}(\|\nabla u_0\|^2 + \|u_0\|^2 + 2(u_0,u_1)) - 2(p+1)E(0).$$

Then, by taking advantage of (6.1), we can get

$$\xi(t) \geq \sigma > 0. \tag{6.16}$$

Therefore, by (6.12)–(6.16) and (6.7), we have

$$B''(t)B(t) - \frac{\lambda+3}{4}(B'(t))^2 > \rho\sigma > 0, \quad t \in [0, T_0].$$

Set $y(t) = B(t)^{-\frac{\lambda-1}{4}}$. Then this inequality becomes

$$y''(t) < -\frac{\lambda-1}{4}\rho\sigma y(t)^{\frac{\lambda+7}{\lambda-1}}, \quad t \in [0, T_0],$$

where $\lambda = p - \frac{C(p-1)}{C+2}$. This proves that $y(t)$ reaches 0 in finite time, say $t \to T_*$. Since T_* is independent of the initial choice of T_0, we may assume that $T_* < T_0$. This tells us that $\lim_{t\to T_*} B(t) = +\infty$, which completes the proof. □

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