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Positive supersolutions of fourth-order nonlinear elliptic equations: explicit estimates and Liouville theorems

Asadollah Aghajani [a,b], Craig Cowan [c], Vicenţiu D. Rădulescu [d,e,f,*]

[a] *School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran*
[b] *School of Mathematics, Institute for Research in Fundamental Sciences (IPM), P.O. Box 19395-5746, Tehran, Iran*
[c] *Department of Mathematics, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada*
[d] *Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland*
[e] *Department of Mathematics, University of Craiova, 200585 Craiova, Romania*
[f] *'Simion Stoilow' Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania*

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Abstract

In this paper, we consider positive supersolutions of the semilinear fourth-order problem

$$\begin{cases} (-\Delta)^2 u = \rho(x) f(u) & \text{in } \Omega, \\ \quad -\Delta u > 0 & \text{in } \Omega, \end{cases}$$

where Ω is a domain in $\mathbb{R}^N$ (bounded or not), $f : D_f = [0, a_f) \to [0, \infty)$ $(0 < a_f \leqslant +\infty)$ is a non-decreasing continuous function with $f(u) > 0$ for $u > 0$ and $\rho : \Omega \to \mathbb{R}$ is a positive function. Using a maximum principle-based argument, we give explicit estimates on positive supersolutions that can easily be applied to obtain Liouville-type results for positive supersolutions either in exterior domains, or in unbounded domains Ω with the property that $\sup_{x\in\Omega} \operatorname{dist}(x, \partial\Omega) = \infty$. In particular, we consider the above problem with $f(u) = u^p$ $(p > 0)$ and with different weights $\rho(x) = |x|^a$, e^{ax_1} or x_1^m (m is an even integer). Also, when f is convex and $\rho : \Omega \to (0, \infty)$ is smooth with $\Delta(\sqrt{\rho}) > 0$, then under an extra condition between f and ρ we show that every positive supersolution u of this problem with $u = 0$ on $\partial\Omega$ (Ω bounded) satisfies the inequality $-\Delta u \geq \sqrt{2\rho(x)F(u)}$ for all $x \in \Omega$, where $F(t) := \int_0^t (f(s) - f(0))ds$.

* Corresponding author at: Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland.

E-mail addresses: aghajani@iust.ac.ir (A. Aghajani), craig.cowan@umanitoba.ca (C. Cowan), radulescu@inf.ucv.ro (V.D. Rădulescu).

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1. Introduction and main results

Our purpose in the present paper is to obtain explicit pointwise estimates on positive classical supersolutions of the problem

$$\begin{cases} (-\Delta)^2 u = \rho(x) f(u) & \text{in } \Omega, \\ -\Delta u > 0 & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a domain in $\mathbb{R}^N$ $(N \geq 1)$ (bounded or not) and f, ρ satisfy

$(\mathcal{C})$ $f : D_f = [0, a_f) \to [0, \infty)$ $(0 < a_f \leqslant +\infty)$ is a non-decreasing continuous function and $\rho : \Omega \to \mathbb{R}$ is a positive smooth function. Also we assume that $f(u) > 0$ for $u > 0$.

By a positive classical solution of (1) we mean a positive function $u \in C^4(\Omega)$, verifying $(-\Delta)^2 u \geq \rho(x) f(u)$ and $-\Delta u > 0$ in Ω pointwise.

In this paper, we give explicit estimates on positive classical supersolutions u of problem (1) at each point $x \in \Omega$. As we shall see, the simplicity and robustness of our maximum principle-based estimates provide their applicability to many fourth-order elliptic inequalities on arbitrary domains in $\mathbb{R}^N$, either bounded or unbounded. We are mainly interested in applications to Liouville-type theorems related to (1) with different weights in unbounded domains with the property that

$$\sup_{x \in \Omega} \operatorname{dist}(x, \partial\Omega) = \infty.$$

In this way, our applications extend to $\mathbb{R}^N$, $\mathbb{R}^N_+$, exterior domains, or cone-like domains, as well as for obtaining upper bounds for the extremal parameter of fourth-order nonlinear eigenvalue problem under Navier boundary conditions on bounded domains.

Existence or nonexistence of solutions to some classes of higher order differential equations and systems on $\mathbb{R}^N$ have received a great deal of attention in recent years. For instance, a differential equation or inequality of the form

$$(-\Delta)^m u \geq f(u) \;\text{ in }\; \Omega, \tag{2}$$

where $\Omega = \mathbb{R}^N$ or an exterior domain in $\mathbb{R}^N$. A relevant special case of (2) is when $f(u) = u^p$ with $p > 0$, that is $(-\Delta)^m u \geq u^p$. It is well known that if $1 < p < \frac{N}{N-2m}$ then the latter inequality in the whole space does not admit any nonnegative *polysuperharmonic* solution u, that is, $(-\Delta)^i u \geq 0$ in Ω, $i = 1, \ldots, m$; see for example Corollary 3.6 in Caristi, D'Ambrosio and Mitidieri [9], where the authors have proved Liouville theorems for supersolutions of the polyharmonic Hénon-Lane-Emden system and also explored its connection with the Hardy-Littlewood-Sobolev systems. Also, for the Liouville theorems for the polyharmonic Lane-Emden equation $(-\Delta)^m u = u^p$ in $\Omega = \mathbb{R}^N$, see Lin [30] and Wei and Xu in [34] for the subcritical Sobolev exponent that is $1 < p < \frac{N+2m}{N-2m}$, $N > 2m$.

Recently, Perez, Melian and Quaas [8] studied the existence and nonexistence of positive supersolutions to the biharmonic problem

$$(-\Delta)^2 u = g(u) \ \text{ in } \mathbb{R}^N \setminus B_{R_0}, \tag{3}$$

where B_{R_0} stands for the ball of radius R_0 centered at the origin and g is continuous and non-decreasing in $[0, \infty)$. They proved that for $1 \le N \le 4$, problem (3) does not admit any positive classical supersolution u verifying

$$-\Delta u > 0 \ \text{ in } \mathbb{R}^N \setminus B_{R_0}. \tag{4}$$

They also proved that if $N \ge 5$, such supersolutions exist if and only if

$$\int_0^\delta \frac{g(t)}{t^{\frac{2N-4}{N-4}}} dt < \infty, \tag{5}$$

for any $\delta > 0$. To prove the results above, they employed the maximum principle and the method of sub and supersolutions, by showing that the existence of a positive supersolution u of problem (3) with the additional property (4) implies the existence of a radially symmetric positive solution of the same problem with the same property.

We also refer to Guo and Liu [24], where the authors established the nonexistence of non-trivial nonnegative classical solutions for problem (3) with $g(u) = u^p$ and Dirichlet boundary condition $u = \frac{\partial u}{\partial \nu} = 0$ on ∂B_{R_0}, where ν is the unit outward normal vector of ∂B_{R_0} relative to B_{R_0} whenever $1 < p < \frac{N+4}{N-4}$, or the Navier boundary condition $u = \Delta u = 0$ on ∂B_{R_0} when $1 < p \le \frac{N+4}{N-4}$. The study of this type of equations plays an important role in conformal geometry [11,18,28] and other related fields [20]. For more results on the structure of positive solutions or classification of positive entire solutions via Morse index of the equation (3) with $g(u) = u^p$, or some related problems, we refer to [12,15–17,23,27,26,31,35,36] and the references therein. It is worth mentioning here that nonlinear Liouville theorems for second order equations of the form $-\Delta u = f(u)$ have been frequently discussed in the literature, and there are general results on nonexistence for both positive solutions and supersolutions (see for instance the references in [1,2]).

In this paper, by just using the maximum principle for the Laplace operator, we estimate the solutions of (1) in any ball $B_r(x) \subset \Omega$. As we shall see, our estimates can be easily applied to obtain Liouville-type results for solutions of the general equation (1) in unbounded domains (see section 2).

In order to formulate our main estimates, we need to introduce some notation as follows. Define, for a given positive supersolution u of problem (1),

$$m_x(r) = \inf_{y \in B_r(x)} u(y) \ \text{ and } \ \rho_x(r) = \inf_{y \in B_r(x)} \rho(y) \ \text{ for } \ 0 < r < d_\Omega(x) := \operatorname{dist}(x, \partial\Omega).$$

We set $d_\Omega(x) = +\infty$ if $\Omega = \mathbb{R}^N$.

Theorem 1. *Let u be a positive classical supersolution of problem (1) with f, ρ satisfying condition ($\mathcal{C}$). Then for all $x \in \Omega$ we have*

$$\int_{m_x(r)}^{u(x)} \frac{ds}{f(s)} \geq \frac{1}{N^2(N+2)} \int_0^r s^3 \rho_x(s)ds, \quad 0 < r < d_\Omega(x). \tag{6}$$

In particular, when $\rho \equiv 1$ we have

$$\int_{m_x(r)}^{u(x)} \frac{ds}{f(s)} \geq \frac{r^4}{4N^2(N+2)}, \quad 0 < r < d_\Omega(x). \tag{7}$$

Remark 1. (a) Notice that if $\frac{1}{f} \in L^1(0, a)$ with $0 < a < a_f$, then the above result provides an explicit lower estimate for $u(x)$ in terms of $d_\Omega(x)$. Indeed, in this case, taking $H(t) := \int_0^t \frac{ds}{f(s)}$ we get from (6) that

$$u(x) \geq H^{-1}\Big(\frac{1}{N^2(N+2)} \int_0^{d_\Omega(x)} s^3 \rho_x(s)ds\Big), \quad x \in \Omega,$$

where H^{-1} is the inverse function of H. Also, when $\frac{1}{f} \notin L^1(0, a)$ for $0 < a < a_f$, then the estimate (6) gives an upper bound for $\inf_{y \in B_r(x)} u(y)$ for any $x \in \Omega$ and $0 < r < d_\Omega(x)$. This estimate, together with Lemma 1 below, will be used to obtain Liouville-type results on exterior domains in $\mathbb{R}^N$.

(b) The requirement that the supersolutions verify the inequality $-\Delta u > 0$, in order to obtain Liouville theorems, is by no means superfluous (see a discussion in [9]), and examples of supersolutions not enjoying this property can be constructed.

Another interesting problem related to equation (1) is to find pointwise inequalities for $-\Delta u$, provided that u is a positive classical supersolution. In the case of the fourth-order Lane-Emden equation

$$(-\Delta)^2 u = u^p \ \text{ in } \ \mathbb{R}^N, \tag{8}$$

Souplet [33] proved that the following pointwise inequality holds for nonnegative solutions of problem (8):

$$-\Delta u \geq \sqrt{\frac{2}{p+1}} u^{\frac{p+1}{2}} \ \text{ in } \ \mathbb{R}^N. \tag{9}$$

Indeed, if we set $v = -\Delta u$ then, from the fact that $-\Delta u \geq 0$, we can consider (8) as a special case (when $q = 1$) of the Lane-Emden system

$$\begin{cases} -\Delta u = v^q & \text{in } \mathbb{R}^N, \\ -\Delta v = u^p & \text{in } \mathbb{R}^N, \end{cases} \tag{10}$$

where $p \geq q \geq 1$. Then from Lemma 2.7 in [33] one has

$$\frac{u^{p+1}}{p+1} \leq \frac{v^{q+1}}{q+1} \ \text{ in } \mathbb{R}^N,$$

for nonnegative solutions u and v of (10) when $pq > 1$. Applying the above inequality, we see that the pointwise inequality (9) holds for nonnegative solutions of (8).
Recently, Fazly, Wei and Xu [21] improved the above result and established that every bounded positive solution u of the fourth-order Hénon equation

$$(-\Delta)^2 u = |x|^a u^p \ \text{ in } \mathbb{R}^N, \tag{11}$$

satisfies the following pointwise inequality

$$-\Delta u \geq \sqrt{\frac{2}{p+1-c_N}} |x|^{\frac{a}{2}} u^{\frac{p+1}{2}} + \frac{2}{N-4} \frac{|\nabla u|^2}{u} \ \text{ in } \mathbb{R}^N, \tag{12}$$

where $c_N = \frac{8}{N-2}$ and $0 \leq a \leq \inf_{k \geq 0} A_k$ (where A_k is defined in [21, relation (4.28)]). For the proof of this estimate, motivated by Moser's proof of the Harnack inequality as well as by Moser iteration type arguments in the regularity theory, the authors in [21] developed an iteration method to establish the above pointwise inequality.

We also refer to Cowan, Esposito and Ghoussoub [13] who proved that if u is a positive solution of the fourth-order autonomous problem

$$\begin{cases} (-\Delta)^2 u = f(u) & x \in \Omega, \\ u = \Delta u = 0 & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and f is smooth, increasing and convex with $f(0) > 0$ then

$$-\Delta u \geq \sqrt{2\tilde{F}(u)}, \ \ x \in \Omega,$$

where

$$\tilde{F}(t) := \int_0^t \tilde{f}(s)ds, \ \ \tilde{f}(t) := f(t) - f(0). \tag{13}$$

In this paper, we prove the following pointwise inequality for $-\Delta u$, for any positive supersolution u of the non-autonomous problem (1) in any bounded domain.

Theorem 2. *Let u be a positive classical supersolution of problem (1) in a bounded domain $\Omega \subset \mathbb{R}^N$ with $u = 0$ on $\partial\Omega$. We assume that $f : [0, a_f) \to [0, \infty)$ is smooth, increasing and strictly convex, and $\rho : \Omega \to (0, \infty)$ is smooth with $\Delta(\sqrt{\rho}) \ge 0$. Moreover, we assume that*

$$\frac{4f'(t)\tilde{F}(t)}{\tilde{f}(t)^2} \ge \tau_\rho := \sup_{x\in\Omega} \frac{\Delta\rho}{\sqrt{\rho}\Delta(\sqrt{\rho})} < \infty \text{ for all } t \in (0, ||u||_\infty), \tag{14}$$

where $\tilde{F}(t)$ defined in (13). Then u satisfies the pointwise inequality

$$-\Delta u \ge \sqrt{2\rho(x)\tilde{F}(u)}, \quad x \in \Omega. \tag{15}$$

For example, when $f(u) = u^p$ and $\rho(x) = |x|^a$ ($a \in \mathbb{R}$) so that ρ is smooth and subharmonic in a domain Ω (which depends on a and $0 \in \Omega$ or not), then we have

$$\tau_\rho = \frac{4(a + N - 2)}{a + 2N - 4} < \infty,$$

which is independent of Ω and estimate (14) is equivalent to

$$p > \frac{N - 2 + a}{N - 2}.$$

If in problem (1) the functions ρ, f additionally satisfy the conditions of Theorem 2 one can use this result to improve the estimate of Theorem 1 as follows. We just consider the case when $\rho(x) \equiv 1$ and will apply it to bound the extremal parameter of semilinear biharmonic elliptic problems under Navier boundary conditions.

Theorem 3. *Let u be a positive classical supersolution of the problem*

$$\begin{cases} (-\Delta)^2 u = f(u) & \text{in } \Omega, \\ -\Delta u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

where Ω is a bounded domain in $\mathbb{R}^N$ and $f : [0, a_f) \to [0, \infty)$ is smooth, increasing and strictly convex. Then u satisfies the pointwise inequality

$$\int_{m_{x_0}(r)}^{u(x_0)} \frac{ds}{f(s) + \frac{N(N+2)}{\sqrt{2}r_\Omega^2}\sqrt{\tilde{F}(s)}} \ge \frac{r^4}{4N^2(N+2)}, \quad \text{for} \quad 0 < r < d_\Omega(x), \tag{17}$$

where $r_\Omega := \sup_{x\in\Omega} d_\Omega(x)$ is the radius of the largest ball contained in Ω. Also, we have

$$-\Delta u \ge \sqrt{2\tilde{F}(u)}, \quad x \in \Omega. \tag{18}$$

In order to apply the above estimates to get Liouville-type results we also need the following auxiliary property.

Lemma 1. *Suppose that $u > 0$ is a smooth function such that*

$$-\Delta u > 0 \ \textit{and} \ (-\Delta)^2 u > 0,$$

in an exterior domain $\Omega \subset \mathbb{R}^N$ ($N \geq 5$). Then there exists a positive constant c, depending only on u, Ω and N, so that

$$u(x) \geq c|x|^{4-N}, \quad x \in \Omega, \tag{19}$$

and

$$\liminf_{|x|\to\infty} u(x) \leq C \ \textit{and} \ \liminf_{|x|\to\infty} -\Delta u(x) \leq C. \tag{20}$$

Proof. Since $(-\Delta)^2 u = -\Delta(-\Delta u) \geq 0$ then $-\Delta u$ is a positive superharmonic function in Ω. Then it is well-known (see [32] or [4, Lemma 2.5]) that we have

$$-\Delta u \geq C|x|^{2-N} \quad \text{in } \Omega.$$

Fix $r_0 > 0$ such that $\mathbb{R}^N \setminus B_{r_0} \subset \Omega$. Select $c > 0$ so small that $c < \frac{C}{2(N-4)}$ and also $u \geq c|x|^{4-N}$ in a neighborhood of ∂B_{r_0}. Then for each $\varepsilon > 0$, there exists $R_\varepsilon > r_0$ such that $u + \varepsilon \geq \varepsilon \geq c|x|^{4-N}$ in $\mathbb{R}^N \setminus B_{R_\varepsilon}$. Now note that we have

$$-\Delta u \geq C|x|^{2-N} \geq 2c(N-4)|x|^{2-N} = -\Delta(c|x|^{4-N}).$$

Applying the maximum principle in $B_R \setminus B_{r_0}$, for each $R > R_\varepsilon$ we get

$$u + \varepsilon \geq c|x|^{4-N} \quad \text{in } \mathbb{R}^N \setminus B_{R_0}.$$

Letting $\varepsilon \to 0$ we obtain $u \geq c|x|^{4-N}$ in $\mathbb{R}^N \setminus B_{r_0}$ that proves (19). Also, since u and $-\Delta u$ are positive superharmonic functions then the inequality (20) is a consequence of Lemma 2.5 in [4]. $\square$

Finally, we point out that the main results included in this paper can be generalized to the higher order differential inequality

$$(-\Delta)^m u \geq \rho(x) f(u) \ \text{ in } \Omega, \tag{21}$$

where $u \in C^{2m}(\Omega)$ verifies

$$(-\Delta)^i u \geq 0 \quad \text{in} \ \ \Omega, \quad i = 1, ..., m. \tag{22}$$

2. Applications

In this section we give some applications of our main estimates.

2.1. Liouville-type results

Proposition 1. *Consider the problem*

$$\begin{cases} (-\Delta)^2 u = |x|^a u^p & \text{in } \Omega, \\ -\Delta u > 0 & \text{in } \Omega, \end{cases} \tag{23}$$

where $p > 0$ and Ω is a domain in $\mathbb{R}^N$. Then

(a) *if $p > 1$ and Ω is an exterior domain in $\mathbb{R}^N$, $N \geq 5$, and*

$$p \leq \frac{N+a}{N-4}, \tag{24}$$

then the above problem does not admit any positive, classical supersolution in Ω. Also, the same property is true when $p = 1$ and $a > -4$.

(b) *Let $p < 1$ and $a > -4$. If Ω is an exterior domain in $\mathbb{R}^N$, $N \geq 5$, then problem (23) does not admit any positive, classical supersolution in Ω. Also, the same nonexistence result holds for bounded classical supersolutions if Ω is an unbounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with the property that $\sup_{x\in\Omega} d_\Omega(x) = \infty$.*

(c) *Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$. When $0 \notin \Omega$ we assume $a \geq 0$, and $a \geq 4$ when $0 \in \Omega$. If u is a positive, classical supersolution of problem (23) with $u = 0$ in $\partial\Omega$, then we have*

$$-\Delta u \geq \sqrt{\frac{2}{p+1}}\,|x|^{\frac{a}{2}} u^{\frac{p+1}{2}}, \quad \text{in } \Omega, \tag{25}$$

provided that $p > \frac{N-2+a}{N-2}$.

Proof. (a) First note that with $f(u) = u^p$ we have

$$\int\limits_{m_x(r)}^{u(x)} \frac{ds}{f(s)} = \frac{u(x)^{1-p} - m_x(r)^{1-p}}{1-p}, \quad 0 < r < d_\Omega(x), \quad x \in \Omega, \quad p \neq 1 \tag{26}$$

and when $p = 1$

$$\int\limits_{m_x(r)}^{u(x)} \frac{ds}{f(s)} = \ln \frac{u(x)}{m_x(r)}, \quad 0 < r < d_\Omega(x), \quad x \in \Omega. \tag{27}$$

To prove (a), for simplicity take $\Omega := \mathbb{R}^N - B_1$. Then for $\rho(x) = |x|^a$ when $a \geq 0$ we have

$$\rho_x(r) = \inf_{B_r(x)} \rho(y) = (|x| - r)^a, \quad 0 < r < |x| - 1.$$

By Theorem 1, for $x \in \Omega$ and $0 < r < |x| - 1$ we obtain, when $p \neq 1$

$$\frac{u(x)^{1-p} - m_x(r)^{1-p}}{1-p} \geq \frac{1}{N^2(N+2)} \int_0^r s^3(|x|-s)^a ds$$

$$= \frac{|x|^{4+a}}{N^2(N+2)} \int_0^{\frac{r}{|x|}} t^3(1-t)^a dt. \tag{28}$$

Similarly, for $a < 0$, we get (noticing that $\rho_x(r) = (|x|+r)^a$ in this case)

$$\frac{u(x)^{1-p} - m_x(r)^{1-p}}{1-p} \geq \frac{|x|^{4+a}}{N^2(N+2)} \int_0^{\frac{r}{|x|}} t^3(1+t)^a dt, \quad 0 < r < |x| - 1. \tag{29}$$

Now let $p > 1$ and $a \in \mathbb{R}$. From inequalities (28) and (29), for $\frac{|x|}{2} < r < |x| - 1$ we obtain

$$m_x(r) \leq C|x|^{\frac{-(4+a)}{p-1}}, \tag{30}$$

where

$$C := \left(\frac{p-1}{N^2(N+2)} \int_0^{\frac{1}{2}} t^3(1-(\operatorname{sgn} a)t)^a dt \right)^{\frac{-1}{p-1}},$$

in which sgn is the signum function, and note that C is a constant independent of x, r.
On the other hand, by Lemma 1 we have $m_x(r) \geq c|x|^{4-N}$ when $\frac{|x|}{2} < r < |x| - 1$. This latter inequality together with (30) implies that $N - 4 \geq \frac{4+a}{p-1}$ or $p \geq \frac{N+a}{N-4}$. Thus, there is no any positive supersolution if $p < \frac{N+a}{N-4}$.
To prove the result when $p = \frac{N+a}{N-4}$, note that in this case we have $\frac{4+a}{p-1} = N - 4$. It then follows by (30) that

$$m_x(r) \leq C|x|^{4-N}, \quad \frac{|x|}{2} < r < |x| - 1.$$

Also by Lemma 1 we have $m_x(r) \geq c|x|^{4-N}$, when $\frac{|x|}{2} < r < |x| - 1$. Thus, taking $\beta(r) := \inf_{\mathbb{R}^N \setminus B_r} \frac{u(x)}{|x|^{4-N}}$, we must have $c \leq \beta(r) \leq C$. Now from the fact that $-\Delta u \geq C|x|^{2-N}$, using Lemma 2.2 in [4] and similar to end of the proof of Theorem 2.1 in [4] we can show that $\beta(r) \to \infty$, which is a contradiction.
We now consider the case $p = 1$. By (27) and the estimate obtained above on $\int_0^r s^3 \rho_x(s) ds$, we obtain that

$$u(x) \geq m_x(r) e^{C|x|^{4+a}}.$$

Next, by Lemma 1 we obtain

$$u(x) \geq C_1 |x|^{4-N} e^{C|x|^{4+a}}.$$

Hence, when $a > -4$, we have $u(x) \to \infty$ as $|x| \to \infty$, which contradicts (20) in Lemma 1.

(b) Now assume $p < 1$. Then from the inequalities in part (a) we get

$$u(x) \geq C d_\Omega(x)^{\frac{4+a}{1-p}}.$$

If $a > -4$ and Ω is an exterior domain then the above inequality implies that $\liminf_{|x|\to\infty} u(x) = \infty$, which contradicts Lemma 1. Also, if $a > -4$ and $\sup_{x\in\Omega} d_\Omega(x) = \infty$ then u can not be bounded by the above inequality.

(c) Now let Ω be a bounded domain and $u = 0$ on $\partial\Omega$. By the assumption on Ω the function $\rho(x) = |x|^a$ is smooth with $\Delta(\sqrt{\rho}) \geq 0$ and

$$\tau_\rho = \frac{4(a+N-2)}{a+2N-4},$$

where τ_ρ defined in (14). Also we have, $\tilde{F}(t) = \frac{t^{p+1}}{p+1}$, thus (14) is equivalent to

$$\frac{4f'(t)\tilde{F}(t)}{\tilde{f}(t)^2} = \frac{4p}{p+1} \geq \frac{4(a+N-2)}{a+2N-4},$$

or

$$p \geq \frac{a+N-2}{N-2}.$$

And in this range of p we have from Theorem 2

$$-\Delta u \geq \sqrt{2\rho(x)\tilde{F}(u)} = \sqrt{\frac{2}{p+1}} |x|^{\frac{a}{2}} u^{\frac{p+1}{2}}.$$

The proof is now complete. $\square$

Remark 2. It is worth mentioning that the above results can be also obtained for the more general problem

$$\begin{cases} (-\Delta)^2 u = |x|^a f(u) & \text{in } \Omega, \\ -\Delta u \geq 0 & \text{on } \partial\Omega, \end{cases} \tag{31}$$

where f satisfies $(\mathcal{C})$. One can also establish that the nonexistence results for positive supersolutions depend on the behavior of $f(t)$ near zero, as follows. Consider for example the case when Ω is an exterior domain in $\mathbb{R}^N$, $(N \geq 5)$. From (20) in Lemma 1, there exists a sequence $x_j \in \Omega$

so that $|x_j| \to \infty$ as $j \to \infty$ and $u(x_j) \le C < \infty$. Then using Theorem 1 and the computations we did above, we get

$$\int_{m_{x_j}(r)}^{C} \frac{ds}{f(s)} \ge \int_{m_{x_j}(r)}^{u(x_j)} \frac{ds}{f(s)} \ge C_1 |x_j|^{4+a}, \tag{32}$$

for $\frac{|x_j|}{2} < r < d_{x_j}(\Omega)$ and j large, also by Lemma 1, $m_{x_j}(r) \ge c|x_j|^{4-N}$. Then from (32) we infer that

$$\int_{|x_j|^{4-N}}^{C} \frac{ds}{f(s)} \ge C_1 |x_j|^{4+a},$$

or

$$|x_j|^{-(4+a)} \int_{|x_j|^{4-N}}^{C} \frac{ds}{f(s)} \ge C_1, \quad \text{for } j \text{ large}. \tag{33}$$

But (33) fails if $a > -4$ and for some $C < \infty$

$$\lim_{t \to 0} t^{\frac{4+a}{N-4}} \int_{t}^{C} \frac{ds}{f(s)} = 0. \tag{34}$$

Hence, there exists no positive supersolution for problem (31) in exterior domains if (34) holds.

We present in what follows some examples illustrating that Theorem 1 can be used to deal with other related problems. For instance, this can occur when the weight $|x|^a$ is replaced by $|x_1|^a$, x_1^m, e^{ax_1} or even more general functions.

We first consider the problem

$$\begin{cases} (-\Delta)^2 u = e^{ax_1} u^p & \text{in } \Omega \\ -\Delta u > 0 & \text{in } \Omega, \end{cases} \tag{35}$$

where $p > 0$, $a > 0$ and Ω is an unbounded domain in $\mathbb{R}^N$.
We have the following nonexistence result for the supersolutions of (35).

Proposition 2. *Consider the problem* (35).

(a) *When $p \ge 1$ and Ω is an exterior domain, then problem* (35) *does not admit any positive, classical supersolution.*

(b) *When $p<1$, and Ω is an unbounded domain in $\mathbb{R}^N$ with the property that*

$$\sup\{x_1;\ \ x=(x_1,...,x_N)\in\Omega\}=\infty,$$

then the above problem does not have any bounded classical supersolution.

Proof. (a) For $\rho(x)=e^{ax_1}$ we have

$$\rho_x(r)=\inf_{B_r(x)} e^{ay_1}=e^{a(x_1-r)},\ \ 0<r<d_\Omega(x),\ \ x\in\Omega.$$

Then we compute

$$\begin{aligned}\int_0^r s^3\rho_x(s)ds &= e^{ax_1}\int_0^r s^3e^{-as}ds\\ &= e^{ax_1}\left(\frac{6}{a^4}-e^{-ar}\left(\frac{r^3}{a}+\frac{3r^2}{a^2}+\frac{6r}{a^3}+\frac{6}{a^4}\right)\right)\\ &\ge \frac{5}{a^4}e^{ax_1},\ \ \text{for } r \text{ sufficiently large.}\end{aligned}$$

As before, by Theorem 1, for $x\in\Omega$ and $0<r<d_\Omega(x)$ with $|x|$ large, we obtain, when $p\neq 1$,

$$\frac{u(x)^{1-p}-m_x(r)^{1-p}}{1-p}\ge\frac{5}{a^4}e^{ax_1},\ \ \text{for } r \text{ sufficiently large.} \tag{36}$$

Similarly, for $p=1$ we get

$$u(x)\ge m_x(r)e^{\frac{5}{a^4}e^{ax_1}},\ \ \text{for } r \text{ sufficiently large.} \tag{37}$$

If $p>1$ we obtain from (36)

$$m_x(r)\le Ce^{\frac{-ax_1}{p-1}},\ \ \text{for } 0<r<d_\Omega(x) \text{ sufficiently large.}$$

Now for all points $x=(x_1,x_1,...,x_1)$ with $x_1>0$, the above estimate implies

$$m_x(r)\le Ce^{\frac{-a|x|}{(p-1)\sqrt{N}}},\ \ \text{for } 0<r<d_\Omega(x) \text{ sufficiently large,}$$

which is impossible as we know that $m_x(r)\ge C|x|^{4-N}$ for $|x|$ large and $\frac{r}{|x|}>\frac{1}{2}$, $r<d_\Omega(x)$. When $p=1$ and Ω is an exterior domain we get from (37) and Lemma 1

$$u(x)\ge|x|^{4-N}e^{\frac{5}{a^4}e^{ax_1}},\ \ \text{for } |x| \text{ sufficiently large.}$$

Then for all points $\bar{x}=(x_1,x_1,...,x_1)$ with $x_1>0$ the above estimate implies that

$$u(\bar{x}) \geq |\bar{x}|^{4-N} e^{\frac{5}{a^4} e^{a \frac{|\bar{x}|}{\sqrt{N}}}}, \quad \text{for } |\bar{x}| \text{ sufficiently large.}$$

It then follows that $u(\bar{x}) \to \infty$ as $|\bar{x}| \to \infty$, hence u is unbounded.

(b) We consider the case $p < 1$. From (36) we get

$$u(x) \geq \left(\frac{5(1-p)}{a^4}\right)^{\frac{1}{1-p}} e^{\frac{ax_1}{1-p}}, \quad \text{for } |x| \text{ sufficiently large.}$$

We deduce that if $\sup\{x_1;\ x = (x_1, ..., x_N) \in \Omega\} = \infty$, then u cannot be bounded. $\square$

Now we consider problem (1) with the weight $\rho(x) = x_1^m$ and $f(u) = u^p$. In this regard, we mention that the following problem for the Laplacian case

$$-\Delta u = x_1^m u^p \ \text{ in } \ \mathbb{R}^N,$$

where m is a positive integer, has been already considered in previous literature, see for example [7], [29], [19]. However, in all these works only odd integers are allowed. Our methods enable us to obtain a Liouville theorem for positive supersolutions in the complementary case where m is an even integer.

Consider the problem

$$\begin{cases} (-\Delta)^2 u = x_1^m u^p & \text{in } \Omega \\ -\Delta u > 0 & \text{in } \Omega. \end{cases} \tag{38}$$

We have the following nonexistence result for the supersolutions of (38).

Proposition 3. *Consider problem (38), where $p > 0$, $m > 0$ is an even integer and Ω is an unbounded domain in $\mathbb{R}^N$. Then the following properties hold.*

(a) *If Ω is an exterior domain in $\mathbb{R}^N$, $N \geq 5$ and $1 \leq p \leq \frac{N+m}{N-4}$, then the above problem does not admit any positive, classical supersolution in Ω.*
(b) *When $p < 1$, and Ω is an unbounded domain in $\mathbb{R}^N$ with the property that*

$$\sup\{x_1;\ \ x = (x_1, ..., x_N) \in \Omega\} = \infty,$$

then the above problem does not have any bounded positive classical supersolution.

Proof. We first apply the estimates in Theorem 1 to $\bar{x} \in \Omega$ where $\bar{x} = (x_1, ..., x_1)$ $(x_1 > 0)$. For the function $\rho(x) = x_1^m$ and $r = \frac{x_1}{2}$, for which $B_r(\bar{x}) \subset \Omega$, we have

$$\rho_{\bar{x}}(r) = \inf_{B_r(\bar{x})} y_1^m = (x_1 - r)^m \geq \frac{x_1^m}{2^m}.$$

Then we compute

$$\int_0^r s^3 \rho_{\bar{x}}(s)ds \geq \frac{x_1^{m+4}}{2^{m+2}} = C|\bar{x}|^{m+4}.$$

By the above estimate and as before, using Theorem 1 and Lemma 1, for $\bar{x} \in \Omega$ and $r = \frac{x_1}{2}$ with $|\bar{x}|$ large, we obtain

$$m_{\bar{x}}(r) \leq C|\bar{x}|^{\frac{m+4}{p-1}}, \quad p > 1, \tag{39}$$

$$u(\bar{x}) \geq C|x|^{4-N} e^{|\bar{x}|^{m+4}}, \quad p = 1 \tag{40}$$

and

$$u(\bar{x}) \geq C|\bar{x}|^{\frac{m+4}{1-p}}, \quad p < 1. \tag{41}$$

Now the rest of the proof uses the same ideas as in the proof of Propositions 1 and 2. □

Remark 3. Notice that using our main estimates one can also extend the above nonexistence results to the more general problem

$$\begin{cases} (-\Delta)^2 u \geq \rho(x_1) f(u) & \text{in } \Omega, \\ -\Delta u > 0 & \text{in } \Omega, \end{cases} \tag{42}$$

where $\rho : \mathbb{R} \to [0, \infty)$ is a smooth function, and will see that the results depend on the behavior of $\rho(t)$ at infinity, and not to the monotonicity property of ρ. To see similar problems to (42) for the Laplacian case we refer to [29], [19] and the references therein. We also notice that for the above problems (35) and (38) we cannot apply our Theorem 2 to get a pointwise inequality for $-\Delta u$, because for the function $\rho(x) = e^{ax_1}$ we have $\tau_\rho = 4$, and for $\rho(x) = x_1^m$ we have $\tau_\rho > 4$, while for $f(u) = u^p$ we have $\frac{4f'(t)\tilde{F}(t)}{\tilde{f}(t)^2} = \frac{4p}{p+1} < 4$. Thus, relation (14) in Theorem 2 does not hold in these cases. However, if we take in (42), for example

$$\rho(x_1) = e^{ax_1^2} \text{ and } f(u) = u^p \ (p > 1),$$

then we see that Theorem 2 can be applied on some bounded domains Ω. Indeed, in this case we have $\tau_\rho = \sup_{x \in \Omega} \left(4 - \frac{2}{1+ax_1^2}\right)$ then (14) holds if one has $\sup_{x \in \Omega} x_1^2 \leq \frac{p-1}{2a}$. Hence, every positive classical supersolution u with $u = 0$ on $\partial\Omega$ satisfies the differential inequality

$$-\Delta u \geq \sqrt{\frac{2}{p+1}} e^{\frac{ax_1^2}{2}} u^{\frac{p+1}{2}} \text{ in } \Omega,$$

provided that Ω satisfies

$$\Omega \subset \{x = (x_1, ..., x_N); \ \ |x_1| \leq \sqrt{\frac{p-1}{2a}}\}.$$

2.2. Fourth-order nonlinear eigenvalue problems

We now consider the semilinear biharmonic elliptic problem under Navier boundary conditions

$$(P_\lambda)\begin{cases}(-\Delta)^2 u=\lambda\rho(x)f(u) & \text{in } \Omega,\\ u=\Delta u=0 & \text{in } \partial\Omega,\end{cases}$$

where $\lambda\geq 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$, $N\geq 2$, $\rho(x)\geq 0$ and smooth in Ω, and where $f:[0,a_f)$ $(0<a_f\leq\infty)$ is smooth, increasing, convex with $f(0)>0$. We define

$$\lambda_*:=\sup\{\lambda\geq 0:(P_\lambda)\text{ has a classical solution}\}.$$

This problem with well-known nonlinearities $f(u)=e^u$, $(1+u)^p$ $(p>1)$ and $\rho(x)=1$, or $f(u)=(1-u)^{-2}$ and $0\leq\rho(x)\leq 1$ with $\rho(x)>0$ on a set of positive Lebesgue measure, has been widely considered in the literature, under both Navier or Dirichlet boundary conditions, see for example [3,10,13,14,22,25] and the references cited therein. It is an interesting problem to estimate λ_* both from below and above, when $\rho(x)\equiv 1$ we refer the reader to see Theorem 3 in Arioli-Gazzola-Grunau-Mitidieri [3] for the exponential nonlinearity, and Theorem 1 in Ferrero-Grunau [22] with power-type nonlinearity. Also, in this regard, Berchio-Gazzola [6] proved that if $\rho(x)\equiv 1$ then the extremal parameter λ_* of (P_λ) satisfies

$$0<\lambda_*<\frac{\lambda_1}{\alpha_f},\qquad \alpha_f:=\max\{\alpha>0:f(s)\geq\alpha s,\ \text{for } s\geq 0\},\tag{43}$$

where λ_1 denotes the first eigenvalue of $(-\Delta)^2$ in Ω under Navier boundary conditions.

Here, as a consequence of Theorem 1, we obtain an explicit upper bound for λ_* for the general problem (P_λ).

Corollary 1. *The extremal parameter λ_* of (P_λ) satisfies*

$$\lambda_*\leq N^2(N+2)\Big(\int_0^\infty\frac{ds}{f(s)}\Big)\Big(\sup_{x\in\Omega}\int_0^{d_\Omega(x)}s^3\rho_x(s)ds\Big)^{-1}.\tag{44}$$

In particular, when $\rho(x)\equiv 1$, we have

$$\lambda_*\leq\frac{4N^2(N+2)}{r_\Omega^4}\int_0^\infty\frac{ds}{f(s)},\tag{45}$$

where $r_\Omega:=\sup_{x\in\Omega}d_\Omega(x)$.
Also, when ρ and f satisfy the conditions of Theorem 2 then any classical solution u_λ of (P_λ) satisfies

$$-\Delta u_\lambda\geq\sqrt{2\rho(x)\tilde{F}(u_\lambda)},\quad x\in\Omega.\tag{46}$$

Example 1. Consider problem (P_λ) with $f(u) = e^u$, $\rho(x) = |x|^a$ $(a \geq 0)$ and let $\Omega = B_R$ be the ball of radius R centered at origin. Computing $\int_0^{d_\Omega(x)} s^3 \rho_x(s)$ for $x \in B_R$ we see that for $a > 0$ the supremum is attained at $|x| = \frac{R}{2}$ with

$$\sup_{x\in\Omega} \int_0^{d_\Omega(x)} s^3 \rho_x(s) ds = (\frac{R}{2})^{4+a} \int_0^1 t^3(1-t)^a dt.$$

Thus, since $\int_0^\infty \frac{ds}{e^s} = 1$, from (45) we obtain

$$\lambda_* \leq N^2(N+2)(\frac{2}{R})^{4+a}\Big(\int_0^1 t^3(1-t)^a dt\Big)^{-1}. \tag{47}$$

Also, when $a = 0$, namely if $\rho(x) \equiv 1$, we obtain

$$\lambda_*(e^u) \leq \frac{4N^2(N+2)}{R^4}. \tag{48}$$

For the same problem with $\rho(x) \equiv 1$ and $f(u) = (1+u)^p$ or the singular nonlinearity $f(u) = \frac{1}{(1-u)^p}$ $(p > 1)$ we obtain

$$\lambda_*((1+u)^p) \leq \frac{4N^2(N+2)}{(p-1)R^4} \text{ and } \lambda_*(\frac{1}{(1-u)^p}) \leq \frac{4N^2(N+2)}{(p+1)R^4}. \tag{49}$$

Comparing our upper bounds for λ_* in the example above (or the general formula given in (45)) with (43), we see that in small dimensions relation (43) gives a better estimate. However, in large dimension, relation (45) gives a better upper bound. Indeed, to use (43) one needs an estimate for λ_1, for example the one obtained by Benedikt-Drábek in [5] shows that

$$\lambda_1 \leq \frac{4N^2}{R^4}\frac{2\Gamma(3+\frac{N}{2})}{N\Gamma(\frac{N}{2})\Gamma(3)} = \frac{N^2(N+2)(N+4)}{R^4}. \tag{50}$$

Then we see that the upper bound given by (43) is $O(N^4)$ for large N but (45) is $O(N^3)$.

Now consider the biharmonic problem

$$(-\Delta)^2 u = \frac{\lambda}{(1-u)^2} \text{ in } B_R \subset \mathbb{R}^N$$

which models a simple micro-electromechanical system (MEMS) device, under Dirichlet boundary conditions $u = \partial_\nu u = 0$ on ∂B_R. Cowan, Esposito, Ghoussoub, and Moradifam [14] proved that for large dimensions (actually, $N \geq 31$) we have $\lambda_* \leq \frac{H_N}{2} := \frac{N^2(N-4)^2}{32}$ (see [14, Theorem 4.2]), which is again $O(N^4)$.

Note also that if we use Theorem 3 to estimate λ_*, we then get a better upper bound, but less explicit, for λ_*. Indeed, we have the following result.

Corollary 2. *Consider problem (P_λ) with $\rho(x) \equiv 1$. Then the extremal parameter λ_* satisfies the inequality*

$$\int_0^\infty \frac{ds}{\lambda_* f(s) + \frac{N(N+2)}{\sqrt{2} r_\Omega^2} \sqrt{\lambda_* \tilde{F}(s)}} \geq \frac{r_\Omega^4}{4N^2(N+2)}, \tag{51}$$

where $r_\Omega := \sup_{x\in\Omega} d_\Omega(x)$.

Example 2. Consider (P_λ) with $f(u) = e^u$, $\rho(x) \equiv 1$ and $\Omega = B_R$. Then from (51), after some simplifications, we obtain

$$\frac{1}{\lambda_*} \int_0^\infty \frac{ds}{e^s + \beta\sqrt{e^s - 1}} \geq \frac{R^4}{4N^2(N+2)}, \quad \text{where } \beta := \frac{N(N+2)}{\sqrt{2\lambda_*} R^2}. \tag{52}$$

Changing the variable $e^s - 1 \to t^2$ we have

$$\begin{aligned} \int_0^\infty \frac{ds}{e^s + \beta\sqrt{e^s - 1}} &= \frac{2}{\beta} \int_0^\infty \frac{t\,dt}{(t^2 + \beta t + 1)(t^2 + 1)} \\ &= \frac{2}{\beta}\Big(\frac{\pi}{2} - \int_0^\infty \frac{dt}{t^2 + \beta t + 1}\Big), \end{aligned}$$

and using this in (52) we get

$$\lambda_*(e^u) \leq \frac{128N^2}{R^4}\Big(\frac{\pi}{2} - \int_0^\infty \frac{dt}{t^2 + \beta t + 1}\Big)^2. \tag{53}$$

Now, by computing the integral term on the right-hand side of (53) (which depends on $\beta^2 \geq 4$ or $\beta^2 < 4$) one can get a better upper bound than (48). However, it is interesting to see that even using a weaker form of (53), that is, without considering the integral term, we get

$$\lambda_*(e^u) \leq \frac{32\pi^2 N^2}{R^4}, \tag{54}$$

which is $O(N^2)$ for large dimension N.

3. Proofs of the main estimates

3.1. *Proof of Theorem 1*

Let u be a positive supersolution of (1). Fix $x_0 \in \Omega$ and $0 < r < d_\Omega(x_0)$. Then we have

$$(-\Delta)^2 u \geq \rho_{x_0}(r) f(m_{x_0}(r)) \quad \text{in } B_r(x_0). \tag{55}$$

Set

$$w_r(y) := \frac{\rho_{x_0}(r) f(m_{x_0}(r))}{2N}\Big(r^2 - |y - x_0|^2\Big). \tag{56}$$

Then from (55) we have

$$-\Delta(-\Delta u) \geq -\Delta w_r(y) \ \text{ in } B_r(x_0) \text{ and } w_r \equiv 0 \ \text{ on } \partial B_r(x_0).$$

Applying the maximum principle we obtain that $-\Delta u \geq w_r(y)$, in $B_r(x_0)$. Also note that we have

$$w_r(y) = -\Delta \Lambda_r(y),$$

where

$$\Lambda_r(y) = \frac{\rho_{x_0}(r) f(m_{x_0}(r))}{2N}\Big(\frac{r^2(r^2 - |y - x_0|^2)}{2N} - \frac{r^4 - |y - x_0|^4}{4(N+2)}\Big).$$

Hence, $-\Delta u \geq -\Delta \Lambda_r$ in $B_r(x_0)$ with $\Lambda_r(y) = 0$ on $\partial B_r(x_0)$. Then by the maximum principle

$$u(y) - m_{x_0}(r) \geq \Lambda_r(y), \quad y \in B_r(x_0).$$

Now let $0 < h < r$ and $y \in B_{r-h}(x_0) \subset B_r(x_0)$. Since the function $\gamma(t) := \frac{r^2(r^2-t^2)}{2N} - \frac{r^4-t^4}{4(N+2)}$ is decreasing for $t \in [0, r]$ we then get from the inequality above

$$\begin{aligned} u(y) - m_{x_0}(r) &\geq \Lambda_r(y) \\ &\geq \frac{\rho_{x_0}(r) f(m_{x_0}(r))}{2N}\Big(\frac{r^2(r^2 - (r-h)^2)}{2N} - \frac{r^4 - (r-h)^4}{4(N+2)}\Big), \quad y \in B_{r-h}(x_0), \end{aligned}$$

and taking infimum over $B_{r-h}(x_0)$ and then dividing by h we obtain

$$\frac{m_{x_0}(r-h) - m_{x_0}(r)}{h} \geq \frac{\rho_{x_0}(r) f(m_{x_0}(r))}{2N}\Big(\frac{r^2(r^2 - (r-h)^2)}{2Nh} - \frac{r^4 - (r-h)^4}{4(N+2)\,h}\Big), \quad 0 < h < r.$$

Letting $h \to 0$ in the above inequality, we arrive at the following ordinary differential inequality with initial value condition

$$\begin{cases} -m'_{x_0}(r) \geq \dfrac{r^3}{N^2(N+2)}\, \rho_{x_0}(r) f(m_{x_0}(r)) & \text{for a.e. } r \in (0, d_\Omega(x_0)), \\ m_{x_0}(0) = u(x_0) \end{cases} \tag{57}$$

where "$' = \frac{d}{dr}$". Dividing inequality (57) by $f(m_{x_0}(r))$ we can rewrite (57) as

$$G'(r) \geq \frac{r^3}{N^2(N+2)}\, \rho_{x_0}(r), \ \text{ a.e. in } (0, d_\Omega(x_0)), \tag{58}$$

where $G:(0,d_\Omega(x_0))\to\mathbb{R}$ defined by

$$G(r):=\int\limits_{m_{x_0}(r)}^{u(x_0)}\frac{ds}{f(s)},\quad r\in(0,d_\Omega(x_0)).$$

Now note that since $m_{x_0}(r)$ is decreasing and f is positive, G is a nondecreasing function. So, by the Lebesgue differentiation theorem

$$\int\limits_0^r G'(s)ds\le G(r)-G(0)=G(r).$$

Thus, by integrating (58) from 0 to r, we get

$$\int\limits_{m_{x_0}(r)}^{u(x_0)}\frac{ds}{f(s)}\ge\frac{1}{N^2(N+2)}\int\limits_0^r s^3\rho_{x_0}(s)ds,$$

which proves the estimate (6). □

3.2. *Proof of Theorem 2*

Set

$$b:=\sqrt{\rho},\quad g(u):=\sqrt{2\tilde F(u)}.$$

First notice that we have $g'(t),g''(t)\ge 0$ for $t>0$. Indeed, we have $g'(t)=\frac{\tilde f(t)}{\sqrt{2\tilde F(t)}}>0$ for $t>0$, and

$$\sqrt{2}g''(t)=\frac{f'(t)\sqrt{\tilde F(t)}-\frac{\tilde f(t)^2}{2\sqrt{\tilde F(t)}}}{\tilde F(t)}=\frac{2f'(t)\tilde F(t)-\tilde f(t)^2}{2\tilde F(t)\sqrt{\tilde F(t)}}>0\quad\text{for } t>0,$$

because for the function $h(t):=2f'(t)\tilde F(t)-\tilde f(t)^2$ we have $h'(t)=2f''(t)\tilde F(t)\ge 0$ and also $h(0)=0$ that implies $h(t)>0$ for $t>0$. Now we set

$$v:=-\Delta u-b(x)g(u),$$

We have $v\ge 0$ on $\partial\Omega$ and

$$-\Delta v=(-\Delta)^2u+\Delta(b(x)g(u))\ge$$
$$\rho(x)f(u)+(\Delta b)g(u)+2g'(u)\nabla b.\nabla u+bg''(u)|\nabla u|^2-bg'(u)v-b^2g'(u)g(u),$$

which implies that

$$-\Delta v + bg'(u)v \geq \rho(x)\big[f(u) - g(u)g'(u)\big] + \Big[bg''(u)|\nabla u|^2 + (\Delta b)g(u) - 2g'(u)|\nabla b||\nabla u|\Big]. \tag{59}$$

Now note that the first term in the right-hand side of (59) is nonnegative because $f(u) - g(u)g'(u) = f(0) > 0$. Also, using the inequality $a^2 + b^2 \geq 2ab$ we see that the last term in the right-hand side of (59) is larger than

$$\Big(2\sqrt{b(\Delta b)g(u)g''(u)} - 2g'(u)|\nabla b|\Big)|\nabla u|,$$

which is nonnegative if we have

$$g'(u)^2|\nabla b|^2 \leq b\Delta b\; g(u)g''(u),$$

or

$$\frac{g(u)g''(u)}{g'(u)^2} \geq \frac{|\nabla b|^2}{b\Delta b},$$

or equivalently,

$$\frac{2f'(u)\tilde{F}(u)}{\tilde{f}(u)^2} - 1 \geq \frac{|\nabla b|^2}{b\Delta b}.$$

Using the formula $\Delta(b^2) = 2|\nabla b|^2 + 2b\Delta b$ we can rewrite the above as

$$\frac{4f'(u)\tilde{F}(u)}{f(u)^2} \geq \frac{\Delta(b^2)}{b\Delta b} = \frac{\Delta\rho}{\sqrt{\rho}\Delta\sqrt{\rho}},$$

which holds by the assumption that

$$\frac{4f'(t)\tilde{F}(t)}{\tilde{f}(t)^2} \geq \tau_\rho \quad \text{for all } t > 0.$$

Thus, we proved that $-\Delta v + bg'(u)v \geq 0$ in Ω, then the maximum principle implies that $v \geq 0$ in Ω, hence (15). □

3.3. *Proof of Theorem 3*

Let u be a positive supersolution of (16). Fix $x_0 \in \Omega$ and $0 < r < d_\Omega(x_0)$. Using the same ideas as in the proof of Theorem 1 we obtain

$$-\Delta(-\Delta u) \geq -\Delta w_r(y) \;\text{ in } B_r(x_0) \text{ and } w_r \equiv 0 \;\text{ on } \partial B_r(x_0),$$

where w_r is given in (56) (with $\rho \equiv 1$). Then by the maximum principle we get

$$-\Delta u \geq w_r(y) + \min_{\partial B_r}(-\Delta u). \tag{60}$$

Now note that by the result of Theorem 2 we have $-\Delta u \geq \sqrt{2\tilde{F}(u)}$ in Ω, which also implies that

$$\min_{\partial B_r}(-\Delta u) \geq \sqrt{2\tilde{F}(m_{x_0}(r))}. \tag{61}$$

Using (61) in (60) we arrive at

$$-\Delta u \geq w_r(y) + \sqrt{2\tilde{F}(m_{x_0}(r))} = -\Delta V_r, \tag{62}$$

where

$$\begin{aligned} V_r(y) := \frac{f(m_{x_0}(r))(r))}{2N}\Big(\frac{r^2(r^2-|y-x_0|^2)}{2N} - \frac{r^4-|y-x_0|^4}{4(N+2)}\Big) \\ + \sqrt{2\tilde{F}(m_{x_0}(r))}\,(\frac{r^2-|y-x_0|^2}{2N}). \end{aligned}$$

We then proceed quite similar as in the proof of Theorem 1 to arrive at the ordinary differential inequality

$$\begin{cases} -m'_{x_0}(r) \geq \dfrac{r^3}{N^2(N+2)} f(m_{x_0}(r)) + \dfrac{r}{\sqrt{2}N}\sqrt{\tilde{F}(m_{x_0}(r))} & \text{for a.e. } r \in (0, d_\Omega(x_0)), \\ m_{x_0}(0) = u(x_0) \end{cases} \tag{63}$$

Using the fact that $r < d_\Omega(x_0) \leq r_\Omega$ and thus $r \geq \frac{r^3}{r_\Omega^2}$, we can estimate the RHS of (63) to arrive at

$$-m'_{x_0}(r) \geq \frac{r^3}{N^2(N+2)}\Big(f(m_{x_0}(r)) + \frac{N(N+2)}{\sqrt{2}\tau_\Omega^2}\sqrt{\tilde{F}(m_{x_0}(r))}\Big).$$

By an argument similar to the end of the proof of Theorem 1, dividing inequality (63) by the term in the RHS of the inequality and then integrating over $[0, r]$ we get

$$\int_{m_{x_0}(r)}^{u(x_0)} \frac{ds}{f(s) + \frac{N(N+2)}{\sqrt{2}r_\Omega^2}\sqrt{\tilde{F}(s)}} \geq \frac{r^4}{4N^2(N+2)},$$

which proves estimate (17). $\square$

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