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The Stationary Navier-Stokes Equations in Variable Exponent Spaces of Clifford-Valued Functions

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Abstract. In the frame of variable exponent spaces of Clifford-valued functions and using the Banach fixed-point theorem, we obtain the existence and uniqueness of solutions to the stationary Navier-Stokes equations and Navier-Stokes equations with heat conduction under certain assumptions. In a sense, we extend some results of P. Cerejeiras and U. Kähler [P. Cerejeiras and U. Kähler, Elliptic boundary value problems of fluid dynamics over unbounded domains, Mathematical Methods in the Applied Sciences, 23(2000), 81-101].

Keywords. Clifford analysis; variable exponent; Navier-Stokes equations; heat conduction.

1. Introduction

Since Kováčik and Rákosník first thoroughly studied the spaces $L^{p(x)}$ and $W^{k,p(x)}$ in [25], Lebesgue and Sobolev variable exponent spaces have attracted more and more attention, see for instance [7-10] for recent properties of variable exponent spaces and [23] for an overview of differential equations with variable growth. In particular, one of the reasons that forced the rapid expansion of the theory of variable exponent function spaces has been the models of electrorheological fluids introduced by Rajagopal and Růžička [28, 29], which can be described by the boundary-value problem for the generalized Navier-Stokes equations. Diening, Lengeler and Růžička [4] proved the existence and uniqueness of strong and weak solutions of the Stokes system and Poisson equations for bounded domains in the setting of variable exponent spaces. Of course, the study of these spaces has been stimulated by problems in elastic mechanics, calculus of variations and differential equations with variable growth conditions, see [11, 12, 32, 33] and references therein.

As a powerful tool for solving elliptic boundary value problems in the plane, the methods of complex function theory play an important role. One

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way to extend these ideas to higher dimension is to start with a generalization of algebraic and geometrical properties of the complex numbers. In this way, Hamilton studied the algebra of quaternion in 1843. Further generalizations were introduced by Clifford in 1878. He initiated the so-called geometric algebras or Clifford algebras, which are generalizations of the complex numbers, the quaternions, and the exterior algebras, see [21]. Clifford analysis is usually the study of Dirac equation or of a generalized Cauchy-Riemann system, in which solutions are defined on domains in the Euclidean space and take values in Clifford algebras, see [1, 30]. In particular, Gürlebeck and Sprößig [16, 17] developed an operator calculus, which is analogous to the known complex analytic approach in the plane and based on three operators: a Cauchy-Riemann-type operator, a Teodorescu transform, and a generalized Cauchy-type integral operator, to investigate elliptic boundary value problems of fluid dynamics over bounded and unbounded domains, especially the Navier-Stokes equations and related equations. Of course, there are a number of unsolved basic problems involving the Navier-Stokes equations. This is mainly due to the problem concerning the solvability of the corresponding linear Stokes equations over domains, see [2, 18].

Our goal in this article is to investigate the existence and uniqueness of solutions for Navier-Stokes equations in the variable exponent context. For this purpose, an iteration method introduced by Gürlebeck and Sprößig [16, 17, 19], which requires the solution of a Stokes-problem in every step of iteration, is used to treat this kind of problems. Besides being of interest in its own as a generalization of classical results, the results are of importance in the study of fluid dynamics. Of course, the whole treatment applies to a much larger class of elliptic problems.

This paper is organized as follows. In Section 2, we begin with a brief summary of basic knowledge of Clifford algebras and variable exponent spaces of Clifford-valued functions. Especially, we require the existence and uniqueness of the Stokes equations in the context of variable exponent spaces, which will be needed later. In Section 3, we investigate an iterative method for the solution of the stationary Navier-Stokes equations. Using the Banach fixed-pointed theorem, we prove the existence and uniqueness of a solution to the Navier-Stokes equations in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega, \mathbb{R})$ under certain hypotheses. In Section 4, with the help of the Banach fixed-pointed theorem, we prove the existence and uniqueness of a solution in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega, \mathbb{R})$ of the Navier-Stokes problem with heat conduction under some conditions.

2. Preliminaries

2.1. Clifford algebras

We first recall some related notions and results concerning Clifford algebras. For a detailed account we refer to [1, 5, 20, 26, 27, 30].

Let $\mathrm{C}\ell_n$ be the real universal Clifford algebras over $\mathbb{R}^n$. Denote $\mathrm{C}\ell_n$ by

$$\mathrm{C}\ell_n = \mathrm{span}\{\mathrm{e}_0, \mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n, e_1 e_2, \ldots, \mathrm{e}_{n-1}\mathrm{e}_n, \ldots, \mathrm{e}_1\mathrm{e}_2 \ldots \mathrm{e}_n\}$$

where $\mathrm{e}_0 = 1$(the identity element in $\mathbb{R}^n$), $\{\mathrm{e}_1, \mathrm{e}_2, \ldots, \mathrm{e}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $\mathrm{e}_i\mathrm{e}_j + \mathrm{e}_j\mathrm{e}_i = -2\delta_{ij}\mathrm{e}_0$. Thus the dimension of $\mathrm{C}\ell_n$ is 2^n. For $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ with $1 \leq i_1 < i_2 < \ldots < i_n \leq n$, put $\mathrm{e}_I = \mathrm{e}_{i_1}\mathrm{e}_{i_2} \ldots \mathrm{e}_{i_r}$, while for $I = \emptyset$, $\mathrm{e}_\emptyset = \mathrm{e}_0$. For $0 \leq r \leq n$ fixed, the space $\mathrm{C}\ell_n^r$ is defined by

$$\mathrm{C}\ell_n^r = \mathrm{span}\{\mathrm{e}_I : |I| := \mathrm{card}(I) = r\}.$$

The Clifford algebras $\mathrm{C}\ell_n$ is a graded algebra as

$$\mathrm{C}\ell_n = \bigoplus_r \mathrm{C}\ell_n^r.$$

Any element $a \in \mathrm{C}\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \ldots + [a]_n$$

where $[\ ]_r : \mathrm{C}\ell_n \to \mathrm{C}\ell_n^r$ denotes the projection of $\mathrm{C}\ell_n$ onto $\mathrm{C}\ell_n^r$. In particular, by $\mathrm{C}\ell_n^2 = \mathbb{H}$ we denote the algebra of real quaternion. It is customary to identify $\mathbb{R}$ with $\mathrm{C}\ell_n^0$ and identify $\mathbb{R}^n$ with $\mathrm{C}\ell_n^1$ respectively. This means that each element x of $\mathbb{R}^n$ may be represented by

$$x = \sum_{i=1}^{n} x_i \mathrm{e}_i.$$

For $u \in \mathrm{C}\ell_n$, we denotes by $[u]_0$ the scalar part of u, that is the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{\mathrm{e}_{i_1}\mathrm{e}_{i_2} \ldots \mathrm{e}_{i_r}} = (-1)^{\frac{r(r+1)}{2}} \mathrm{e}_{i_1}\mathrm{e}_{i_2} \ldots \mathrm{e}_{i_r}.$$

We denote

$$(A, B) = \left[\overline{A}B\right]_0.$$

Then an inner product is thus obtained, give rising to the norm $|\cdot|$ on $\mathrm{C}\ell_n$ given by

$$|A|^2 = \left[\overline{A}A\right]_0.$$

For all what follows let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a bounded domain with a sufficiently smooth boundary $\partial\Omega$. A Clifford-valued function $u : \Omega \to \mathrm{C}\ell_n$ can be written as $u = \Sigma_I u_I \mathrm{e}_I$, where the coefficients $u_I : \Omega \to \mathbb{R}$ are real-valued functions.

The Dirac operator on Euclidean space used here is introduced by

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j} := \sum_{j=1}^{n} e_j \partial_j.$$

If u is a real-valued function defined on a domain Ω in $\mathbb{R}^n$, then $Du = \nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_n u)$. Moreover, $D^2 = -\Delta$, where Δ is the Laplace operator which operates only on coefficients. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right monogenic function. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_n}\frac{\overline{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. Basic properties of left monogenic functions one can refer to [14, 19, 30] and references therein.

2.2. Variable exponent spaces of Clifford-valued functions

Next we recall some basic properties of variable exponent spaces of Clifford-valued functions. Note that in what follows, we use the short notation $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, etc., instead of $L^{p(x)}(\Omega,\mathbb{R})$, $W^{1,p(x)}(\Omega,\mathbb{R})$, etc.

Throughout this paper we always assume (unless declare specially)

$$p \in \mathcal{P}^{\log}(\Omega) \text{ and } 1 < p_- := \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq \sup_{x\in\overline{\Omega}} p(x) =: p_+ < \infty. \tag{2.1}$$

where $\mathcal{P}^{\log}(\Omega)$ is the set of exponent p satisfying the so-called log-Hölder continuity, i.e.,

$$|p(x) - p(y)| \leq \frac{C}{\log(e + |x-y|^{-1})}$$

holds for all $x, y \in \Omega$, see [4, 13]. Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1,\infty)$. Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p'(x) \in \mathcal{P}(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x)-1} \quad \text{for each } x \in \Omega.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \{u \in \mathcal{P}(\Omega) : \int_\Omega |u|^{p(x)}dx < \infty\},$$

with the norm

$$||u||_{L^{p(x)}(\Omega)} = \inf\{t > 0 : \int_\Omega |\frac{u}{t}|^{p(x)}dx \leq 1\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

with the norm

$$||u||_{W^{1,p(x)}(\Omega)} = ||\nabla u||_{L^{p(x)}(\Omega)} + ||u||_{L^{p(x)}(\Omega)}. \tag{2.2}$$

Denote $W_0^{1,p(x)}(\Omega)$ by the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm (2.2). The space $W^{-1,p(x)}(\Omega)$ is defined as the dual of the space $W_0^{1,p'(x)}(\Omega)$. For more details we refer to [3, 7-10] and reference therein.

In the following, we say that $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ can be understood coordinate wisely. For example, $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ means that $\{u_I\} \subset L^{p(x)}(\Omega)$ for $u = \Sigma_I u_I e_I \in \mathrm{C}\ell_n$ with the norm $||u||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} = \sum_I ||u_I||_{L^{p(x)}(\Omega)}$. In this way, spaces $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $C_0^\infty(\Omega, \mathrm{C}\ell_n)$, etc., can be understood similarly. In particular, the space $L^2(\Omega, \mathrm{C}\ell_n)$ can be converted into a right Hilbert $\mathrm{C}\ell_n$-module by defining the following Clifford-valued inner product (see [16, Definition 3.74])

$$(f, g)_{\mathrm{C}\ell_n} = \int_\Omega \overline{f(x)} g(x) dx. \tag{2.3}$$

Remark 2.1. A simple calculation yields that the norm $||u||_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$ is equivalent to the norm $||\,|u|\,||_{L^{p(x)}(\Omega)}$. Furthermore, we also have that for every $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $||Du||_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$ is an equivalent norm of $||u||_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)}$ (see [14]).

Lemma 2.1. (see [14]) *If $p(x) \in \mathcal{P}(\Omega)$, then $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ and $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ are reflexive Banach spaces.*

Definition 2.1. (see [16]) Let $u \in C(\Omega, \mathrm{C}\ell_n)$. The Teodorescu operator is defined by

$$Tu(x) = \int_\Omega G(x - y) u(y) dy,$$

where $G(x)$ is the generalized Cauchy kernel mentioned above.

Lemma 2.2. *The following operators are continuous linear operators:*

(i) $T : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

(ii) $\widetilde{T} : W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n) \to L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.

Proof. (i) See Theorem 2.7 in [14].

(ii) In view of Proposition 12.3.2 in [3], we know that for each $f \in W^{-1,p(x)}(\Omega)$, there exists $f_k \in L^{p(x)}(\Omega), k = 0, 1, \ldots, n$, such that

$$< f, \varphi >= \sum_{k=0}^{n} \int_\Omega f_k \frac{\partial \varphi}{\partial x_k} dx, \tag{2.4}$$

for all $\varphi \in W_0^{1,p'(x)}(\Omega)$. Moreover, $\|f\|_{W^{-1,p(x)}(\Omega)}$ and $\sum_{k=0}^{n} \|f_k\|_{L^{p(x)}(\Omega)}$ are equivalent. Obviously, for every $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$ the equality (2.4) still holds for $f_k \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, $k = 0, 1, \ldots, n$. Moreover, $\|f\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)}$ is equivalent to $\sum_{k=0}^{n} \|f_k\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}$. On the other hand, by Proposition 12.3.4 in [3], it is easy to show that $C_0^\infty(\Omega, Cl_n)$ is dense in $W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Thus we may choose

$$u^j = u_0^j + \sum_{k=1}^{n} \frac{\partial u_k^j}{\partial x_k},$$

where $u_0^j, u_k^j \in C_0^\infty(\Omega, \mathrm{C}\ell_n)$, such that $\|u^j - f\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)} \to 0$ and $\|u_k^j - f_k\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} \to 0$ as $j \to \infty$, where $k = 0, 1, \ldots, n$. Here, we are using the fact that $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$(see [14]). Then we consider

$$Tu^j = \int_\Omega G(x - y) u^j(y) dy,$$

where $G(x)$ is the above-mentioned generalized Cauchy kernel. Further, we have

$$Tu^j = \int_\Omega G(x-y)\big(u_0^j(y) + \sum_{k=1}^n \frac{\partial}{\partial y_k} u_k^j(y)\big)dy$$
$$= \int_\Omega G(x-y)u_0^j(y)dy + \sum_{k=1}^n \int_\Omega \frac{\partial}{\partial x_k} G(x-y)u_k^j(y)dy.$$

Since

$$\big|\int_\Omega G(x-y)u_0^j(y)dy\big| \leq \int_\Omega \frac{1}{|x-y|^{n-1}}\big|u_0^j(y)\big|dy.$$

By Remark 2.1, Lemma 2.7 and Lemma 2.8, there exists a constant $C_0 > 0$ such that

$$\Big|\Big|\int_\Omega G(x-y)u_0^j(y)dy\Big|\Big|_{L^{p(x)}(\Omega,Cl_n)} \leq C_0||u_0^j||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{2.5}$$

Now let us extend $u_k^j(x)$ by zero to $\mathbb{R}^n \setminus \Omega$. Then it is easy to show that $\frac{\partial}{\partial x_k}G(x-y)$ satisfies the conditions of Calderón-Zygmund kernel on $\mathbb{R}^n \times \mathbb{R}^n$ (see [14]). In view of Lemma 2.9, there exist positive constant $C_k (k = 1, \ldots, n)$ such that

$$\Big|\Big|\int_\Omega \frac{\partial}{\partial x_k} G(x-y)u_k^j(y)\Big|\Big|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \leq C_k\big\|u_k^j\big\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{2.6}$$

Combining (2.5) with (2.6), we have

$$\big|\big|Tu^j\big|\big|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \leq C_0\big|\big|u_0^j\big|\big|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} + \sum_{k=1}^n C_k\big|\big|u_k^j\big|\big|_{L^{p(x)}(\Omega,Cl_n)}.$$

Letting $j \to \infty$, by means of the Continuous Linear Extension Theorem, the operator T can be uniquely extended to a bounded linear operator $\widetilde{T}$ such that for all $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$, there exists a constant $\widetilde{C} > 0$ such that

$$\begin{aligned}\big\|\widetilde{T}f\big\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} &\leq C\Big(\big\|f_0\big\|_{L^{p(x)}(\Omega,Cl_n)} \\ &+ \textstyle\sum_{k=1}^n \big\|f_k\big\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\Big) \leq \widetilde{C}\big\|f\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}.\end{aligned}$$

Now we complete the proof of the lemma 2.2. □

Lemma 2.3. *The following operators are continuous linear operators:*

(i) $D : W^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \to L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.

(ii) $\widetilde{D} : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

Proof. (i) See Lemma 2.6 in [14].

(ii) We consider the following Dirichlet problem of the Poisson equation with homogeneous boundary data

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

It is easy to see that for all $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$ the problem (2.7) still has a unique weak solution $u \in W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, see L. Diening, D. Lengeler and M. Ružička [4]. We denote by Δ_0^{-1} the solution operator. On the other hand, it is clear that the operator

$$\Delta : W^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$$

is continuous, so we obtain from Lemma 2.2 that the operator $\widetilde{D} = -\Delta T : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is continuous, where the operator $\widetilde{D}$ can be considered as a unique continuous linear extension of the operator D. $\square$

Lemma 2.4. *Let $p(x) \in \mathcal{P}(\Omega)$.*

(i) *If $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then the equation $TDu(x) = u(x)$ holds for all $x \in \Omega$.*

(ii) *If $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, then the equation $DTu(x) = u(x)$ holds for all $x \in \Omega$.*

Proof. By Remark 4.21 in [16], (i) and (ii) can be deduced from the continuous embedding $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \hookrightarrow W^{1,p_-}(\Omega, \mathrm{C}\ell_n)$ and $L^{p(x)}(\Omega, \mathrm{C}\ell_n) \hookrightarrow L^{p_-}(\Omega, \mathrm{C}\ell_n)$. $\square$

Lemma 2.5. *Let $p(x)$ satisfies (2.1).*

(i) *If $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, then the equation $\widetilde{T}\widetilde{D}u(x) = u(x)$ holds for all $x \in \Omega$.*

(ii) *If $u \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then the equation $\widetilde{D}\widetilde{T}u(x) = u(x)$ holds for all $x \in \Omega$.*

Proof. (i) follows from Lemma 2.4 (i) and the density of $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.

(ii) follows from Lemma 2.4 (ii) and the density of $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ in space $W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. $\square$

Gürlebeck and Sprößig [16, 17] showed that the orthogonal decomposition of the space $L^2(\Omega)$ holds in the hyper-complex function theory:

$$L^2(\Omega, \mathrm{C}\ell_n) = (\ker D \cap L^2(\Omega, \mathrm{C}\ell_n)) \oplus DW_0^{1,2}(\Omega, \mathrm{C}\ell_n) \tag{2.8}$$

with respect to the Clifford-valued product (2.3). Note that $\ker D$ denotes the set of all monogenic functions on Ω. This decomposition has a number of applications, especially to the theory of partial differential equations, see [6] for the complex case and [30] for the hyper-complex case. Kähler [24] extended the orthogonal decomposition (2.8) to the spaces $L^p(\Omega)$ in form of a direct decomposition in the case of Clifford analysis. Similar to the proof of Theorem 6 in [24], we can generalize the direct decomposition into the case of variable exponent Lebesgue spaces as follows:

$$L^{p(x)}(\Omega, \mathrm{C}\ell_n) = (\ker\widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n)) \oplus DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \tag{2.9}$$

with respect to the Clifford-valued product (2.3).

From this decomposition we can get the following projections

$$P : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to \ker\widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n)$$

$$Q : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n).$$

Moreover, we have

$$Q = D\Delta_0^{-1}\widetilde{D}, \quad P = I - Q.$$

Lemma 2.6. *$L^{p(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q$ is a closed subspace of $L^{p(x)}(\Omega, C\ell_n)$.*

Proof. Let $u \in \overline{L^{p(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q} = \overline{DW_0^{1,p(x)}(\Omega, C\ell_n)}$. Then there exists $u_k \in W_0^{1,p(x)}(\Omega, C\ell_n)$ such that $||Du_k - u||_{L^{p(x)}(\Omega,C\ell_n)} \to 0$ as $k \to \infty$. In view of Lemma 2.1, we can extract a subsequence of $\{u_k\}$(still labeled by $\{u_k\}$), such that $u_k \rightharpoonup v$ weakly in $W_0^{1,p(x)}(\Omega, C\ell_n)$. Since the norm in a Banach space is weakly lower semicontinuous, we obtain from Lemma 2.3

$$||Dv - u||_{L^{p(x)}(\Omega,C\ell_n)} \leq \liminf_{n\to\infty} ||Du_n - u||_{L^{p(x)}(\Omega,C\ell_n)} = 0.$$

Thus $u = Dv$, which completes the proof of Lemma 2.6. □

Lemma 2.7. $\left(L^{p(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q\right)^* = L^{p'(x)}(\Omega, C\ell_n) \cap \mathrm{im}Q$. *Namely, the linear operator*

$$\Phi : DW_0^{1,p'(x)}(\Omega, C\ell_n) \to \left(DW_0^{1,p(x)}(\Omega, C\ell_n)\right)^*$$

given by

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} := \int_\Omega \left[\overline{D\varphi} Du\right]_0 dx$$

is a Banach space isomorphism.

Proof. In terms of Lemma 2.6, $DW_0^{1,p(x)}(\Omega, C\ell_n)$ and $DW_0^{1,p'(x)}(\Omega, C\ell_n)$ are reflexive Banach spaces since they are closed in spaces $L^{p(x)}(\Omega, C\ell_n)$ and spaces $L^{p'(x)}(\Omega, C\ell_n)$ respectively. Linearity of Φ is clear. For injectivity, suppose

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} = 0 \tag{2.10}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega, C\ell_n)$ and some $u \in W_0^{1,p'(x)}(\Omega, C\ell_n)$. For any $\omega \in L^{p(x)}(\Omega, C\ell_n)$, according to (2.9), we may write $\omega = \alpha + \beta$ with $\alpha \in \ker\widetilde{D} \cap L^{p(x)}(\Omega, C\ell_n)$ and $\beta \in DW_0^{1,p(x)}(\Omega, C\ell_n)$. Thus we obtain

$$(\omega, Du)_{Sc} = (\alpha + \beta, Du)_{Sc} = (\alpha, Du)_{Sc} + (\beta, Du)_{Sc} = (\beta, Du)_{Sc}.$$

This together with (2.10) gives $(\omega, Du)_{Sc} = 0$. This results in $Du = 0$. It follows that Φ is injective. To get surjectivity, let $f \in \left(DW_0^{1,p(x)}(\Omega, C\ell_n)\right)^*$. By the Hahn-Banach Theorem, there is $F \in \left(L^{p(x)}(\Omega, C\ell_n)\right)^*$ with $||F|| = ||f||$ and $F|_{DW_0^{1,p(x)}(\Omega,C\ell_n)} = f$. Moreover, there exists $\varphi \in L^{p'(x)}(\Omega, C\ell_n)$ such that $F(u) = (u, \varphi)_{Sc}$ for any $u \in L^{p(x)}(\Omega, C\ell_n)$. According to (2.9), we can write $\varphi = \eta + D\alpha$, where $\eta \in \ker\widetilde{D} \cap L^{p'(x)}(\Omega, C\ell_n)$, $D\alpha \in DW_0^{1,p'(x)}(\Omega, C\ell_n)$. For any $Du \in DW_0^{1,p(x)}(\Omega, C\ell_n)$, we have

$$f(Du) = (Du, \varphi)_{Sc} = (Du, \eta)_{Sc} + (Du, D\alpha)_{Sc} = (Du, D\alpha)_{Sc} = \Phi(D\alpha)(Du).$$

Consequently, $\Phi(D\alpha) = f$. It follows that Φ is surjective. By Theorem 3.1 in [13] we have

$$|\Phi(Du)(D\varphi)| = |(D\varphi, Du)_{Sc}| \leq C||D\varphi||_{L^{p(x)}(\Omega,C\ell_n)} ||Du||_{L^{p'(x)}(\Omega,C\ell_n)}.$$

This means that Φ is continuous. Furthermore, it is immediate that Φ^{-1} is continuous from the Inverse Function Theorem. This finishes the proof of Lemma 2.7. $\square$

In the following, we consider the Stokes system which consists in finding a solution (u, π) :

$$-\Delta u + \frac{1}{\mu}\nabla\pi = \frac{\rho}{\mu}f \quad \text{in } \Omega, \tag{2.11}$$

$$\text{div } u = f_0 \quad \text{in } \Omega, \tag{2.12}$$

$$u = v_0 \qquad \text{on } \partial\Omega. \tag{2.13}$$

With $\int_\Omega f_0 dx = \int_{\partial\Omega} n \cdot v_0 dx$ the necessary condition for the solvability is given. Here, u is the velocity, π the hydrostatic pressure, ρ the density, μ the viscosity, f the vector of the external forces and the scalar function f_0 a measure of the compressibility of fluid. The boundary condition (2.13) describes the adhesion at the boundary of the domain Ω for $v_0 = 0$. This system describes the stationary flow of a homogeneous viscous fluid for small Reynold's numbers. For more details, we refer to [2, 16-18, 22].

In this paper, for $f = \sum_{i=1}^n f_i \mathrm{e}_i$ and $u = \sum_{i=1}^n u_i \mathrm{e}_i$, let us consider the following Stokes system in the hyper-complex formulation (see [16, 17]):

$$\widetilde{D}Du + \frac{1}{\mu}D\pi = \frac{\rho}{\mu}f \quad \text{in } \Omega, \tag{2.14}$$

$$[Du]_0 = 0 \quad \text{in } \Omega, \tag{2.15}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{2.16}$$

Definition 2.2. We call $(u, \pi) \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega)$ a solution of (2.14)-(2.16) provided that it satisfies the system (2.14)-(2.16) for every $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$.

Definition 2.3. The operator $\widetilde{\nabla} : L^{p(x)}(\Omega) \to (W^{-1,p(x)}(\Omega))^n$ is defined by

$$< \widetilde{\nabla} f, \varphi >= - < f, \text{div}\varphi >:= - \int_\Omega f \ \text{div}\varphi dx$$

for all $f \in L^{p(x)}(\Omega)$ and $\varphi \in (C_0^\infty(\Omega))^n$.

Lemma 2.8. *Let Ω be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$. Let $f \in (W^{-1,p(x)}(\Omega))^n$ satisfies:*

$$< f, \varphi >= \int_\Omega f \cdot \varphi dx = 0,$$

for any $\varphi \in \mathscr{W}(\Omega) := \left\{v \in (W_0^{1,p'(x)}(\Omega))^n : \text{div } v = 0\right\}$. Then there exists $q \in L^{p(x)}(\Omega)$ such that $f = \widetilde{\nabla} q$.

Proof. We first show that the range space $R(\widetilde{\nabla})$ is a closed subspace of $(W^{-1,p(x)}(\Omega))^n$. Let $E_1 = L^{p(x)}(\Omega), E_2 = (W^{-1,p(x)}(\Omega))^n, E_3 = W^{-1,p(x)}(\Omega)$ $A = \widetilde{\nabla}$, $B = I$ (the identity operator). Since the domain is bounded, the canonical imbedding B of E_1 into E_3 is compact (see [4]). On the other hand, by means of Hölder inequality, we have for every $f \in L^{p(x)}(\Omega)$

$$\begin{aligned}\|\widetilde{\nabla} f\|_{W^{-1,p(x)}(\Omega)} &= \sup_{||g||_{W_0^{1,p'(x)}(\Omega)} \le 1} \left| < \widetilde{\nabla} f, g > \right| \\ &= \sup_{||g||_{W_0^{1,p'(x)}(\Omega)} \le 1} \left| \int_\Omega f \ \mathrm{div} g dx \right| \le C\|f\|_{L^{p(x)}(\Omega)}.\end{aligned}$$

Then it is easy to show that

$$\|\widetilde{\nabla} f\|_{W^{-1,p(x)}(\Omega)} + \|f\|_{W^{-1,p(x)}(\Omega)} \le C\|f\|_{L^{p(x)}(\Omega)} \quad \textit{for all } f \in L^{p(x)}(\Omega). \tag{2.17}$$

On the other hand, by Theorem 14.3.18 in [3] we have

$$\|f\|_{L^{p(x)}(\Omega)} \le C\big(\|\widetilde{\nabla} f\|_{W^{-1,p(x)}(\Omega)} + \|f\|_{W^{-1,p(x)}(\Omega)}\big), \tag{2.18}$$

holds for every $f \in L^{p(x)}(\Omega)$. Then (2.17) and (2.18) yield the equivalence of the norms of both sides above. Thus, the desired conclusion follows immediately from Lemma 11.1 in [31].

Since $< -\widetilde{\nabla} u, g >=< u, \mathrm{div}\ g >$ for every $u \in L^{p(x)}(\Omega)$ and $g \in (W_0^{1,p'(x)}(\Omega))^n$, the operator $-\widetilde{\nabla} \in \mathscr{L}(L^{p(x)}(\Omega), (W^{-1,p(x)}(\Omega))^n)$ is the adjoint of the operator div $\in \mathscr{L}((W_0^{1,p'(x)}(\Omega))^n, L^{p'(x)}(\Omega))$. Then the Closed Range Theorem of Banach implies that

$$R(\widetilde{\nabla}) = (\ker(\mathrm{div}))^\perp := \{y \in (W^{-1,p(x)}(\Omega))^n : < y, v >= 0, \forall\ v \in \mathscr{W}(\Omega)\}.$$

This implies the desired conclusion. □

Theorem 2.1. *Suppose $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Then the Stokes system* (2.14)–(2.16) *has a unique solution $(u, \pi) \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega)$ in the form*

$$u + \frac{1}{\mu} TQ\pi = \frac{\rho}{\mu} TQ\widetilde{T} f,$$

with respect to the estimate

$$||Du||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} + \frac{1}{\mu}||Q\pi||_{L^{p(x)}(\Omega)} \le C\frac{\rho}{\mu}||Q\widetilde{T} f||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

Here, $C \ge 1$ is a constant and the hydrostatic pressure π is unique up to a constant.

Proof. We first prove that if $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then we have the following representation:

$$TQ\widetilde{T} f = u + TQ\omega.$$

Indeed, let $\varphi_n \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ with $\varphi_n \to \varphi$ in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. By Lemma 2.4, we have

$$TQT(D\varphi_n) = TQ\varphi_n.$$

Since $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, it follows that $TQ\widetilde{T}\widetilde{D}\varphi = TQ\varphi$. Thus, for $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ and $\pi \in L^{p(x)}(\Omega)$ we obtain

$$TQ\widetilde{T}(\frac{\rho}{\mu}f) = TQ\widetilde{T}(\widetilde{D}Du + \frac{1}{\mu}\widetilde{D}\pi) = u + \frac{1}{\mu}TQ\pi.$$

This implies that our system (2.14)-(2.15) is equivalent to the system

$$u + \frac{1}{\mu}QTQ\pi = \frac{\rho}{\mu}TQ\widetilde{T}f, \tag{2.19}$$

$$[Q\pi]_0 = [Q\widetilde{T}f]_0. \tag{2.20}$$

Obviously, the equality (2.14) is equivalent to the following equality

$$Du + \frac{1}{\mu}Q\pi = \frac{\rho}{\mu}Q\widetilde{T}f. \tag{2.21}$$

Now we need to show that for each $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n^1)$, the function QTf can be decomposed into two functions Du and $Q\pi$. Suppose $Du + Q\pi = 0$ for $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n^1) \cap \ker \operatorname{div}$ and $\pi \in L^{p(x)}(\Omega)$. Then (2.15) gives $[Q\pi]_0 = 0$. Thus, $Q\pi = 0$. Hence, $Du = Q\pi = 0$. This means that $Du + Q\pi$ is a direct sum, which is a subset of $\mathrm{im}Q$.

Next we have to ask about the existence of functional $\mathcal{F}$ with $\mathcal{F}(Du) = 0$ and $\mathcal{F}(Q\pi) = 0$ but $\mathcal{F}(Q\widetilde{T}f) \neq 0$, here $\mathcal{F} \in (L^{p(x)}(\Omega, \mathrm{C}\ell_n^1) \cap \mathrm{im}Q)^*$. This is equivalent to ask if there exists $g \in W^{-1,p'(x)}(\Omega, \mathrm{C}\ell_n^1)$, such that for all $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n^1) \cap \ker \operatorname{div}$ and $\omega \in L^{p(x)}(\Omega)$

$$(Du, Q\widetilde{T}g)_{Sc} = 0, \tag{2.22}$$

$$(Q\pi, Q\widetilde{T}g)_{Sc} = 0, \tag{2.23}$$

but $(Q\widetilde{T}f, Q\widetilde{T}g)_{Sc} \neq 0$. Here, Lemma 2.7 is used.

Thus, let us consider (2.22) and (2.23) with $g \in W^{-1,p'(x)}(\Omega, \mathrm{C}\ell_n^1)$. Notice that, with the help of Lemma 2.5, (2.22) yields

$$(Du, Q\widetilde{T}g)_{Sc} = (u, \widetilde{D}Q\widetilde{T}g)_{Sc} = (u, \widetilde{D}\widetilde{T}g - \widetilde{D}P\widetilde{T}g)_{Sc} = (u, g)_{Sc} = 0,$$

which implies $g = \widetilde{\nabla}h = \widetilde{D}h$ with $h \in L^{p'(x)}(\Omega)$ because of Lemma 2.8. Thus we obtain from (2.23) and Lemma 2.7

$$(Q\pi, Q\widetilde{T}g)_{Sc} = (Q\pi, Q\widetilde{T}\widetilde{D}h)_{Sc} = (Q\pi, Qh)_{Sc} = 0$$

holds for each $\pi \in L^{p(x)}(\Omega)$. Hence, $Q\pi = |Qh|^{p'(x)-2}Qh$ gives $Qh = 0$. Then we obtain

$$g = \widetilde{D}h = \widetilde{D}Qh + \widetilde{D}Ph = 0.$$

Furthermore, we get

$$(Q\widetilde{T}f, Q\widetilde{T}g)_{Sc} = 0, \quad \forall f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n^1).$$

Finally, (2.21) yields

$$||Du||_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} + \frac{1}{\mu}||Q\pi||_{L^{p(x)}(\Omega)} \geq \frac{\rho}{\mu}||Q\widetilde{T}f||_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}.$$

By the Norm Equivalence Theorem, we obtain

$$||Du||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} + \frac{1}{\mu}||Q\pi||_{L^{p(x)}(\Omega)} \le C\frac{\rho}{\mu}||Q\widetilde{T}f||_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

By Remark 2.1, Lemma 2.2 and the boundedness of the operator Q, we get

$$||u||_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} + \frac{1}{\mu}||Q\pi||_{L^{p(x)}(\Omega)} \le C\frac{\rho}{\mu}||f||_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{2.24}$$

From (2.24) the uniqueness of the solution follows. Notice that $Q\pi = 0$ implies $\pi \in \ker\widetilde{D}$. Therefore, π is unique up to a constant. Now we complete the proof of Theorem 2.1. □

3. The Steady Navier-Stokes Equations

Our aim in this section is to seek after an iterative method for the solution of the time-independent Navier-Stokes equations:

$$-\Delta u + \frac{\rho}{\mu}(u\cdot\nabla)u + \frac{1}{\mu}\nabla\pi = \frac{\rho}{\mu}f \quad \text{in } \Omega, \tag{3.1}$$

$$\mathrm{div}u = f_0 \quad \text{in } \Omega, \tag{3.2}$$

$$u = v_0 \quad \text{on } \partial\Omega. \tag{3.3}$$

In addition to the case of the Stokes system, the main difference from the above-mentioned Stokes equations is the appearance of the non-linear convection term $(u\cdot\nabla)u$. In 1928, Oseen showed that one can get relatively good results if the convection term $(u\cdot\nabla)u$ is replaced by $(v\cdot\nabla)u$, where v is a solution of the corresponding Stokes equations. In 1965, Finn [15] proved the existence of solutions for small external forces with a spatial decreasing to infinity of order $|x|^{-1}$ for the case of $n = 3$, and used the Banach fixed-pointed theorem. Gürlebeck and Sprößig [16, 17, 19] solved this system by a reduction to a sequence of Stokes problems provided the external force f belongs to $L^p(\Omega,\mathbb{H})$ for a bounded domain Ω and $\frac{6}{5} < p < \frac{3}{2}$. Cerejeiras and Kähler [7] obtained the similar results provided the external force f belongs to $W^{-1,p}(\Omega,\mathrm{C}\ell_n)$ for an unbounded domain Ω and $\frac{n}{2} \le p < \infty$. Our interest in this section is to extend these results to the setting of variable exponent spaces.

In this paper, for $f = \sum_{i=1}^n f_ie_i$, $u = \sum_{i=1}^n u_ie_i$, let us consider the following steady Navier-Stokes equations in the hyper-complex notation:

$$\widetilde{D}Du + \frac{1}{\mu}D\pi = \frac{\rho}{\mu}F(u) \quad \text{in } \Omega, \tag{3.4}$$

$$\big[Du\big]_0 = 0 \quad \text{in } \Omega, \tag{3.5}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{3.6}$$

with the non-linear part $F(u) = f - \big[uD\big]_0u$.

We first give the following lemma, which is crucial to the convergence of the iteration method.

Lemma 3.1. *Let $p(x)$ satisfy* (2.1) *and* $\frac{n}{2} \leq p_- \leq p(x) \leq p_+ < \infty$. *Then the operator* $F : W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n^1) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n^1)$ *is a continuous operator and we have*

$$\big\| [uD]_0 u \big\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)} \leq C_1 \big\| u \big\|^2_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)},$$

where $C_1 = C_1(n, p, \Omega)$ is a positive constant.

Proof. Let $u = \sum_{i=1}^n u_i e_i \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n^1)$. Then

$$\begin{aligned}\Big\| [uD]_0 u \Big\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)} &= \Big\| \sum_{j=1}^n (\sum_{i=1}^n u_i \partial_i u_j) e_j \Big\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)} \\ &\leq \sum_{i,j=1}^n \Big\| u_i \partial_i u_j \Big\|_{W^{-1,p(x)}(\Omega)}.\end{aligned}$$

In view of the continuous embedding $L^{s(x)}(\Omega) \hookrightarrow W^{-1,p(x)}(\Omega)$ for $s(x) = \frac{np(x)}{n+p(x)}$ (see [3]), we have

$$\big\| u_i \partial_i u_j \big\|_{W^{-1,p(x)}(\Omega)} \leq C(n, p, \Omega) \big\| u_i \partial_i u_j \big\|_{L^{s(x)}(\Omega)}.$$

According to Theorem 2.3 in [25] and Hölder's inequality, we obtain

$$\begin{aligned}\big\| u_i \partial_i u_j \big\|_{L^{s(x)}(\Omega)} &\leq C \sup_{\|\varphi_j\|_{L^{s'(x)}(\Omega)} \leq 1} \int_\Omega \big| u_i \partial_i u_j \big| \big| \varphi_j \big| dx \\ &\leq C \big\| u_i \big\|_{L^n(\Omega)} \big\| \partial_i u_j \big\|_{L^{p(x)}(\Omega)} \big\| \varphi_j \big\|_{L^{s'(x)}(\Omega)} \\ &\leq C \big\| u_i \big\|_{L^n(\Omega)} \big\| u_j \big\|_{W_0^{1,p(x)}(\Omega)}.\end{aligned}$$

In terms of the continuous embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^n(\Omega)$ for $\frac{n}{2} \leq p_- \leq p(x) \leq p_+ < \infty$ (see [3]), we get

$$\Big\| u_i \partial_i u_j \Big\|_{L^{s(x)}(\Omega)} \leq C(n, p, \Omega) \big\| u_i \big\|_{W_0^{1,p(x)}(\Omega)} \big\| u_j \big\|_{W_0^{1,p(x)}(\Omega)}.$$

Finally, it is immediate to get the desired estimate from inequalities mentioned above. Hence, it is easy to get the continuity of the operator F from $F(u) = f - [uD]_0 u$. □

Remark 3.1. Actually, $\frac{n}{2} \leq p_-$ means $p_- \in (1, +\infty)$ for $n = 2$ while $p_- \in [\frac{n}{2}, +\infty)$ for $n > 2$. Obviously, Lemma 3.1 is a direct generalization of Lemma 4.1 in [2] to the variable exponent context.

We are now in position to show our main result in this section as follows.

Theorem 3.1. *Let $p(x)$ satisfy* (2.1) *and* $\frac{n}{2} \leq p_- \leq p(x) \leq p_+ < \infty$. *Then the system* (3.4)–(3.6) *has a unique solution* $(u, \pi) \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega, \mathbb{R})$ *(π is unique up to a real constant) if the right-hand side f satisfies the condition*

$$\|f\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)} < \frac{\nu^2}{4C_1 C_4^2}, \tag{3.7}$$

with $\nu = \frac{\mu}{\rho}$, $C_4 = C_2(1 + C_3)$, *where* $C_3 \geq 1$ *indicated in* (3.11) *and*

$$C_2 = \|T\|_{[L^{p(x)} \cap \mathrm{im} Q, W_0^{1,p(x)}]} \|Q\|_{[L^{p(x)}, L^{p(x)} \cap \mathrm{im} Q]} \|\widetilde{T}\|_{[W^{-1,p(x)}, L^{p(x)} \cap \mathrm{im} Q]}.$$

For any function $u_0 \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ *with*

$$||u_0||_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)} \leq \frac{\nu}{2C_1C_2} - \mathcal{W}, \tag{3.8}$$

here, $\mathcal{W} = \sqrt{\frac{\nu^2}{4C_1^2C_4^2} - \frac{1}{C_1}||f||_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)}}$ *, the iteration procedure*

$$u_k + \frac{1}{\mu} TQ\pi_k = \frac{\rho}{\mu} TQ\widetilde{T}F(u_{k-1}), \ k = 1, 2, \ldots \tag{3.9}$$

$$\frac{1}{\mu}\left[Q\pi_k\right]_0 = \frac{\rho}{\mu}\left[Q\widetilde{T}F(u_{k-1})\right]_0, \tag{3.10}$$

converges in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega, \mathbb{R})$.

Proof. The proof is similar to the theorem 4.6.8 in [17], and for the reader's convenience, here we will give the full details of the proof. Replacing f by $F(u_{k-1})$ in the proof of Theorem 2.1 we obtain the unique solvability of the Stokes equations (3.9)-(3.10) which we have to solve in each step. Moreover, we have the following estimate:

$$\left\|Du_k\right\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} + \frac{1}{\mu}\left\|Q\pi_k\right\|_{L^{p(x)}(\Omega)} \leq C_3 \frac{\rho}{\mu}\left\|Q\widetilde{T}F(u_{k-1})\right\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)} \tag{3.11}$$

where $C_3 \geq 1$ is a constant. The only remaining problem is the convergence of our iteration procedure. From Theorem 2.1 we know

$$Du_k + \frac{1}{\mu}Q\pi_k = \frac{\rho}{\mu}Q\widetilde{T}F(u_{k-1}).$$

Then we deduce from (3.11)

$$\frac{1}{\mu}\left\|Q(\pi_k - \pi_{k-1})\right\|_{L^{p(x)}(\Omega)} \leq \frac{C_3}{\nu}\left\|Q\widetilde{T}(F(u_{k-1}) - F(u_{k-2}))\right\|_{L^{p(x)}(\Omega, \mathrm{C}\ell_n)}.$$

Hence we have

$$\begin{aligned} &\left\|u_k - u_{k-1}\right\|_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)} \\ \leq &\frac{1}{\mu}\left\|TQ(\pi_k - \pi_{k-1})\right\|_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)} + \frac{\rho}{\mu}\left\|TQ\widetilde{T}(F(u_{k-1})\right. \\ &\left. - F(u_{k-2}))\right\|_{W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)} \\ \leq &\frac{C_2(1 + C_3)}{\nu}\left\|F(u_{k-1}) - F(u_{k-2})\right\|_{W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)}. \end{aligned}$$

By means of Lemma 3.1 we deduce

$$\begin{aligned}
&\big\|F(u_{k-1})-F(u_{k-2})\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
=&\big\|[u_{k-1}D]_0u_{k-1}-[u_{k-2}D]_0u_{k-2}\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
=&\big\|((u_{k-1}-u_{k-2})\cdot\nabla)u_{k-1}\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
&+\big\|(u_{k-2}\cdot\nabla)(u_{k-1}-u_{k-2})\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
\le&C_1\big\|u_{k-1}-u_{k-2}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}(\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
&+\big\|u_{k-2}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}).
\end{aligned}$$

Let $L_k=\frac{C_1C_4}{\nu}(\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}+\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)})$ with $C_4=C_2(1+C_3)$. Then we obtain

$$\big\|u_k-u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\le L_k\big\|u_{k-1}-u_{k-2}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{3.12}$$

On the other hand, by (3.11) and Lemma 3.1, we have

$$\begin{aligned}
\big\|u_k\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}&\le\frac{1}{\mu}\big\|TQ\pi_k\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
&\quad+\frac{\rho}{\mu}\big\|TQ\widetilde{T}F(u_{k-1})\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\\
&\le\frac{C_1C_4}{\nu}\big\|u_{k-1}\big\|^2_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}+\frac{C_4}{\nu}\big\|f\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}.
\end{aligned}$$

Now we have to ensure that $\big\|u_k\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\le\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$. For this we notice that

$$\frac{C_1C_4}{\nu}\big\|u_{k-1}\big\|^2_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}+\frac{C_4}{\nu}\big\|f\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}\le\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)},$$

which is equivalent to

$$\big\|u_{k-1}\big\|^2_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}-\frac{\nu}{C_1C_4}\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,Cl_n)}+\frac{1}{C_1}\big\|f\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}\le 0,$$

which is equivalent to

$$(\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}-\frac{\nu}{2C_1C_4})^2\le\frac{\nu^2}{(2C_1C_4)^2}-\frac{1}{C_1}\big\|f\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

According to the assumption (3.7), we have

$$\Big|\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}-\frac{\nu}{2C_1C_4}\Big|\le\mathcal{W}$$

with $\mathcal{W}=\sqrt{\frac{\nu^2}{4C_1^2C_4^2}-\frac{1}{C_1}\big\|f\big\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}}$. This leads to the following condition for $\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$

$$\frac{\nu}{2C_1C_4}-\mathcal{W}\le\big\|u_{k-1}\big\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\le\frac{\nu}{2C_1C_4}+\mathcal{W}.$$

Now assume $\|u_{k-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} - \mathcal{W}$. Then it follows

$$\begin{aligned}\|u_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} &\le \frac{1}{\mu}\|TQ\pi_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\quad + \frac{\rho}{\mu}\|TQ\widetilde{T}F(u_{k-1})\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\le \frac{C_1C_4}{\nu}\|u_{k-1}\|^2_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} + \frac{C_4}{\nu}\|f\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\le \frac{C_1C_4}{\nu}(\frac{\nu}{2C_1C_4} - \mathcal{W})^2 + \frac{C_4}{\nu}\|f\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\le \frac{\nu}{2C_1C_4} - \mathcal{W}.\end{aligned}$$

Consequently, using the inequality $\|u_{k-2}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} - \mathcal{W}$ and (3.12) we have

$$\begin{aligned}&\|u_k - u_{k-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\qquad \le \frac{2C_1C_4}{\nu}(\frac{\nu}{2C_1C_4} - \mathcal{W})\|u_{k-1} - u_{k-2}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\ &\qquad \le (1 - \frac{2C_1C_4}{\nu}\mathcal{W})\|u_k - u_{k-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}.\end{aligned}$$

And

$$L_k \le 1 - \frac{2C_1C_4}{\nu}\mathcal{W} := L < 1.$$

In the case one has

$$\|u_k - u_{k-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \le L\|u_{k-1} - u_{k-2}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \tag{3.13}$$

with $0 < L < 1$ and fixed. The convergence of the sequence $\{u_k\}$ is therefore obtained by making use of the Banach fixed-point theorem, and hence the convergence of the sequence $\{\pi_k\}$ immediately follows from (3.9). □

Remark 3.2. Here, ν is the kinematic viscosity of the fluid. Theorem 3.1 states that under certain smallness condition of the external force, there exists a unique solution to the stationary Navier-Stokes equations.

Corollary 3.1. *Under the same hypotheses as in Theorem* 3.1*, we have the a-priori estimate*

$$\|u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} - \mathcal{W}. \tag{3.14}$$

An a-priori estimate for the term $\|Q\pi\|_{L^{p(x)}(\Omega)}$ *is easy to obtain.*

Proof. (3.14) immediately follows from $\|u_{k-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \le \frac{\nu}{2C_1C_4} - \mathcal{W}$. It is easy to obtain the estimate for $\|Q\pi\|_{L^{p(x)}(\Omega)}$ from (3.11), (3.14) and Lemma 3.1. □

Now it is immediate to obtain the following result from (3.13).

Corollary 3.2. *There exists the error estimate*

$$\|u_k - u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \leq \frac{L^k}{1-L}\|u_0 - u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}.$$

In the case of $u_0 = 0$ *we have*

$$\|u_k - u\|_{W_0^{1,p(x)}(\Omega,Cl_n)} \leq \frac{L^k}{1-L}\Big(\frac{\nu}{2C_1C_4} - \mathcal{W}\Big).$$

4. The Steady Navier-Stokes Equations with Heat Conduction

In this section we will consider the flow of a viscous fluid under the influence of temperature. Similar to [17] the above used method for treating the steady Navier-Stokes equations can be applied to more complicated problems. More precisely, we consider the following problem:

$$-\Delta u + \frac{\rho}{\mu}(u\cdot\nabla)u + \frac{1}{\mu}\nabla\pi + \frac{\gamma}{\mu}gw = -f \quad \text{in } \Omega, \tag{4.1}$$

$$-\Delta w + \frac{m}{\kappa}(u\cdot\nabla)w = \frac{1}{\kappa}h \quad \text{in } \Omega, \tag{4.2}$$

$$\mathrm{div}u = 0 \quad \text{in } \Omega, \tag{4.3}$$

$$u, w = 0 \quad \text{on } \partial\Omega. \tag{4.4}$$

In addition to the case of Navier-Stokes equations, w denotes the temperature, γ the Grasshof number, m the Prandtl number, κ the number of temperature conductivity and g the vector $(0,0,\ldots,-1)^T$, where only the nth component is different from zero.

Remark 4.1. The detailed account for the Grasshof number, the Prandtl number and the Reynolds number we refer to [18].

Remark 4.2. In the case of Ω a bounded domain and space $W_0^{k,2}(\Omega,\mathbb{H})$ the problem (4.1)-(4.4) was already studied by Gürlebeck and Sprößig [18]. In the case of Ω a unbounded domain and space $W_0^{1,p}(\Omega,\mathrm{C}\ell_n)$ the problem (4.1)-(4.4) was already investigated by Cerejeiras and Kähler [2].

In analogy to the case of the Navier-Stokes equations, we consider the following equivalent hyper-complex problem:

$$u + \frac{1}{\mu}TQ\pi = -TQ\widetilde{T}(F(u) - \frac{\gamma}{\mu}e_n w) \ \text{ in } \Omega, \tag{4.5}$$

$$\frac{1}{\mu}\big[Q\pi\big]_0 = \big[Q\widetilde{T}(F(u) - \frac{\gamma}{\mu}e_n w)\big]_0 \ \text{ in } \Omega, \tag{4.6}$$

$$w = -\frac{m}{\kappa}TQ\widetilde{T}\big[uD\big]_0 w + \frac{1}{\kappa}TQ\widetilde{T}h \ \text{ in } \Omega, \tag{4.7}$$

with $F(u) := f + \frac{\rho}{\mu}\left[uD\right]_0 u$. Then the problem can be solved by the following iteration procedure:

$$u_k + \frac{1}{\mu} TQ\pi_k = -TQ\widetilde{T}(F(u_{k-1}) - \frac{\gamma}{\mu} e_n w_{k-1}) \text{ in } \Omega, \tag{4.8}$$

$$\frac{1}{\mu}\left[Q\pi_k\right]_0 = \left[Q\widetilde{T}(F(u_{k-1}) - \frac{\gamma}{\mu} e_n w_{k-1})\right]_0 \text{ in } \Omega, \tag{4.9}$$

$$w_k = -\frac{m}{\kappa} TQ\widetilde{T}\left[u_k D\right]_0 w_k + \frac{1}{\kappa} TQ\widetilde{T}h \text{ in } \Omega. \tag{4.10}$$

The equations (4.8) and (4.9) represent an iteration similar to the case of the Navier-Stokes equations. Therefore we have to study the solvability of equation (4.10). For this purpose, in analogy to [17], we propose the following "inner" iteration:

$$w_k^i = -\frac{m}{\kappa} TQ\widetilde{T}(u_k \cdot \nabla) w_k^{i-1} + \frac{1}{\kappa} TQ\widetilde{T}h. \tag{4.11}$$

Theorem 4.1. *Let $u_k \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, where $p(x)$ satisfies (2.1) and $\frac{n}{2} \leq p_- \leq p(x) \leq p_+ < \infty$. Furthermore, suppose*

(i) $\|u_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} < \kappa/mC_1C_2$;

(ii) $m\nu < 2\kappa(1 + C_3)$.

Then the iteration procedure (4.11) converges in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ to a unique solution of (4.10) and we have a-priori estimate

$$\|w_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \leq \frac{2(1 + C_3)C_2}{2\kappa(1 + C_3) - m\nu} \|h\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}. \tag{4.12}$$

Proof. The proof is similar to the Theorem 4.7.1 in [17], for the reader's convenience we sketch the key steps of the proof. From the assumption (i), (4.10) and Lemma 3.1 it follows

$$\|w_k^i\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \leq \frac{C_2\|h\|_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}}{\kappa - mC_1C_2\|u_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}}$$

for each $i \in \mathbb{N}$. Thus, the sequence $\{w_k^i\}_{i\in\mathbb{N}}$ is bounded in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. In view of Lemma 3.1 we get

$$\begin{aligned}
&\|w_k^i - w_k^{i-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\
=&\|\frac{m}{\kappa} TQ\widetilde{T}(u_k \cdot \nabla)(w_k^{i-1} - w_k^{i-2})\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\
\leq&\frac{m}{\kappa} C_1C_2\|u_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}\|w_k^i - w_k^{i-1}\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}.
\end{aligned}$$

Therefore, if $\frac{m}{\kappa} C_1C_2\|u_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} < 1$, then we obtain the convergence of the sequence $\{w_k^i\}_{i\in\mathbb{N}}$ in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Similar to the proof of Theorem

3.1, from (4.8) we can obtain

$$\begin{aligned}
&\|u_k\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\
\leq& \frac{\nu}{2C_1C_4} - \sqrt{\frac{\nu^2}{4C_1^2C_4^2} - \frac{\nu}{C_1} \parallel f \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} - \frac{\gamma}{\mu C_1} \parallel w_{k-1} \parallel_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}} \\
\leq& \frac{\nu}{2C_1C_4}. \qquad (4.13)
\end{aligned}$$

This implies that the condition (ii) is a sufficient convergence condition. □

Theorem 4.2. *Let* $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$*,* $h \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$*, where* $p(x)$ *satisfies* (2.1) *and* $\frac{n}{2} \leq p_- \leq p(x) \leq p_+ < \infty$ *. Furthermore, assume*

(a) $\nu \parallel f \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} + C_5 \parallel h \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} < C_6$*;*
(b) $\parallel h \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} < C_7$*;*
(c) $m\nu < 2\kappa(1 + C_3)$*,*

where

$$C_5 = \frac{2\gamma(1 + C_3)C_2}{\mu(2\kappa(1 + C_3) - m\nu)}, \quad C_6 = \frac{3\nu^2}{16C_1C_4^2}, \quad C_7 = \frac{\mu(2\kappa(1 + C_3) - m\nu)^2}{8\gamma m C_1 C_2^3(1 + C_3)^3}.$$

Then the boundary value problem (4.5)–(4.7) *has a solution* (u, w, π) *in spaces* $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \times L^{p(x)}(\Omega, \mathbb{R})$*, where* u *and* w *are uniquely defined and* π *uniquely up to a constant. Our iteration procedure* (4.8)–(4.10) *converges to the solution of* (4.5)–(4.7*.*

Proof. The proof is similar to the theorem 4.7.4 in [17], here we sketch the key steps of the proof for the reader's convenience. It is immediate to get the following estimate from the assumption (a)

$$\frac{\nu}{C_1} \parallel f \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} + \frac{C_5}{C_1} \parallel h \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} < \frac{\nu^2}{4C_1^2C_4^2}.$$

In virtue of the a-priori estimate (4.12), Similar to the proof of Theorem 3.1, we obtain the following a-priori estimate

$$\parallel u_k \parallel_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \leq \frac{\nu}{2C_1C_4} - M \qquad (4.14)$$

with

$$M = \sqrt{\frac{\nu^2}{4C_1^2C_4^2} - \frac{\nu}{C_1} \parallel f \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} - \frac{C_5}{C_1} \parallel h \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)}}.$$

So the sequence $\{u_k\}_{k\in\mathbb{N}}$ and $\{w_k\}_{k\in\mathbb{N}}$ are bounded in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Now (4.10) yields

$$\begin{aligned}
&\parallel w_k - w_{k-1} \parallel_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \\
\leq& \frac{m}{\kappa} C_2 \parallel (u_k \cdot \nabla)(w_k - w_{k-1}) + ((u_k - u_{k-1}) \cdot \nabla) w_{k-1} \parallel_{W^{-1,p(x)}(\Omega,\mathrm{C}\ell_n)} .
\end{aligned}$$

Then Lemma 3.1, (4.12) and (4.13) give

$$\begin{aligned} &\| w_k - w_{k-1} \|_{W_0^{1,p(x)}(\Omega,C\ell_n)} \\ \leq &\frac{\mu}{2\gamma C_4 C_7} \| h \|_{W^{-1,p(x)}(\Omega,C\ell_n)} \| u_k - u_{k-1} \|_{W_0^{1,p(x)}(\Omega,C\ell_n)} . \end{aligned}$$

Consequently, we obtain from (4.8)

$$\| u_k - u_{k-1} \|_{W_0^{1,p(x)}(\Omega,C\ell_n)} \leq L_k \| u_{k-1} - u_{k-2} \|_{W_0^{1,p(x)}(\Omega,C\ell_n)},$$

with

$$\begin{aligned} L_k = &\frac{C_1 C_4}{\nu} \left(\| u_{k-1} \|_{W_0^{1,p(x)}(\Omega,C\ell_n)} + \| u_{k-2} \|_{W_0^{1,p(x)}(\Omega,C\ell_n)} \right) \\ &+ \frac{1}{2C_7} \| h \|_{W^{-1,p(x)}(\Omega,C\ell_n)} . \end{aligned}$$

Obviously, we can get from the conditions (a), (b) and (4.14)

$$L_k \leq L < 1,$$

where $0 < L < 1$ is a constant independent of k. Now we can apply the Banach fixed-pointed theorem. The proof is thus completed. □

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