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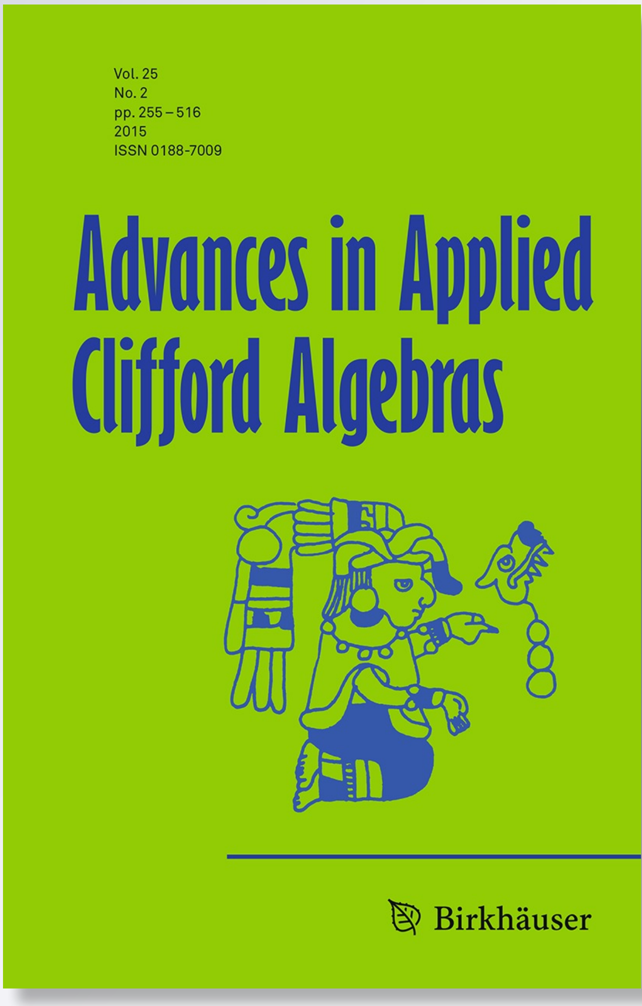

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Existence of Stationary States for A-Dirac Equations with Variable Growth

Giovanni Molica Bisci, Vicenţiu D. Rădulescu* and Binlin Zhang

Abstract. In this paper, using a Hodge-type decomposition of variable exponent Lebesgue spaces of Clifford-valued functions and variational methods, we study the properties of weak solutions to the homogeneous and nonhomogeneous A-Dirac equations with variable growth in the setting of variable exponent Sobolev spaces of Clifford-valued functions.

Keywords. Clifford analysis; A-Dirac equation; variable exponent; Caccioppoli estimates; Hodge-type decomposition.

1. Introduction

The Dirac equation arises in the study of nonlinear spinor fields in the unified theory of elementary particles, see Heisenberg [26] and Weyl [39]. The stationary states of the nonlinear Dirac field have been proposed as a model for elementary extended fermions and nucleons, see Thaller [38].

The paper is primarily concerned with the existence of weak solutions to the homogeneous A-Dirac equations

$$DA(x, Du) = 0, \tag{1.1}$$

and the non-homogenous A-Dirac equations

$$DA(x, Du) = Df, \tag{1.2}$$

where u is a function valued in the universal Clifford algebra $\mathrm{C}\ell_n$ over a bounded domain Ω with a sufficiently smooth boundary $\partial\Omega$ in $\mathbb{R}^n (n \geq 2)$, D is the usual Euclidean Dirac operator, $f \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$ and the operator $A : \Omega \times \mathrm{C}\ell_n \to \mathrm{C}\ell_n$ satisfies the following conditions with variable growth:

(A1) $A(x, \xi)$ is measurable with respect to x for $\xi \in \mathrm{C}\ell_n$ and continuous with respect to ξ for a.e. $x \in \Omega$;

(A2) $|A(x, \xi)| \leq C_1|\xi|^{p(x)-1}+g(x)$ for a.e. $x \in \Omega$ and $\xi \in \mathrm{C}\ell_n$;

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(A3) $\left[\overline{A(x,\xi)}\xi\right]_0 \geq C_2|\xi|^{p(x)}+h(x)$ for a.e. $x \in \Omega$ and $\xi \in \mathrm{C}\ell_n$;

(A4) $\left[\overline{(A(x,\xi_1)-A(x,\xi_2))}(\xi_1-\xi_2)\right]_0 \geq 0$ for a.e. $x \in \Omega$ and $\xi_1,\xi_2 \in \mathrm{C}\ell_n$,

where C_i $(i=1,2)$ are positive constants, $g(x)$ is bounded in $L^{p'(x)}(\Omega)$ and $h(x) \in L^1(\Omega)$. Of course, $DA(x,Du)=0$ and $DA(x,Du)=Df$ are meant in the distributional sense.

Clifford algebras have important applications in a variety of fields including geometry, theoretical physics and digital image processing. They are named after the English geometer William Kingdon Clifford, see [4].

A study of the conformal p-Dirac equations, a special case of A-Dirac equations, appears in [34]. These equations are nonlinear generalizations of the Dirac Laplace equation as well as generalizations of elliptic equations of A-harmonic type div $A(x,\nabla u)=0$. The study of these equations is partially motivated by the fact that some arise as the Euler-Lagrange equations to variational integrals.

In [32, 33], Nolder first introduced A-Dirac equations (1.1) and developed some tools for the study of weak solutions to nonlinear A-Dirac equations in the space $W_0^{1,p}(\Omega,\mathrm{C}\ell_n)$. In [3], Wang and Chen considered the nonhomogeneous A-Dirac equations $DA(x,Du)=f(x,Du)$ in space $W_0^{1,p}(\Omega,\mathrm{C}\ell_n)$. The authors proved that under certain conditions, the solutions to the inhomogeneous A-harmonic equations if f satisfies the controllable growth condition is in fact the scalar part of weak solutions to the corresponding inhomogeneous A-Dirac equations. In [31], Lu and Bao were concerned with the regularity properties of weak solutions to the obstacle problem for homogeneous A-Dirac equations, such as a global reverse Hölder inequality and stability. However, the existence of weak solutions to the A-Dirac equations has not been showed under the conditions (A1)–(A3) as $p(x) \equiv p$. Inspired by their works, Fu and Zhang [15, 16, 40] were interested in the the existence of weak solutions for A-Dirac equations with variable growth. Until now they have proved the existence of weak solutions to the scalar part of homogeneous and non-homogeneous A-Dirac equations under the assumptions (A1)–(A4). Recently, Fu, Zhang and Rădulescu [17] established a Hodge-type decomposition of variable exponent Lebesgue spaces of Clifford-valued functions. By using this decomposition, together with the Minty-Browder Theorem, existence and uniqueness of a weak solution to the A-Dirac equations $DA(Du)=0$ were obtained under the following assumptions:

(H1) $|A(\xi)-A(\eta)| \leq C_1'(|\xi|+|\eta|)^{p(x)-2}|\xi-\eta|$;

(H2) $\left[\overline{(A(\xi)-A(\eta))}(\xi-\eta)\right]_0 \geq C_2'(|\xi|+|\eta|)^{p(x)-2}|\xi-\eta|^2$;

(H3) $A(0) \in L^{p'(x)}(\Omega,\mathrm{C}\ell_n)$,

where ξ and η are arbitrary elements from $\mathrm{C}\ell_n$, both C_1' and C_2' are positive constants independent of ξ and η. Obviously, conditions (A1)–(A4) are weaker than conditions (H1)–(H3).

It is worth pointing out that an A-harmonic equation $\mathrm{div}A(x,\nabla u)=0$ is the scalar part of the equation (1.1) under appropriate identifications, see [32, 33]. When u is a real function, Du can be identified with ∇u. Hence the equation (1.2) corresponds to the nonhomogeneous A-harmonic equation

$$\mathrm{div}\, A(x,\nabla u)=\mathrm{div} f, \tag{1.3}$$

Iwaniec and Sbordone [27] introduced the definition of very weak solutions for the equation (1.3) in the Sobolev space $W_0^{1,p}(\Omega)$ and studied the existence and uniqueness of such solutions under the conditions (H1), (H2) and the homogeneity condition as $p(x)\equiv p$. Many results have been obtained concerning existence, uniqueness and regularity results for this kinds of equations (1.3), for example, see [2, 5, 13, 18, 23, 27] and the references therein.

In [20], Diening, Kaplicky and Schwarzacher showed BMO estimates of p–Laplace system given by $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = -\mathrm{div} f$. In [5], Diening and Kaplicky studied the interior regularity of the local weak solutions $u\in W^{1,\varphi}(\Omega)$ and $\pi\in L^{\varphi^*}(\Omega)$ of the following stationary generalized Stokes system:

$$\begin{cases} -\mathrm{div}A(Du)+\nabla\pi=-\mathrm{div}G \quad \text{in } \Omega \\ \mathrm{div}\ u=0 \quad \text{in } \Omega, \end{cases}$$

where D is the symmetric part of the gradient, the extra stress tensor A determines properties of the fluid. The system originates in fluid mechanics. The authors showed optimal BMO and Campanato estimates for $A(Du)$. In [7], Diening and Kaplicky proceeded to study regularity theory of solution to the stationary generalized Stokes system, then apply these estimates to the stationary generalized Navier–Stokes system. Therefore, it is reasonable to consider the nonhomogeneous A-Dirac equations $DA(x,Du) = Df$ as a national extension.

This paper is organized as follows. In section 2, we begin with a brief summary of basic knowledge of Clifford algebras and variable exponent spaces of Clifford-valued functions, which will be needed later. In section 3, appealing to a Hodge-type decomposition as well as variational methods, we prove the existence and uniqueness of solutions to the homogeneous A-Dirac equations with variable growth in $W_0^{1,p(x)}(\Omega, \mathcal{C}\ell_n)$, and a Caccioppoli-type estimate for weak solutions is obtained in the variable exponent context. In section 4, we study the existence, uniqueness and stability of solutions to the nonhomogeneous A-Dirac equations with variable growth in $W_0^{1,p(x)}(\Omega, \mathcal{C}\ell_n)$.

2. Preliminaries

2.1. Clifford algebra

We first recall some related notions and results from Clifford algebra. The most important Clifford algebras are those over real and complex vector spaces equipped with nondegenerate quadratic forms. For a detailed account we refer to [1, 8, 10, 11, 19, 21, 22, 28, 34, 37]. Let $C\ell_n$ be the real universal Clifford algebra over $\mathbb{R}^n$. Denote

$$C\ell_n = \text{span}\{e_0, e_1, e_2, \ldots, e_n, e_1e_2, \ldots, e_{n-1}e_n, \ldots, e_1e_2\ldots e_n\}$$

where $e_0 = 1$ (the identity element in $\mathbb{R}^n$), $\{e_1, e_2, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{R}^n$ with the relation $e_ie_j + e_je_i = -2\delta_{ij}e_0$. Thus, the dimension of $C\ell_n$ is 2^n. In particular, by $\mathbb{H} := C\ell_2$ we denote the algebra of real quaternions, see [20] for further details about the algebra of real quaternions. For $I := \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ with $1 \le i_1 < i_2 < \ldots < i_n \le n$, put $e_I = e_{i_1}e_{i_2}\ldots e_{i_r}$, while for $I = \emptyset$, $e_\emptyset = e_0$. For $0 \le r \le n$ fixed, the space $C\ell_n^r$ is defined by

$$C\ell_n^r = \text{span}\{e_I : |I| = r\},$$

where $|I|$ denotes cardinal number of the set I. The Clifford algebra $C\ell_n$ is a graded algebra as

$$C\ell_n = \bigoplus_{0\le r\le n} C\ell_n^r.$$

Any element $a \in C\ell_n$ may thus be written in a unique way as

$$a = [a]_0 + [a]_1 + \ldots + [a]_n,$$

where $[\ ]_r : C\ell_n \to C\ell_n^r$ denotes the projection of $C\ell_n$ onto $C\ell_n^r$. It is customary to identify $\mathbb{R}$ with $C\ell_n^0$ and identify $\mathbb{R}^n$ with $C\ell_n^1$ respectively. This means that each element x of $\mathbb{R}^n$ may be represented by $x = \sum_{i=1}^n x_ie_i$. From an analysis viewpoint, an important property of the universal Clifford algebra is that every non-zero vector $x \in \mathbb{R}^n$ has a multiplicative inverse given by $-x/|x|^2$. Up to a sign, this inverse corresponds to the Kelvin inverse of a vector in Euclidean space.

We point out that *Hamilton's quaternions* are constructed as the even sub algebra of the Clifford algebra, while *dual quaternions* are constructed as the even Clifford algebra of real four dimensional space with a degenerate quadratic form.

For $u \in C\ell_n$, we denote by $[u]_0$ the scalar part of u, that is, the coefficient of the element e_0. We define the Clifford conjugation as follows:

$$\overline{e_{i_1}e_{i_2}\ldots e_{i_r}} = (-1)^{\frac{r(r+1)}{2}} e_{i_1}e_{i_2}\ldots e_{i_r}.$$

For $A \in C\ell_n$, $B \in C\ell_n$, we have

$$\overline{AB} = \overline{B}\ \overline{A}, \quad \overline{\overline{A}} = A.$$

We denote

$$(A, B) = [\overline{A}B]_0.$$

Then an inner product is thus obtained, give rising to the norm $|\cdot|$ on $C\ell_n$ given by

$$|A|^2 = [\overline{A}A]_0.$$

By [22], we know that this norm is submultiplicative:

$$|AB| \le C_n|A||B|, \tag{2.1}$$

where C_n is a positive constant depending only on n and smaller than $2^{n/2}$.

A Clifford-valued function $u : \Omega \to C\ell_n$ can be written as $u = \sum_I u_I e_I$, where the coefficients $u_I : \Omega \to \mathbb{R}$ are real-valued functions.

The Dirac operator on Euclidean space used here is introduced by

$$D = \sum_{j=1}^{n} e_j \frac{\partial}{\partial x_j}.$$

This is a special case of the Atiyah-Singer-Dirac operator acting on sections of a spinor bundle. We also point out that the most famous Dirac operator describes the propagation of a free fermion in three dimensions.

If u is a real-valued function defined on a domain Ω in $\mathbb{R}^n$, then $Du = \nabla u$. Moreover, $D^2 = -\Delta$, where Δ is the Laplace operator which operates only on coefficients. A function is left monogenic if it satisfies the equation $Du(x) = 0$ for each $x \in \Omega$. A similar definition can be given for right monogenic function. An important example of a left monogenic function is the generalized Cauchy kernel

$$G(x) = \frac{1}{\omega_n} \frac{\overline{x}}{|x|^n},$$

where ω_n denotes the surface area of the unit ball in $\mathbb{R}^n$. This function is a fundamental solution of the Dirac operator. Basic properties of left monogenic functions one can refer to [8, 19, 21, 22].

2.2. Variable exponent spaces of Clifford-valued functions

Next we investigate some basic properties of variable exponent spaces of Clifford-valued functions. Note that in what follows, we use the short notation $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, instead of $L^{p(x)}(\Omega, \mathbb{R})$, $W^{1,p(x)}(\Omega, \mathbb{R})$. Throughout this paper we always assume (unless declared specially)

$$p \in \mathcal{P}^{\log}(\Omega) \text{ and } 1 < p_- := \inf_{x \in \overline{\Omega}} p(x) \le p(x) \le \sup_{x \in \overline{\Omega}} p(x) =: p_+ < \infty, \quad (2.2)$$

where $\mathcal{P}^{\log}(\Omega)$ is the set of exponent p satisfying the so-called log-Hölder continuity, that is,

$$|p(x) - p(y)| \le \frac{C}{\log(e + |x - y|^{-1})}$$

holds for all $x, y \in \Omega$, see [29, 6]. Let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1, \infty)$. Given $p \in \mathcal{P}(\Omega)$ we define the conjugate function $p'(x) \in \mathcal{P}(\Omega)$ by

$$p'(x) = \frac{p(x)}{p(x) - 1}, \qquad \text{for all } x \in \Omega.$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega) = \Big\{ u \in \mathcal{P}(\Omega) : \int_\Omega |u(x)|^{p(x)} dx < \infty \Big\},$$

with the norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\Big\{t > 0 : \int_\Omega \Big|\frac{u(x)}{t}\Big|^{p(x)} dx \leq 1\Big\},$$

and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|\nabla u\|_{L^{p(x)}(\Omega)} + \|u\|_{L^{p(x)}(\Omega)}. \tag{2.3}$$

Let $W_0^{1,p(x)}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm (2.3). The space $W^{-1,p(x)}(\Omega)$ is defined as the dual of the space $W_0^{1,p'(x)}(\Omega)$. For more details we refer to [6, 12, 14, 29] and the references therein.

In the sequel, we say that $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ can be understood coordinatewisely. For example, $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ means that $\{u_I\} \subset L^{p(x)}(\Omega)$ for $u = \Sigma_I u_I e_I \in \mathrm{C}\ell_n$ with the norm $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} = \sum_I \|u_I\|_{L^{p(x)}(\Omega)}$. In this way, spaces $W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, $C_0^\infty(\Omega, \mathrm{C}\ell_n)$, etc., can be understood similarly. In particular, the space $L^2(\Omega, \mathrm{C}\ell_n)$ can be converted into a right Hilbert $\mathrm{C}\ell_n$-module by defining the following Clifford-valued inner product (see [21, Definition 3. 74])

$$(f, g)_{Cl_n} = \int_\Omega \overline{f(x)} g(x) dx. \tag{2.4}$$

Remark 2.1. A simple calculation shows that

$$2^{-\frac{n(1+p_+)}{p_-}} \||u|\|_{L^{p(x)}(\Omega)} \leq \|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \leq 2^n \||u|\|_{L^{p(x)}(\Omega)}.$$

holds for each $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, see [3, Remark 1]. Thus, $\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}$ and $\||u|\|_{L^{p(x)}(\Omega)}$ are equivalent norms on $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Furthermore, we have that $\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}$ and $\|u\|_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, see [40, Definition 2.9].

The following definitions and lemmas will be crucial in the sequel.

Lemma 2.1. (See [6].) *Let* $\rho(u) = \int_\Omega |u(x)|^{p(x)} dx$. *For* $u \in L^{p(x)}(\Omega)$, *we have*

(1) *If* $\|u\|_{L^{p(x)}(\Omega)} \geq 1$, *then* $\|u\|^{p_-}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_+}_{L^{p(x)}(\Omega)}$.
(2) *If* $\|u\|_{L^{p(x)}(\Omega)} \leq 1$, *then* $\|u\|^{p_+}_{L^{p(x)}(\Omega)} \leq \rho(u) \leq \|u\|^{p_-}_{L^{p(x)}(\Omega)}$.

Lemma 2.2. (See [15].) *Assume that* $p(x) \in \mathcal{P}(\Omega)$. *Then the inequality*

$$\int_\Omega |uv| dx \leq C(n,p)\|u\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \|v\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}$$

holds for every $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ *and* $v \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$.

Lemma 2.3. (See [15, 16].) *Assume that $p(x) \in \mathcal{P}(\Omega)$. Then*

(1) *The dual of the space $L^{p(x)}(\Omega, C\ell_n)$ is the space $L^{p'(x)}(\Omega, C\ell_n)$. That is, $\left(L^{p(x)}(\Omega, C\ell_n)\right)^* = L^{p'(x)}(\Omega, C\ell_n)$. Thus, $L^{p(x)}(\Omega, C\ell_n)$ is a reflexive and separable Banach space.*

(2) *The space $W^{1,p(x)}(\Omega, C\ell_n)$ is a reflexive and separable Banach space.*

Definition 2.1. (See [21].)

(i) Let $u \in C(\Omega, C\ell_n)$. The *Teodorescu operator* is defined by

$$Tu(x) = \int_\Omega G(x-y)u(y)dy,$$

where $G(x)$ is the generalized Cauchy kernel above mentioned.

(ii) Let $u \in C^1(\Omega, C\ell_n) \bigcap C(\overline{\Omega}, C\ell_n)$. The *boundary operator* is defined by

$$Fu(x) = \int_{\partial\Omega} G(y-x)\alpha(y)u(y)dS_y,$$

where $\alpha(y)$ denotes the outward normal unit vector at y.

Lemma 2.4. (See [15].) *The operator $D : W^{1,p(x)}(\Omega, C\ell_n) \to L^{p(x)}(\Omega, C\ell_n)$ is bounded.*

Lemma 2.5. (See [16].) *The operator $T : L^{p(x)}(\Omega, C\ell_n) \to W^{1,p(x)}(\Omega, C\ell_n)$ is bounded.*

Lemma 2.6. *Let $p(x) \in \mathcal{P}(\Omega)$. If $u \in W^{1,p(x)}(\Omega, C\ell_n)$, then the Borel-Pompeiu formula $Fu(x) + TDu(x) = u(x)$ holds for all $x \in \Omega$. Moreover, if $u \in L^{p(x)}(\Omega, C\ell_n)$, then the equation $DTu(x) = u(x)$ holds for all $x \in \Omega$.*

Proof. By Remark 4.21 in [21], the conclusions are implied by $W^{1,p(x)}(\Omega, C\ell_n) \hookrightarrow W^{1,p_-}(\Omega, C\ell_n)$ and $L^{p(x)}(\Omega, C\ell_n) \hookrightarrow L^{p_-}(\Omega, C\ell_n)$. □

Lemma 2.7. *The operator $F : \mathrm{tr}\left(W^{1,p(x)}(\Omega, C\ell_n)\right) \to W^{1,p(x)}(\Omega, C\ell_n) \cap \ker D$ is bounded. Here, the trace space $\mathrm{tr}\left(W^{1,p(x)}(\Omega, C\ell_n)\right)$ is defined by*

$$\mathrm{tr}\left(W^{1,p(x)}(\Omega, C\ell_n)\right) = \left\{f \in L^1(\partial\Omega, C\ell_n) : \exists u \in W^{1,p(x)}(\Omega, C\ell_n), s.t.\ u|_{\partial\Omega} = f\right\}.$$

Proof. Let $u \in \mathrm{tr}\left(W^{1,p(x)}(\Omega, C\ell_n)\right)$. Then there exists $v \in W^{1,p(x)}(\Omega, C\ell_n)$ such that $v|_{\partial\Omega} = u$. Using Borel-Pompeiu formula, we know that $Fu = v - TDv$. By Lemma 2.4 and Lemma 2.5, $Fu \in W^{1,p(x)}(\Omega, C\ell_n)$ and $I - TD$ is continuous in $W^{1,p(x)}(\Omega, C\ell_n)$. Since $DFu = Du - DTDu = 0$, we have $Fu \in \ker D$. □

Lemma 2.8. (See [17].) *There exists a unique linear extension $\widetilde{T}$ of the operator T such that the operator $\widetilde{T} : W^{-1,p(x)}(\Omega, C\ell_n) \to L^{p(x)}(\Omega, C\ell_n)$ is bounded.*

Diening, Lengeler and Ružička [9] showed that the Dirichlet problem of the Poisson equation with homogeneous boundary data

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

possesses a unique weak solution $u \in W^{1,p(x)}(\Omega)$ for each $f \in W^{-1,p(x)}(\Omega)$. Moreover, there is the estimate

$$\|u\|_{W^{1,p(x)}(\Omega)} \leq C(n,p,\Omega)\|f\|_{W^{-1,p(x)}(\Omega)}$$

where we call u a weak solution of (2.5) provided that

$$\langle f, \varphi\rangle = \int_\Omega \nabla u \cdot \nabla \varphi dx, \quad \forall\, \varphi \in W_0^{1,p'(x)}(\Omega).$$

Then it is easy to see that for all $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$ the problem (2.5) still has a unique weak solution $u \in W^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. We denote by Δ_0^{-1} the solution operator. On the other hand, it is clear that the operator

$$\Delta : W^{1,p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$$

is continuous, so we obtain that the operator $\widetilde{D} = -\Delta T : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$ is continuous from Lemma 2.5, where the operator $\widetilde{D}$ can be considered as a unique continuous linear extension of the Dirac operator.

Lemma 2.9. (See [17].) *Assume that $p(x)$ satisfies relation* (2.2).

- (i) *If $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, then the equation $\widetilde{T}\widetilde{D}u(x) = u(x)$ holds for all $x \in \Omega$.*
- (ii) *If $u \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$, then the equation $\widetilde{D}\widetilde{T}u(x) = u(x)$ holds for all $x \in \Omega$.*

Lemma 2.10. (See [17].) *The space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ allows the Hodge-type decomposition*

$$L^{p(x)}(\Omega, \mathrm{C}\ell_n) = (\ker\widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n)) \oplus DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$$

with respect to the Clifford-valued product (2.4).

Proof. The proof is given in [17, Theorem 3.1]. We sketch it here for the reader's convenience. First, it is easy to prove that

$$\left(\ker\widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n)\right) \cap DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n) = \{0\}.$$

Now let $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Then we have $u_2 = D\Delta_0^{-1}\widetilde{D}u \in DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Let $u_1 = u - u_2$. Then $u_1 \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Furthermore, we have

$$Du_1 = Du - \widetilde{D}D\Delta_0^{-1}Du = Du + \Delta\Delta_0^{-1}Du = Du - Du = 0.$$

Thus, $u_1 \in \ker\widetilde{D}$. Since $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ is arbitrary, the desired result follows immediately. □

Beginning with this decomposition we can get the following projections

$$P : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to \ker\widetilde{D} \cap L^{p(x)}(\Omega, \mathrm{C}\ell_n),$$

$$Q : L^{p(x)}(\Omega, \mathrm{C}\ell_n) \to DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n).$$

For $p(x) \equiv 2$, these are ortho-projections. Notice that directly from the proof of Theorem 3.1 we obtain

$$Q = D\Delta_0^{-1}\widetilde{D}, P = I - Q. \tag{2.6}$$

It follows from (2.6) that the operator Q as well as P maps the space $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$ into itself.

Lemma 2.11. (See [17].) *The space* $L^{p(x)}(\Omega, \mathrm{C}\ell_n) \cap \mathrm{im}Q$ *is a closed subspace of* $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$, *that is, the space* $DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ *is closed in* $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$.

Lemma 2.12. (See [17].) $\left(L^{p(x)}(\Omega, \mathrm{C}\ell_n) \cap \mathrm{im}Q\right)^* = L^{p'(x)}(\Omega, \mathrm{C}\ell_n) \cap \mathrm{im}Q$, *that is, the linear operator*

$$\Phi : DW_0^{1,p'(x)}(\Omega, \mathrm{C}\ell_n) \to \left(DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)\right)^*$$

given by

$$\Phi(Du)(D\varphi) = (D\varphi, Du)_{Sc} := \int_\Omega [\overline{D\varphi} Du]_0 dx$$

is a Banach space isomorphism.

3. The Homogeneous A-Dirac Equations

In this section, we are interested in the existence of weak solutions for the homogeneous A-Dirac equations (1.1). In order to get the existence of weak solutions to (1.1), we need the following result, see [30, Theorem 2.1] for the proof.

Proposition 3.1. *Let X be a reflexive, separable Banach space, and assume that $G : X \to X^*$ is*

(i) *monotone:* $(Gv - Gw, v - w) \geq 0 \quad \forall\ v, w \in X$;
(ii) *bounded: G maps bounded sets to bounded sets;*
(iii) *demicontinuous:* $(G(u + \lambda v), w) \to (G(u + v), w)$, *as* $\lambda \in \mathbb{R}$, $\lambda \to 0$, $u, v, w \in X$;
(iv) *coercive:* $\lim_{\|v\| \to \infty} \|v\|^{-1}(Gv, v) = \infty$.

Then G is surjective.

Now we are ready to prove our result as follows.

Theorem 3.1. *Under the conditions* (A1)–(A4), *there exists a weak solution* $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ *to the A-Dirac equations* (1.1), *that is to say, there exists a Clifford-valued function* $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ *such that*

$$\int_\Omega \overline{A(x, Du)} Dv dx = 0 \tag{3.1}$$

for any $v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. *Furthermore, the solution to the scalar part of* (1.1) *is unique up to a monogenic function.*

Proof. We divide the proof into four steps:
Step 1. We first claim that $A(x, u) \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$ for every $u \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Indeed, from (A2) we obtain

$$\int_\Omega |A(x, u)|^{p'(x)} dx \leq 2^{n-1} C_1 \int_\Omega |u|^{p(x)} dx + 2^{n-1} \int_\Omega |g|^{p'(x)} dx.$$

This estimate together with Remark 2.1 and Lemma 2.1 yields the previous assertion.

Step 2. By Lemma 2.11, we know that $DW_0^{1,p(x)}(\Omega, C\ell_n)$ is a reflexive and separable Banach space. By Lemma 2.12, we get that $\big(DW_0^{1,p(x)}(\Omega, C\ell_n)\big)^* = DW_0^{1,p'(x)}(\Omega, C\ell_n)$. Obviously, it follows from (2.6) that

$$QA\big(\Omega \times DW_0^{1,p(x)}(\Omega, C\ell_n)\big) \subset DW_0^{1,p'(x)}(\Omega, C\ell_n).$$

Now, we define the nonlinear mapping

$$\mathcal{F} : DW_0^{1,p(x)}(\Omega, C\ell_n) \to DW_0^{1,p'(x)}(\Omega, C\ell_n)$$

as follows:

$$\mathcal{F}(Du) = QA(x, Du), \quad \text{for each } u \in W_0^{1,p(x)}(\Omega, C\ell_n).$$

In the following, to get surjectivity of the operator $\mathcal{F}$, we need to verify the conditions of Proposition 3.1 respectively.

(1) The operator $\mathcal{F}$ is demicontinuous. Obviously, it suffice to show that the operator $\mathcal{F}$ is strongly-weakly continuous. Suppose that $Du_k, Du \in DW_0^{1,p(x)}(\Omega, C\ell_n)$ and $||Du_k - Du||_{L^{p(x)}(\Omega, C\ell_n)} \to 0$ as $k \to \infty$. Then $\{Du_k\}$ is uniformly bounded in $L^{p(x)}(\Omega, C\ell_n)$. By (A1) we can deduce that for each $v \in W_0^{1,p(x)}(\Omega, C\ell_n)$

$$\Big[\overline{QA(x, Du_k)}Dv\Big]_0 \to \Big[\overline{QA(x, Du)}Dv\Big]_0 \quad \text{a.e. on } \Omega, \quad \text{as } k \to \infty.$$

On the other hand, to see equi-continuous integrability of the sequence $\big\{\big[\overline{A(x, Du_k)}Dv\big]_0\big\}$, we take a measurable subset $\Omega^{'} \subset \Omega$, by (2.1), (A2) and the Hölder inequality we have for each $v \in W_0^{1,p(x)}(\Omega, C\ell_n)$

$$\begin{aligned} &\Big|\int_{\Omega'} \Big[\overline{QA(x, Du_k)}Dv\Big]_0 dx\Big| \\ &\le 2C_n \|Q\|_{L^{p'(x)}(\Omega')} \big\|\, |Dv|\, \big\|_{L^{p(x)}(\Omega')} \big(C_1 \big\||Du_k|^{p(x)-1}\big\|_{L^{p'(x)}(\Omega')} + \|g\|_{L^{p'(x)}(\Omega')}\big). \end{aligned} \tag{3.2}$$

In terms of Remark 2.1, Lemma 2.1 and boundedness of the operator Q, we obtain that the third part of (3.2) is uniformly bounded in k. The second norm of (3.2) is arbitrarily small if the measure of $\Omega^{'}$ is chosen small enough. By the Vitali Convergence Theorem, we have for each $v \in W_0^{1,p(x)}(\Omega, C\ell_n)$

$$\begin{aligned} (\mathcal{F}(Du_k), Dv) :&= \int_\Omega \Big[\overline{QA(x, Du_k)}Dv\Big]_0 dx \\ &\to \int_\Omega \Big[\overline{QA(x, Du)}Dv\Big]_0 dx = (\mathcal{F}(Du), Dv) \end{aligned}$$

as $k \to \infty$. That is to say, $\mathcal{F}$ is strongly-weakly continuous.

(2) The operator $\mathcal{F}$ is bounded. In terms of the Hölder inequality, the boundedness of Q and (A2), together Remark 2.1 and Lemma 2.1, we obtain

for each $v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$

$$\begin{aligned}\left|(\mathcal{F}(Du), Dv)\right| &= \left|\int_\Omega \left[\overline{QA(x,Du)}Dv\right]_0 dx\right| \\ &\leq C_3\Big(C_4 \max\Big\{\|Du\|^{p_+-1}_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}, \|Du\|^{p_--1}_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\Big\} \\ &\quad + \|g(x)\|_{L^{p'(x)}(\Omega)}\Big)\|Dv\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)},\end{aligned}$$

where C_3 and C_4 are two positive constants. This implies that $\mathcal{F}$ is bounded.

(3) The operator $\mathcal{F}$ is monotone. In view of Lemma 2.10, we have

$$QA(x,Du) = A(x,Du) - PA(x,Du)$$

for each $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Thus, for any $u, v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$, (3.2) gives

$$\begin{aligned}\big(QA(Du), Dv\big)_{Sc} &= \big(A(Du), Dv\big)_{Sc} - \big(PA(Du), Dv\big)_{Sc} \\ &= \big(A(Du), Dv\big)_{Sc}.\end{aligned} \tag{3.3}$$

Then the condition (A4) yields

$$\begin{aligned}\big(\mathcal{F}(Du) - \mathcal{F}(Dv), Du - Dv\big) &= \big(QA(Du) - QA(Dv), Du - Dv\big)_{Sc} \\ &= \big(A(Du) - A(Dv), Du - Dv\big)_{Sc} \\ &= \int_\Omega \left[\overline{\big(A(Du) - A(Dv)\big)}(Du - Dv)\right]_0 dx \geq 0.\end{aligned}$$

(4) The operator $\mathcal{F}$ is coercive. By means of (3.3) and (A3) we have

$$\begin{aligned}\frac{\big(\mathcal{F}(Du), Du\big)}{|||Du|||_{L^{p(x)}(\Omega)}} = \frac{\big(QA(x,Du), Du\big)_{Sc}}{|||Du|||_{L^{p(x)}(\Omega)}} &= \frac{\big(A(x,Du), Du\big)_{Sc}}{|||Du|||_{L^{p(x)}(\Omega)}} \\ &= \frac{\int_\Omega \left[\overline{A(x,Du)}Du\right]_0 dx}{||||Du|||_{L^{p(x)}(\Omega)}} \\ &\geq \frac{C_2 \int_\Omega |Du|^{p(x)} dx + \int_\Omega h(x) dx}{|||Du|||_{L^{p(x)}(\Omega)}}.\end{aligned}$$

Since

$$\frac{\int_\Omega |Du|^{p(x)} dx}{|||Du|||_{L^{p(x)}(\Omega)}} = \int_\Omega \left(\frac{|Du|}{2^{-1}|||Du|||_{L^{p(x)}(\Omega)}}\right)^{p(x)} \frac{\left(2^{-1}|||Du|||_{L^{p(x)}(\Omega)}\right)^{p(x)}}{|||Du|||_{L^{p(x)}(\Omega)}} dx,$$

when $|||Du|||_{L^{p(x)}(\Omega)} \geq 1$, we have

$$\frac{\int_\Omega |Du|^{p(x)} dx}{|||Du|||_{L^{p(x)}(\Omega)}} \geq 2^{-p_+} |||Du|||^{p_--1}_{L^{p(x)}(\Omega)}.$$

Hence, by Remark 2.1, it easily follows that

$$\frac{(\mathcal{F}(Du), Du)}{\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}} \to \infty$$

as $\|Du\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)} \to \infty$.

Step 3. Let $X = DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. According to Proposition 3.1, we get that the operator $\mathcal{F}$ is surjective. Consequently, there exists $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ such that $\mathcal{F}(Du) = QA(x, Du) = 0$. Furthermore, Lemma 2.10 deduces

$$\begin{aligned} \textstyle\int_\Omega \overline{A(x, Du)} D\varphi dx &= \textstyle\int_\Omega \overline{(QA(x, Du) + PA(x, Du))} D\varphi dx \\ &= \textstyle\int_\Omega \overline{\widetilde{D}PA(x, Du)} \varphi dx = 0, \end{aligned}$$

for any $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Therefore, u is a weak solution of the A-Dirac equations (1.1).

Step 4. If u_1, u_2 are solutions to the A-Dirac equations (1.1), then $\big(A(x, Du_i), D\varphi\big)_{Sc} = 0$ $(i = 1, 2)$ hold for any $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Set $\varphi = u_1 - u_2$, then the condition (A4) yields

$$\begin{aligned} 0 &= \big(A(x, Du_1) - A(x, Du_2), Du_1 - Du_2\big)_{Sc} \\ &= \int_\Omega \Big[\overline{\big(A(x, Du_1) - A(x, Du_2)\big)}(Du_1 - Du_2)\Big]_0 dx \geq 0. \end{aligned}$$

Thus, $[Du_1]_0 = [Du_2]_0$. The proof is now complete. □

As $p(x)$ is a constant function, Nolder [32, Theorem 3.1] proved a Caccioppoli-type estimate for weak solutions for (1.1) under the assumptions (A2) with $g(x) \equiv 0$ and (A3) with $h(x) \equiv 0$. Harjulehto, Hästo and Latvala [25, Lemma 5.3] showed a Caccioppoli-type estimate for weak solutions to the equation $-\mathrm{div}(p(x)|\nabla u|^{p(x)-2}\nabla u) = 0$ as u is a real function. Thus it is natural to study a Caccioppoli-type estimate for weak solutions to (1.1) in the variable exponent setting. Taking the similar approach presented in [25] we prove the following result:

Theorem 3.2. *Let* $p(x) \in \mathcal{P}(\Omega)$ *and* A *satisfies the hypotheses* (A2) *with* $g(x) \equiv 0$ *and* (A3) *with* $h(x) \equiv 0$. *If* u *be a weak solution to* (1.1) *and* $\eta \in C_0^\infty(\Omega)$ *with* $0 < \eta \leq 1$, *then*

$$\int_\Omega |Du|^{p(x)} \eta^{p_+} dx \leq \left(1 + \frac{2C_1 p_+}{C_2}\right)^{p_+} \int_\Omega |u|^{p(x)} |\nabla \eta|^{p(x)} dx.$$

Proof. Choose $\varphi = -u\eta^{p_+}$. Then $D\varphi = -p_+\eta^{p_+-1}(D\eta)u - \eta^{p_+}Du$. Hence, according to (3.1) and (A3) we obtain

$$\begin{aligned} 0 &= \int_\Omega \Big[\overline{A(x, Du)} D\varphi\Big]_0 dx = \int_\Omega \Big[\overline{A(x, Du)}(-p_+\eta^{p_+-1}(D\eta)u - \eta^{p_+}Du)\Big]_0 dx \\ &\leq -C_2 \int_\Omega |Du|^{p(x)} \eta^{p_+} dx + p_+ \int_\Omega |\overline{A(x, Du)}||u||D\eta||\eta|^{p_+-1} dx. \end{aligned}$$

Then from (A2) we have

$$\begin{aligned} C_2 \int_\Omega |Du|^{p(x)} \eta^{p_+} dx &\le p_+ \int_\Omega \left|\overline{A(x,Du)}\right| |u||D\eta||\eta|^{p_+-1} dx \\ &\le C_1 p_+ \int_\Omega \left|Du\right|^{p(x)-1} |u||D\eta||\eta|^{p_+-1} dx. \end{aligned}$$

By the Young's inequality, for any $\varepsilon \in (0,1]$ we have

$$ab \le \left(\frac{1}{\varepsilon}\right)^{p(x)-1} \frac{a^{p(x)}}{p(x)} + \varepsilon \frac{b^{p'(x)}}{p'(x)} \le \left(\frac{1}{\varepsilon}\right)^{p(x)-1} a^{p(x)} + \varepsilon b^{p'(x)}.$$

Take $a = |u||D\eta|\eta^{p_+ - \frac{p_+}{p(x)} - 1}$ and $b = \left|Du\right|^{p(x)-1} \eta^{\frac{p_+}{p(x)}}$, we get

$$\begin{aligned} &\int_\Omega \left|Du\right|^{p(x)-1} |u||D\eta|\eta^{p_+-1} dx \\ &\le \left(\frac{1}{\varepsilon}\right)^{p(x)-1} \int_\Omega |u|^{p(x)} |D\eta|^{p(x)} \eta^{p_+ - p(x)} dx + \varepsilon \int_\Omega |Du|^{p(x)} \eta^{p_+} dx. \end{aligned}$$

Let $\varepsilon = \min\left\{1, \frac{C_2}{2C_1 p_+}\right\}$. Then

$$\int_\Omega |Du|^{p(x)} \eta^{p_+} dx \le \frac{2C_1 p_+}{C_2} \left(1 + \frac{2C_1 p_+}{C_2}\right)^{p_+-1} \int_\Omega |u|^{p(x)} |\nabla \eta|^{p(x)} dx.$$

Thus the proof is complete. $\square$

Remark 3.1. From the proof of Theorem 3.2, we know that the conclusion in Theorem 3.2 still holds if u is just a weak solution to the scalar part of the equations (1.1).

4. The Nonhomogeneous A-Dirac Equations

In this section we are concerned with the existence of solutions for the following nonlinear A-Dirac equations with right hand side in Dirac equations form:

$$DA(x, Du) = Df. \tag{4.1}$$

The natural space in which to consider the weak solutions of (4.1) is the Sobolev spaces $W_0^{1,p(x)}(\Omega, C\ell_n)$ under the conditions (A1)–(A4). Thus, we suppose that f belongs to the dual space $L^{p'(x)}(\Omega, C\ell_n)$. Note that for every $f \in \mathbb{R}^{m\times n}$, the existence and uniqueness of the solution $u : \Omega \to \mathbb{R}^m$ were established by general principles of monotone operators in Iwaniec and Sbordone [27, Proposition 4.1]. For further details we refer to [5, 13, 23, 27] and the references therein.

Theorem 4.1. *Under the assumptions* (A1)–(A4), *for each* $f \in L^{p'(x)}(\Omega, C\ell_n)$, *there exists a weak solution* $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$ *to the* A*-Dirac equations* (4.2), *that is to say, there exists a Clifford-valued function* $u \in W_0^{1,p(x)}(\Omega, C\ell_n)$ *such that*

$$\int_\Omega \overline{A(x,Du)} Dv dx = \int_\Omega \overline{f} Dv dx$$

for any $v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. *Furthermore, the solution to the scalar part of* (4.2) *is unique up to a monogenic function.*

Proof. We notice that *Step* 1, *Step* 2, *Step* 4 of the proof are completely similar to those of the proof in Theorem 3.1. The only difference lies in *Step* 3. In fact, let $X = DW_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. In view of Proposition 3.1, the operator $\mathcal{F}$ is surjective. By means of Lemma 2.10, for each $f \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$, we have the following decomposition:

$$f = f_1 + f_2, \quad f_1 \in \ker\widetilde{D} \cap L^{p'(x)}(\Omega, \mathrm{C}\ell_n), f_2 \in DW_0^{1,p'(x)}(\Omega, \mathrm{C}\ell_n).$$

Then for $f_2 \in DW_0^{1,p'(x)}(\Omega, \mathrm{C}\ell_n)$, there exists $u \in W_0^{1,p(x)}(\Omega, Cl_n)$ such that $\mathcal{F}(Du) = QA(x, Du) = f_2$. Furthermore, Lemma 2.10 gives

$$\begin{aligned} \int_\Omega \overline{A(x,Du)} D\varphi dx &= \int_\Omega \overline{\big(QA(x,Du) + PA(x,Du)\big)} D\varphi dx \\ = \int_\Omega \overline{f - f_1} D\varphi dx &+ \int_\Omega \overline{PA(x,Du)} D\varphi dx \\ = \int_\Omega \overline{f} D\varphi dx &- \int_\Omega \overline{\widetilde{D} f_1} \varphi dx + \int_\Omega \overline{\widetilde{D} PA(x,Du)} \varphi dx = \int_\Omega \overline{f} D\varphi dx, \end{aligned}$$

for any $\varphi \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Therefore, u is a weak solution of the A-Dirac equations (4.1). □

In what follows we first consider the solvability of the following Dirac equation with homogeneous boundary data.

$$\begin{cases} Du = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

Let us show the following results.

Theorem 4.2. *Let* $f \in L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. *Then the equations* (4.2) *is solvable in* $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ *if and only if* $\mathrm{tr}\ Tf = 0$. *Furthermore, if a solution u exists, then it can be represented by* $u = Tf$.

Proof. On the one hand, if the equations (4.2) is solvable, then from the Borel-Pompeiu formula we have

$$u = Fu + TDu = TDu = Tf.$$

Thus we get $\mathrm{tr}\ Tf = 0$ due to $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$.
On the other hand, let $\mathrm{tr}\ Tf = 0$. In view of Lemma 2.10, we know that $f = Pf + Qf$. Then we have

$$\mathrm{tr}\ Tf = \mathrm{tr}\ TPf + \mathrm{tr}\ TQf = 0.$$

From (2.6), it follows that $\mathrm{tr}\ TQf = 0$. Thus, $\mathrm{tr}\ TPf = 0$. Note that $-\Delta TPf = \widetilde{D}DTPf = 0$ due to Lemma 2.6 and the definition of the operator P. According to the uniqueness of solutions of the problem (2.5), we obtain $TPf = 0$. And hence $DTPf = Pf = 0$. Further, $f = Qf$. Therefore, by the definition of the operator Q, there exists $u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ such that $Du = f$. The proof is now completed. □

Corollary 4.1. *Let $f \in W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. Then the equations* (4.2) *is solvable in $W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n)$ if and only if* $\operatorname{tr} \widetilde{T} f = 0$. *Furthermore, if a solution u exists, then it can be represented by $u = \widetilde{T} f$.*

Proof. By means of Proposition 12.3.4 in [6], it is easy to show that that the space $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ is dense in $W^{-1,p(x)}(\Omega, \mathrm{C}\ell_n)$. According to Theorem 2.1 in [31], we know that $C_0^\infty(\Omega, \mathrm{C}\ell_n)$ is dense in $L^{p(x)}(\Omega, \mathrm{C}\ell_n)$. Therefore, by the density arguments it is easy to see that the desired conclusion follows from Lemma 2.8 and Theorem 4.2. □

We would like to point out that the uniqueness of weak solutions for the equations (1.1) in Theorem 3.1 and (4.1) in Theorem 4.1 can be obtained if the condition (A4) is replaced by the following strong monotonicity:

$$(A4') \quad \left[\overline{(A(x,\xi_1) - A(x,\xi_2))}(\xi_1 - \xi_2)\right]_0 \geq C_3 |\xi_1 - \xi_2|^{p(x)},$$

where $C_3 > 0$ is a constant. Indeed, it is easy to deduce the assertion from Step 4 in the proof of Theorem 3.1 and Remark 2.1. Furthermore, we can consider the stability of weak solutions to the non-homogeneous A-Dirac equations. In other words, we have the following result.

Theorem 4.3. *Under the assumptions* (A1)–(A3) *and* (A4′), *for given $f, g \in L^{p'(x)}(\Omega, \mathrm{C}\ell_n)$, each of the two equations*

$$\begin{cases} DA(x, Du) = Df \\ u \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n), \end{cases} \tag{4.3}$$

$$\begin{cases} DA(x, Dv) = Dg \\ v \in W_0^{1,p(x)}(\Omega, \mathrm{C}\ell_n), \end{cases} \tag{4.4}$$

has a unique weak solution and

$$\min\left\{ \|u-v\|^{p_+-1}_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)}, \|u-v\|^{p_--1}_{W_0^{1,p(x)}(\Omega,\mathrm{C}\ell_n)} \right\} \leq C(n,p,\Omega)\|f-g\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}.$$

Proof. In view of Theorem 4.1 and the above arguments, we know that the equations (4.3) and (4.4) has a unique weak solution u and v respectively. Then we have

$$\big(A(x,Du), D(u-v)\big)_{Sc} = (f, D(u-v))_{Sc}, \ \big(A(x,Dv), D(u-v)\big)_{Sc} = (g, D(u-v))_{Sc}.$$

Therefore, we obtain

$$\big(A(x,Du) - A(x,Dv), Du - Dv\big)_{Sc} = (f - g, Du - Dv)_{Sc}.$$

From $(A4')$ and Lemma 2.2 it follows that

$$\begin{aligned}\int_\Omega |Du - Dv|^{p(x)} dx &\le \frac{1}{C_3}\int_\Omega \left[\overline{\big(A(x,Du) - A(x,Dv)\big)}(Du - Dv)\right]_0 dx\\ &\le \frac{1}{C_3}\int_\Omega \left[\overline{(f-g)}(Du - Dv)\right]_0 dx\\ &\le C(n,p,\Omega)\|f-g\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}\|Du - Dv\|_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}.\end{aligned}$$

Using Remark 2.1 and Lemma 2.1 we have

$$\begin{aligned}&\min\left\{\|Du - Dv\|^{p_+-1}_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}, \|Du - Dv\|^{p_--1}_{L^{p(x)}(\Omega,\mathrm{C}\ell_n)}\right\}\\ &\le C(n,p,\Omega)\|f-g\|_{L^{p'(x)}(\Omega,\mathrm{C}\ell_n)}.\end{aligned}$$

Then the desired inequality immediately follows from Remark 2.1. □

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