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Nonlinear problems with boundary blow-up: a Karamata regular variation theory approach

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Abstract. We study the uniqueness and expansion properties of the positive solution of the logistic equation $\Delta u + au = b(x)f(u)$ in a smooth bounded domain Ω, subject to the singular boundary condition $u = +\infty$ on $\partial\Omega$. The absorption term f is a positive function satisfying the Keller–Osserman condition and such that the mapping $f(u)/u$ is increasing on $(0, +\infty)$. We assume that b is non-negative, while the values of the real parameter a are related to an appropriate semilinear eigenvalue problem. Our analysis is based on the Karamata regular variation theory.

1. Introduction and main results

Let $\Omega \subset \mathbf{R}^N$ ($N \geqslant 3$) be a smooth bounded domain.
Consider the semilinear elliptic equation

$$\Delta u + au = b(x)f(u) \quad \text{in } \Omega, \tag{1.1}$$

where $f \in C^1[0, \infty)$, $a \in \mathbf{R}$ is a parameter and $b \in C^{0,\mu}(\overline{\Omega})$ satisfies $b \geqslant 0$, $b \not\equiv 0$ in Ω. Such equations are also known as the stationary version of the Fisher equation [22] and the Kolmogoroff–Petrovsky–Piscounoff equation [32] and they have been studied by Kazdan and Warner [30], Ouyang [44], del Pino [18] and Du and Huang [19].

Note that if $f(u) = u^{(N+2)/(N-2)}$, then (1.1) originates from the Yamabe problem, which is a basic problem in Riemannian geometry (see, e.g., [36]).

The existence of positive solutions of (1.1) subject to the Dirichlet boundary condition, $u = 0$ on $\partial\Omega$, has been intensively studied in the case $f(u) = u^p$, $p > 1$ (see [1,2,15,18,23] and [44]); this problem is a basic population model (see [26]) and it is also related to some prescribed curvature problems in Riemannian geometry (see [30] and [44]). Moreover, if $b > 0$ in $\overline{\Omega}$, then it is referred to as the logistic equation and it has a unique positive solution if and only if $a > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $(-\Delta)$ in $H_0^1(\Omega)$.

In the understanding of (1.1) an important role is played by the interior of the zero set of b:

$$\Omega_0 := \text{int}\{x \in \Omega\colon\ b(x) = 0\}.$$

We assume, throughout this paper, that Ω_0 is connected (possibly empty), $\overline{\Omega}_0 \subset \Omega$ and $b > 0$ in $\Omega \setminus \overline{\Omega}_0$. Note that we allow $b \geqslant 0$ on $\partial\Omega$. Let $\partial\Omega_0$ satisfy an exterior cone condition and $\lambda_{\infty,1}$ be the first Dirichlet eigenvalue of $(-\Delta)$ in $H_0^1(\Omega_0)$ (with $\lambda_{\infty,1} = \infty$ if $\Omega_0 = \emptyset$).

By a *large* (or *blow-up*) solution of (1.1), we mean any non-negative $C^2(\Omega)$-solution of (1.1) such that $u(x) \to \infty$ as $d(x) := \text{dist}(x, \partial\Omega) \to 0$.

Assuming that f satisfies

$$f \in C^1[0,\infty) \text{ is non-negative and } f(u)/u \text{ is increasing on } (0,\infty), \qquad (A_1)$$

then, necessarily $f(0) = 0$, and by the strong maximum principle, any non-negative classical solution of (1.1) is positive in Ω unless it is identically zero. Consequently, any large solution of (1.1) is positive. Moreover, it is well known (see, e.g., Remark 1.1 in [12]) that in this situation, the Keller–Osserman condition

$$\int_1^\infty \frac{\mathrm{d}t}{\sqrt{F(t)}} < \infty, \quad \text{where } F(t) = \int_0^t f(s)\,\mathrm{d}s \qquad (A_2)$$

is necessary for the existence of large solutions of (1.1).

When (A_1) and (A_2) hold, Theorem 1.1 in [12] shows that (1.1) possesses large solutions if and only if $a < \lambda_{\infty,1}$. The hypothesis (A_1) is inspired by [1], where it is developed an exhaustive study of positive solutions of (1.1), subject to $u = 0$ on $\partial\Omega$.

Our major goal is to advance innovative methods to study the uniqueness and asymptotic behavior of large solutions of (1.1). We develop the research line opened up in [13] to gain insight into the two-term asymptotic expansion of the large solution near $\partial\Omega$. Our approach relies essentially on the *regular variation theory* (see [8] and Section 2) not only in the statement but in the proof as well. This enables us to obtain significant information about the qualitative behavior of the large solution to (1.1) in a general framework that removes previous restrictions in the literature.

We point out that, despite a long history and intense research on the large solutions, the regular variation theory arising in probability theory has not been exploited before in this context.

Singular value problems having large solutions have been initially studied for the special case $f(u) = \mathrm{e}^u$ by Bieberbach [7] (if $N = 2$). Problems of this type arise in Riemannian geometry. More precisely, if a Riemannian metric of the form $|\mathrm{d}s|^2 = \mathrm{e}^{2u(x)}|\mathrm{d}x|^2$ has constant Gaussian curvature $-g^2$ then $\Delta u = g^2\,\mathrm{e}^{2u}$. This study was continued by Rademacher [45] (if $N = 3$) in connection with some concrete questions arising in the theory of Riemann surfaces, automorphic functions and in the theory of the electric potential in a glowing hollow metal body.

The question of large solutions was later considered in N-dimensional domains and for other classes of nonlinearities (see [3–6,11–14,17,19,25,31,33–35,38–41,43,46]).

In higher dimensions the notion of Gaussian curvature has to be replaced by the scalar curvature. It turns out that if a metric of the form $|\mathrm{d}s|^2 = u(x)^{4/(N-2)}|\mathrm{d}x|^2$ has constant scalar curvature $-g^2$, then u satisfies (1.1) for $f(u) = u^{(N+2)/(N-2)}$, $a = 0$ and $b(x) = [(N-2)g^2]/[4(N-1)]$. In a celebrated paper, Loewner and Nirenberg [38] described the precise asymptotic behavior at the boundary of large solutions to this equation and used this result in order to establish the uniqueness of the solution. Their main result is derived under the assumption that $\partial\Omega$ consists of the disjoint union of finitely compact C^∞ manifolds, each having codimension less than $N/2+1$. More precisely, the uniqueness of a large solution is a consequence of the fact that every large solution u satisfies

$$u(x)=\mathcal{E}\big(d(x)\big)+\mathrm{o}\big(\mathcal{E}\big(d(x)\big)\big)\quad \text{as } d(x)\to 0, \tag{1.2}$$

where $\mathcal{E}$ is defined by

$$\int_{\mathcal{E}(t)}^{\infty}\frac{\mathrm{d}s}{\sqrt{2F(s)}}=\left(\frac{(N-2)g^2}{4(N-1)}\right)^{1/2}t,\quad \text{for all } t>0. \tag{1.3}$$

Kondrat'ev and Nikishkin [33] established the uniqueness of a large solution for the case $a=0$, $b=1$ and $f(u)=u^p$ ($p\geqslant 3$), when $\partial\Omega$ is a C^2-manifold and Δ is replaced by a more general second-order elliptic operator.

Dynkin [20] showed that there exist certain relations between hitting probabilities for some Markov processes called superdiffusions and maximal solutions of (1.1) with $a=0$, $b=1$ and $f(u)=u^p$ ($1<p\leqslant 2$). By means of a probabilistic representation, a uniqueness result in domains with non-smooth boundary was established by le Gall [37] when $p=2$. We point out that the case $p=2$ arises in the study of the subsonic motion of a gas. In this connection the question of uniqueness is of special interest.

Recently, [25] gives the uniqueness and exact two-term asymptotic expansion of the large solution of (1.1) in the special case $f(u)=u^p$ ($p>1$), $b>0$ in Ω and $b\equiv 0$ on $\partial\Omega$ such that

$$b(x)=C_0\big[d(x)\big]^\gamma+\mathrm{o}\big(\big[d(x)\big]^\gamma\big)\quad \text{as } d(x)\to 0, \text{for some constants } C_0,\gamma>0. \tag{1.4}$$

It was shown there that the degenerate case $b\equiv 0$ on $\partial\Omega$ is a *natural* restriction for b inherited from the logistic equation.

To present our main results, we briefly recall some notions from Karamata's theory (see [8] or [48]); more details are provided in Section 2.

A positive measurable function R defined on $[A,\infty)$, for some $A>0$, is called *regularly varying with index* $q\in\mathbf{R}$, written $R\in RV_q$, provided that

$$\lim_{u\to\infty}\frac{R(\lambda u)}{R(u)}=\lambda^q,\quad \text{for all } \lambda>0.$$

When the index q is zero, we say that the function is *slowly varying*.

Clearly, if $R\in RV_q$, then $L(u):=R(u)/u^q$ is a slowly varying function.

Let $\mathcal{K}$ denote the set of all positive, non-decreasing $k\in C^1(0,\nu)$ that satisfy

$$\lim_{t\searrow 0}\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right):=\ell_0\quad \text{and}\quad \lim_{t\searrow 0}\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)':=\ell_1.$$

Notice that $\ell_0=0$ and $\ell_1\in[0,1]$, for every $k\in\mathcal{K}$. Thus, $\mathcal{K}=\mathcal{K}_{(01]}\cup\mathcal{K}_0$, where

$$\mathcal{K}_{(01]}=\{k\in\mathcal{K}:\ 0<\ell_1\leqslant 1\}\quad \text{and}\quad \mathcal{K}_0=\{k\in\mathcal{K}:\ \ell_1=0\}.$$

The exact characterization of $\mathcal{K}_{(01]}$ and $\mathcal{K}_0$ will be provided in Section 3.

If H is a non-decreasing function on **R**, then we define the (left continuous) inverse of H by

$$H^{\leftarrow}(y) = \inf\{s\colon\ H(s) \geqslant y\}.$$

Our first result establishes the uniqueness of the large solution of (1.1).

Theorem 1.1. *Let* (A_1) *hold and* $f \in RV_{\rho+1}$ *with* $\rho > 0$. *Suppose there exists* $k \in \mathcal{K}$ *such that*

$$b(x) = k^2(d) + \mathrm{o}\big(k^2(d)\big) \quad \text{as } d(x) \to 0. \tag{1.5}$$

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, (1.1) *admits a unique large solution* u_a. *Moreover, the asymptotic behavior is given by*

$$u_a(x) = \big[2(2+\ell_1\rho)/\rho^2\big]^{1/\rho}\varphi(d) + \mathrm{o}\big(\varphi(d)\big) \quad \text{as } d(x) \to 0, \tag{1.6}$$

where φ *is defined by*

$$\frac{f(\varphi(t))}{\varphi(t)} = \frac{1}{\big(\int_0^t k(s)\,\mathrm{d}s\big)^2}, \quad \text{for } t > 0 \text{ small}. \tag{1.7}$$

Under the assumptions of Theorem 1.1, let $r(t)$ satisfy $\lim_{t\searrow 0}(\int_0^t k(s)\,\mathrm{d}s)^2 r(t) = 1$ and $\hat{f}(u)$ be chosen such that $\lim_{u\to\infty} \hat{f}(u)/f(u) = 1$ and $j(u) = \hat{f}(u)/u$ is non-decreasing for $u > 0$ large. Then, $\lim_{t\searrow 0} \varphi(t)/\widehat{\varphi}(t) = 1$, where φ is defined by (1.7) and $\widehat{\varphi}(t) = j^{\leftarrow}(r(t))$ for $t > 0$ small.

The behavior of $\varphi(t)$ for small $t > 0$ will be described in Section 3. In particular, if $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then $\varphi(1/u) \in RV_{2/(\rho\ell_1)}$. In contrast, if $k \in \mathcal{K}$ with $\ell_1 = 0$, then $\varphi(1/u) \notin RV_q$, for all $q \in \mathbf{R}$ (see Remark 3.3).

Remark 1.1. Theorem 1.1 improves the main result in [13], where assuming that $f' \in RV_\rho$ (which yields $f \in RV_{\rho+1}$), we prove

$$u_a(x) = \xi_0 h(d) + \mathrm{o}\big(h(d)\big) \quad \text{as } d(x) \to 0, \tag{1.8}$$

where $\xi_0 = (\frac{2+\ell_1\rho}{2+\rho})^{1/\rho}$ and h is given by

$$\int_{h(t)}^{\infty} \frac{\mathrm{d}s}{\sqrt{2F(s)}} = \int_0^t k(s)\,\mathrm{d}s, \quad \text{for } t > 0 \text{ small}. \tag{1.9}$$

Remark 1.2. Theorem 1.1 recovers the uniqueness results of [38] and [25]. Note that for $k(t) = [(N-2)g^2/4(N-1)]^{1/2}$ in (1.5) and $f(u) = u^{(N+2)/(N-2)}$, (1.6) reduces to relation (1.2), prescribed by Loewner and Nirenberg [38] for their problem. Moreover, if $f(u) = u^p$ (with $p = \rho + 1 > 1$) and $k(t) = \sqrt{C_0}\, t^{\gamma/2}$ ($C_0, \gamma > 0$), then we regain the uniqueness result of [25].

The next objective is to find the two-term blow-up rate of u_a when (1.5) is replaced by

$$b(x) = k^2(d)\big(1 + \tilde{c}d^{\theta} + \mathrm{o}(d^{\theta})\big) \quad \text{as } d(x) \to 0, \tag{1.10}$$

where $\theta > 0$, $\tilde{c} \in \mathbf{R}$ are constants. To simplify the exposition, we assume that $f' \in RV_{\rho}$ ($\rho > 0$), which is equivalent to $f(u)$ being of the form

$$f(u) = Cu^{\rho+1} \exp\left\{\int_B^u \frac{\phi(t)}{t}\,\mathrm{d}t\right\}, \quad \forall u \geqslant B, \tag{1.11}$$

for some constants $B, C > 0$, where $\phi \in C[B, \infty)$ satisfies $\lim_{u\to\infty} \phi(u) = 0$. In this case, $f(u)/u$ is increasing on $[B, \infty)$ provided that B is large enough.

We prove that the two-term asymptotic expansion of u_a near $\partial\Omega$ depends on the chosen subclass for $k \in \mathcal{K}$ and the additional hypotheses on f (by means of ϕ in (1.11)).

Let $-\rho - 2 < \eta \leqslant 0$ and $\tau, \zeta > 0$. We define

$$\mathcal{F}_{\rho\eta} = \big\{f' \in RV_{\rho}\ (\rho > 0)\text{: either } \phi \in RV_{\eta} \text{ or } -\phi \in RV_{\eta}\big\},$$

$$\mathcal{F}_{\rho 0,\tau} = \Big\{f \in \mathcal{F}_{\rho 0}\text{: } \lim_{u\to\infty} (\ln u)^{\tau} \phi(u) = \ell^{\star} \in \mathbf{R}\Big\},$$

$$\mathcal{K}_{(01],\tau} = \left\{k \in \mathcal{K}_{(01]}\text{: } \lim_{t\searrow 0} (-\ln t)^{\tau} \left[\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)' - \ell_1\right] := L_{\sharp} \in \mathbf{R}\right\},$$

$$\mathcal{K}_{0,\zeta} = \left\{k \in \mathcal{K}_0\text{: } \lim_{t\searrow 0} \frac{1}{t^{\zeta}} \left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)' := L_{\star} \in \mathbf{R}\right\}.$$

Further in the paper, η, τ and ζ are understood in the above range.

For the sake of comparison, we state here the following result.

Theorem 1.2. *Suppose* (A_1), (1.10) *with* $k \in \mathcal{K}_{0,\zeta}$, *and one of the following growth conditions at infinity*:

(i) $f(u) = Cu^{\rho+1}$ *in a neighborhood of infinity* (*i.e.*, $\phi \equiv 0$ *in* (1.11));
(ii) $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta \neq 0$;
(iii) $f \in \mathcal{F}_{\rho 0,\tau_1}$ *with* $\tau_1 = \varpi/\zeta$, *where* $\varpi = \min\{\theta, \zeta\}$.

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *the two-term blow-up rate of* u_a *is*

$$u_a(x) = \xi_0 h(d)\big(1 + \chi d^{\varpi} + \mathrm{o}(d^{\varpi})\big) \quad \text{as } d(x) \searrow 0, \tag{1.12}$$

where h *is given by* (1.9), $\xi_0 = [2/(2+\rho)]^{1/\rho}$ *and*

$$\chi = \begin{cases} \dfrac{L_{\star}}{2}\mathrm{Heaviside}(\theta - \zeta) - \dfrac{\tilde{c}}{\rho}\mathrm{Heaviside}(\zeta - \theta) := \chi_1 & \text{if (i) or (ii) holds,} \\ \chi_1 - \dfrac{\ell^{\star}}{\rho}\left[\dfrac{\rho\zeta L_{\star}}{2(1+\zeta)}\right]^{\tau_1}\left(\dfrac{1}{\rho+2} + \ln\xi_0\right) & \text{if } f \text{ obeys (iii).} \end{cases}$$

Theorem 1.2 is a consequence of [14, Theorem 1] and Proposition 3.4.

Theorem 1.3. *Suppose* (A_1), (1.10) *with* $k \in \mathcal{K}_{(01],\tau}$, *and one of the following conditions*:

(i) $f \in \mathcal{F}_{\rho\eta}$ *with* $\eta L_\sharp \neq 0$;
(ii) $f \in \mathcal{F}_{\rho 0,\tau}$ *with* $[\ell^\star(\ell_1 - 1)]^2 + L_\sharp^2 \neq 0$.

Then, for any $a \in (-\infty, \lambda_{\infty,1})$, *the two-term blow-up rate of* u_a *is*

$$u_a(x) = \xi_0 h(d)\big[1 + \widetilde{\chi}(-\ln d)^{-\tau} + \mathrm{o}\big((-\ln d)^{-\tau}\big)\big] \quad \text{as } d(x) \searrow 0, \tag{1.13}$$

where h *is given by* (1.9), $\xi_0 = [(2 + \ell_1\rho)/(2 + \rho)]^{1/\rho}$ *and*

$$\widetilde{\chi} = \begin{cases} \dfrac{L_\sharp}{2 + \rho\ell_1} := \chi_2 & \text{if (i) holds,} \\ \chi_2 - \dfrac{\ell^\star}{\rho}\left(\dfrac{\rho\ell_1}{2}\right)^\tau \left[\dfrac{2(1 - \ell_1)}{(\rho + 2)(\rho\ell_1 + 2)} + \ln \xi_0\right] & \text{if } f \text{ obeys (ii).} \end{cases} \tag{1.14}$$

Remark 1.3. Note that Theorems 1.2 and 1.3 distinguish from Theorem 1 in [25], which treats the particular case $f(u) = u^p$ $(p > 1)$, $\Omega_0 = \emptyset$, $k(t) = \sqrt{C_0 t^\gamma}$ $(C_0, \gamma > 0)$ and $\theta = 1$ in (1.10). The second term in the asymptotic expansion of u_a near $\partial\Omega$ involves in [25] both the distance function $d(x)$ and the mean curvature of $\partial\Omega$.

Theorem 1.2 admits the case $f(u) = u^p$ assuming that $k \in \mathcal{K}_{0,\zeta}$, while the alternative (ii) of Theorem 1.3 includes the case $k(t) = \sqrt{C_0 t^\gamma}$ (when $L_\sharp = 0$) provided that $f \in \mathcal{F}_{\rho 0,\tau}$ with $\ell^\star \neq 0$. Relations (1.12) and (1.13) show how dramatically changes the two-term asymptotic expansion of u_a from the result in [25]. Our approach is completely different from that in [3,4,25,35], as we use essentially Karamata's theory.

We point out that the asymptotic general results stated in the above theorems do not concern the difference or the quotient of $u(x)$ and $\psi(d(x))$, as established in [4,7,35,45] for $a = 0$ and $b = 1$, where ψ is a large solution of

$$\psi''(r) = f\big(\psi(r)\big) \quad \text{on } (0, \infty).$$

For instance, Bieberbach [7] and Rademacher [45] proved that $|u(x) - \psi(d(x))|$ is bounded in a neighborhood of the boundary. Their result was improved by Bandle and Essén [3] who showed that $\lim_{d(x)\to 0}(u(x) - \psi(d(x))) = 0$.

The rest of the paper is organized as follows. In Section 2.1 we collect the notions and properties of regularly varying functions that are invoked in our proofs. In Section 2.2 we prove some auxiliary results including Lemmas 1 and 2 in [14], which have only been stated there. In Section 3 we characterize the class $\mathcal{K}$ as well as its subclasses $\mathcal{K}_{0,\zeta}$ and $\mathcal{K}_{(01],\tau}$ that appear in Theorems 1.2 and 1.3. Sections 4 and 5 are dedicated to the proof of Theorems 1.1 and 1.3.

2. Preliminaries

2.1. Properties of regularly varying function

The theory of regular variation was instituted in 1930 by Karamata [28,29] and subsequently developed by himself and many others. Although Karamata originally introduced his theory in order to use it in Tauberian theorems, regularly varying functions have been later applied in several branches of Analysis: Abelian theorems (asymptotic of series and integrals – Fourier ones in particular), analytic (entire) functions, analytic number theory, etc. The great potential of regular variation for probability theory and its applications was realized by Feller [21] and also stimulated by de Haan [16]. The first monograph on regularly varying functions was written by Seneta [48], while the theory and various applications of the subject are presented in the comprehensive treatise of Bingham, Goldie and Teugels [8].

We give here a brief account of the definitions and properties of regularly varying functions involved in our paper (see [8] or [48] for details).

Definition 2.1. A positive measurable function Z defined on $[A,\infty)$, for some $A>0$, is called *regularly varying* (*at infinity*) *with index* $q\in\mathbf{R}$, written $Z\in RV_q$, provided that

$$\lim_{u\to\infty}\frac{Z(\xi u)}{Z(u)}=\xi^q,\quad \text{for all } \xi>0.$$

When the index q is zero, we say that the function is *slowly varying*.

Remark 2.1. Let $Z:[A,\infty)\to(0,\infty)$ be a measurable function. Then

(1) Z is regularly varying if and only if $\lim_{u\to\infty}Z(\xi u)/Z(u)$ is finite and positive for each ξ in a set $S\subset(0,\infty)$ of positive measure (see [48, Lemma 1.6 and Theorem 1.3]).
(2) The transformation $Z(u)=u^qL(u)$ reduces regular variation to slow variation. Indeed, $\lim_{u\to\infty}Z(\xi u)/Z(u)=u^q$ if and only if $\lim_{u\to\infty}L(\xi u)/L(u)=1$, for every $\xi>0$.

Example 2.1. Any measurable function on $[A,\infty)$ which has a positive limit at infinity is slowly varying. The logarithm $\log u$, its iterates $\log\log u$ $(=\log_2 u)$, $\log_m u$ $(=\log\log_{m-1}u)$ and powers of $\log_m u$ are non-trivial examples of slowly varying functions. Non-logarithmic examples are given by $\exp\{(\log u)^{\alpha_1}\}$, where $\alpha_1\in(0,1)$ and $\exp\{(\log u)/\log\log u\}$.

In what follows L denotes a slowly varying function defined on $[A,\infty)$. For details on Propositions 2.1–2.5, we refer to [8].

Proposition 2.1 (Uniform Convergence Theorem). *The convergence* $\frac{L(\xi u)}{L(u)}\to 1$ *as* $u\to\infty$ *holds uniformly on each compact* ξ*-set in* $(0,\infty)$.

Proposition 2.2 (Representation Theorem). *The function* $L(u)$ *is slowly varying if and only if it can be written in the form*

$$L(u)=M(u)\exp\left\{\int_B^u\frac{y(t)}{t}\,\mathrm{d}t\right\}\quad(u\geqslant B)\tag{2.1}$$

for some $B>A$, *where* $y\in C[B,\infty)$ *satisfies* $\lim_{u\to\infty}y(u)=0$ *and* $M(u)$ *is measurable on* $[B,\infty)$ *such that* $\lim_{u\to\infty}M(u):=\overline{M}\in(0,\infty)$.

The Karamata representation (2.1) is non-unique because we can adjust one of $M(u)$, $y(u)$ and modify properly the other one. Thus, the function y may be assumed arbitrarily smooth, but the smoothness properties of $M(u)$ can ultimately reach those of $L(u)$. If $M(u)$ is replaced by its limit at infinity $\overline{M} > 0$, we obtain a slowly varying function $L_0 \in C^1[B, \infty)$ of the form

$$L_0(u) = \overline{M} \exp\left\{\int_B^u \frac{y(t)}{t}\,\mathrm{d}t\right\} \quad (u \geqslant B),$$

where $y \in C[B, \infty)$ vanishes at infinity. Such a function $L_0(u)$ is called a *normalized* slowly varying function.

As an important subclass of RV_q, we distinguish NRV_q defined as

$$NRV_q = \left\{Z \in RV_q\colon\ \frac{Z(u)}{u^q} \text{ is a } \textit{normalized} \text{ slowly varying function}\right\}. \tag{2.2}$$

Notice that $L(u)$ given by (2.1) is asymptotic equivalent to $L_0(u)$, which has much enhanced properties. For instance, we see that $y(u) = \frac{uL_0'(u)}{L_0(u)}$, for all $u \geqslant B$. Conversely, any function $L_0 \in C^1[B, \infty)$ which is positive and satisfies

$$\lim_{u\to\infty} \frac{uL_0'(u)}{L_0(u)} = 0 \tag{2.3}$$

is a normalized slowly varying. More generally, if the right-hand side of (2.3) is $q \in \mathbf{R}$, then $L_0 \in NRV_q$.

Proposition 2.3 (Elementary properties of slowly varying functions). *If L is slowly varying, then*

(1) *For any $\alpha > 0$, $u^\alpha L(u) \to \infty$, $u^{-\alpha} L(u) \to 0$ as $u \to \infty$;*
(2) *$(L(u))^\alpha$ varies slowly for every $\alpha \in \mathbf{R}$;*
(3) *If L_1 varies slowly, so do $L(u)L_1(u)$ and $L(u) + L_1(u)$.*

From Proposition 2.3(i) and Remark 2.1(ii), $\lim_{u\to\infty} Z(u) = \infty$ (resp., 0) for any function $Z \in RV_q$ with $q > 0$ (resp., $q < 0$).

Remark 2.2. Note that the behavior at infinity for a slowly varying function cannot be predicted. For instance,

$$L(u) = \exp\bigl\{(\log u)^{1/3} \cos\bigl((\log u)^{1/3}\bigr)\bigr\}$$

exhibits infinite oscillation in the sense that

$$\liminf_{u\to\infty} L(u) = 0 \quad \text{and} \quad \limsup_{u\to\infty} L(u) = \infty.$$

Proposition 2.4 (Karamata's Theorem; direct half). *Let $Z \in RV_q$ be locally bounded in $[A, \infty)$. Then*

(1) *for any $j \geqslant -(q+1)$,*

$$\lim_{u\to\infty} \frac{u^{j+1} Z(u)}{\int_A^u x^j Z(x)\,\mathrm{d}x} = j + q + 1; \tag{2.4}$$

(2) *for any* $j < -(q+1)$ (*and for* $j = -(q+1)$ *if* $\int^\infty x^{-(q+1)} Z(x)\,\mathrm{d}x < \infty$)

$$\lim_{u\to\infty} \frac{u^{j+1} Z(u)}{\int_u^\infty x^j Z(x)\,\mathrm{d}x} = -(j+q+1). \tag{2.5}$$

Proposition 2.5 (Karamata's Theorem; converse half). *Let* Z *be positive and locally integrable in* $[A, \infty)$.

(1) *If* (2.4) *holds for some* $j > -(q+1)$, *then* $Z \in RV_q$.
(2) *If* (2.5) *is satisfied for some* $j < -(q+1)$, *then* $Z \in RV_q$.

For a non-decreasing function H on **R**, we define the (left continuous) inverse of H by

$$H^{\leftarrow}(y) = \inf\{s\colon\ H(s) \geqslant y\}.$$

Proposition 2.6 (see Proposition 0.8 in [47]). *We have*

(1) *If* $Z \in RV_q$, *then* $\lim_{u\to\infty} \log Z(u)/\log u = q$.
(2) *If* $Z_1 \in RV_{q_1}$ *and* $Z_2 \in RV_{q_2}$ *with* $\lim_{u\to\infty} Z_2(u) = \infty$, *then*

$$Z_1 \circ Z_2 \in RV_{q_1 q_2}.$$

(3) *Suppose* Z *is non-decreasing,* $Z(\infty) = \infty$, *and* $Z \in RV_q$, $0 < q < \infty$. *Then*

$$Z^{\leftarrow} \in RV_{1/q}.$$

(4) *Suppose* Z_1, Z_2 *are non-decreasing and q-varying,* $0 < q < \infty$. *Then for* $c \in (0, \infty)$

$$\lim_{u\to\infty} \frac{Z_1(u)}{Z_2(u)} = c \quad \textit{if and only if} \quad \lim_{u\to\infty} \frac{Z_1^{\leftarrow}(u)}{Z_2^{\leftarrow}(u)} = c^{-1/q}.$$

2.2. *Auxiliary results*

Based on regular variation theory, we prove here two results that have only been stated in [14].

Remark 2.3. If $f \in RV_{\rho+1}$ ($\rho > 0$) is continuous, then

$$\Xi(u) := \frac{\sqrt{F(u)}}{f(u)\int_u^\infty [F(s)]^{-1/2}\,\mathrm{d}s} \to \frac{\rho}{2(\rho+2)} \quad \text{as } u \to \infty, \tag{2.6}$$

where F stands for an antiderivative of f. Indeed, by Proposition 2.4, we have

$$\lim_{u\to\infty} \frac{F(u)}{uf(u)} = \frac{1}{\rho+2} \quad \text{and} \quad \lim_{u\to\infty} \frac{u[F(u)]^{-1/2}}{\int_u^\infty [F(s)]^{-1/2}\,\mathrm{d}s} = \frac{\rho}{2}. \tag{2.7}$$

Lemma 2.1 (Properties of h). *If* $f \in RV_{\rho+1}$ ($\rho > 0$) *is continuous and* $k \in \mathcal{K}$, *then* h *defined by* (1.9) *is a* C^2*-function satisfying the following*:

(i) $\lim_{t\searrow 0} \dfrac{h''(t)}{k^2(t)f(h(t)\xi)} = \dfrac{2+\rho\ell_1}{\xi^{\rho+1}(2+\rho)}$, *for each* $\xi > 0$;

(ii) $\lim_{t\searrow 0} \dfrac{h(t)h''(t)}{[h'(t)]^2} = \dfrac{2+\rho\ell_1}{2}$ *and* $\lim_{t\searrow 0} \dfrac{\ln k(t)}{\ln h(t)} = \dfrac{\rho(\ell_1-1)}{2}$;

(iii) $\lim_{t\searrow 0} \dfrac{h'(t)}{th''(t)} = -\dfrac{\rho\ell_1}{2+\rho\ell_1}$ *and* $\lim_{t\searrow 0} \dfrac{h(t)}{t^2h''(t)} = \dfrac{\rho^2\ell_1^2}{2(2+\rho\ell_1)}$;

(iv) $\lim_{t\searrow 0} \dfrac{h(t)}{th'(t)} = \lim_{t\searrow 0} \dfrac{\ln t}{\ln h(t)} = -\dfrac{\rho\ell_1}{2}$;

(v) $\lim_{t\searrow 0} t^j h(t) = \infty$, *for all* $j > 0$, *provided that* $k \in \mathcal{K}_0$. *If, in addition,* $k \in \mathcal{K}_{0,\zeta}$ *then*

$$\lim_{t\searrow 0} \frac{1}{-\zeta t^\zeta \ln h(t)} = \lim_{t\searrow 0} \frac{h'(t)}{t^{\zeta+1}h''(t)} = \frac{-\rho L_\star}{2(\zeta+1)}.$$

Proof. By (1.9), the function $h \in C^2(0,\nu)$, for some $\nu > 0$, and $\lim_{t\searrow 0} h(t) = \infty$.
For any $t \in (0,\nu)$, we have $h'(t) = -k(t)\sqrt{2F(h(t))}$ and

$$h''(t) = k^2(t)f\big(h(t)\big)\left\{1 + 2\Xi\big(h(t)\big)\left[\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)' - 1\right]\right\}. \tag{2.8}$$

Using Remark 2.3 and $f \in RV_{\rho+1}$, we reach (i).
(ii) By (i) and (2.7), we get

$$\lim_{t\searrow 0} \frac{h(t)h''(t)}{[h'(t)]^2} = \lim_{t\searrow 0} \frac{h''(t)}{k^2(t)f(h(t))}\frac{h(t)f(h(t))}{2F(h(t))} = \frac{2+\rho\ell_1}{2}, \tag{2.9}$$

respectively

$$\lim_{t\searrow 0} \frac{k'(t)}{k(t)}\frac{h(t)}{h'(t)} = \lim_{t\searrow 0} \frac{h(t)f(h(t))}{F(h(t))}\frac{-k'(t)\big(\int_0^t k(s)\,\mathrm{d}s\big)}{k^2(t)}\Xi\big(h(t)\big) = \frac{\rho(\ell_1-1)}{2}. \tag{2.10}$$

(iii) Using (i) and Remark 2.3, we find

$$\lim_{t\searrow 0} \frac{h'(t)}{th''(t)} = \frac{-2(2+\rho)}{2+\rho\ell_1}\lim_{t\searrow 0} \frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)}\Xi\big(h(t)\big) = \frac{-\rho\ell_1}{2+\rho\ell_1},$$

which, together with (2.9), implies that

$$\lim_{t\searrow 0} \frac{h(t)}{t^2h''(t)} = \lim_{t\searrow 0} \frac{h(t)h''(t)}{[h'(t)]^2}\left[\frac{h'(t)}{th''(t)}\right]^2 = \frac{\rho^2\ell_1^2}{2(2+\rho\ell_1)}.$$

(iv) If $\ell_1 \neq 0$, then by (iii), we have

$$\lim_{t\searrow 0} \frac{h(t)}{th'(t)} = \lim_{t\searrow 0} \frac{h(t)}{t^2h''(t)}\frac{th''(t)}{h'(t)} = \frac{-\rho\ell_1}{2}.$$

If $\ell_1 = 0$, then we derive

$$\lim_{t\searrow 0} \frac{k(t)}{tk'(t)} = \lim_{t\searrow 0} \frac{k^2(t)}{k'(t)\left(\int_0^t k(s)\,\mathrm{d}s\right)} \frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)} = 0. \tag{2.11}$$

This and (2.10) yield $\lim_{t\searrow 0} \frac{h(t)}{th'(t)} = 0$, which concludes (iv).

(v) If $k \in \mathcal{K}_0$, then using (iv), we obtain $\lim_{t\searrow 0} \ln[t^j h(t)] = \infty$, for all $j > 0$.

Suppose $k \in \mathcal{K}_{0,\zeta}$, for some $\zeta > 0$. Then, $\lim_{t\searrow 0} \frac{\int_0^t k(s)\,\mathrm{d}s}{t^{\zeta+1}k(t)} = \frac{L_\star}{\zeta+1}$ and

$$\frac{L_\star}{\zeta+1} = \lim_{t\searrow 0} \frac{\int_0^t k(s)\,\mathrm{d}s}{t^{\zeta+1}k(t)} \frac{k^2(t)}{k'(t)\left(\int_0^t k(s)\,\mathrm{d}s\right)} = \lim_{t\searrow 0} \frac{k(t)}{t^{\zeta+1}k'(t)} = \frac{-1}{\zeta} \lim_{t\searrow 0} \frac{1}{t^\zeta \ln k(t)}. \tag{2.12}$$

By (2.9), (2.10) and (2.12), we deduce

$$\lim_{t\searrow 0} \frac{h'(t)}{t^{\zeta+1}h''(t)} = \lim_{t\searrow 0} \frac{h(t)}{h'(t)t^{\zeta+1}} = \lim_{t\searrow 0} \frac{k'(t)h(t)}{k(t)h'(t)} \frac{k(t)}{t^{\zeta+1}k'(t)} = \frac{-\rho L_\star}{2(\zeta+1)}.$$

This completes the proof of the lemma. □

Let $\tau > 0$ be arbitrary and f be as in Remark 2.3. For $u > 0$ sufficiently large, we define

$$T_{1,\tau}(u) = \left[\frac{\rho}{2(\rho+2)} - \varXi(u)\right](\ln u)^\tau \quad \text{and} \quad T_{2,\tau}(u) = \left[\frac{f(\xi_0 u)}{\xi_0 f(u)} - \xi_0^\rho\right](\ln u)^\tau. \tag{2.13}$$

Remark 2.4. When $f(u) = Cu^{\rho+1}$, we have $T_{1,\tau}(u) = T_{2,\tau}(u) = 0$.

Lemma 2.2. *Assume that $f \in \mathcal{F}_{\rho\eta}$ (where $-\rho - 2 < \eta \leqslant 0$). The following hold*:

(i) *If $f \in \mathcal{F}_{\rho 0,\tau}$, then*

$$\lim_{u\to\infty} T_{1,\tau}(u) = \frac{-\ell^\star}{(\rho+2)^2} \quad \textit{and} \quad \lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \ell^\star \ln \xi_0.$$

(ii) *If $f \in \mathcal{F}_{\rho\eta}$ with $\eta \neq 0$, then*

$$\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} T_{2,\tau}(u) = 0.$$

Proof. Using the second limit in (2.7), we obtain

$$\lim_{u\to\infty} T_{1,\tau}(u) = \frac{\rho}{2} \lim_{u\to\infty} \frac{\frac{\rho}{2(\rho+2)} \int_u^\infty [F(s)]^{-1/2}\,\mathrm{d}s - \frac{\sqrt{F(u)}}{f(u)}}{u[F(u)]^{-1/2}(\ln u)^{-\tau}}.$$

By L'Hospital's rule, we arrive at

$$\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} \left[\frac{\rho+1}{\rho+2} - \frac{F(u)f'(u)}{f^2(u)}\right](\ln u)^\tau := \lim_{u\to\infty} Q_{1,\tau}(u).$$

A simple calculation shows that, for $u > 0$ large,

$$Q_{1,\tau}(u) = \frac{(\ln u)^\tau}{\rho+2}\left[\rho+1-\frac{uf'(u)}{f(u)}\right] + \frac{uf'(u)}{f(u)}\left[\frac{1}{\rho+2} - \frac{F(u)}{uf(u)}\right](\ln u)^\tau$$
$$=: \frac{1}{\rho+2}Q_{2,\tau}(u) + \frac{uf'(u)}{f(u)}Q_{3,\tau}(u).$$

Since (1.11) holds with $\phi \in RV_\eta$ or $-\phi \in RV_\eta$, we can assume $B > 0$ such that $\phi \neq 0$ on $[B,\infty)$. For any $u > B$, we have $Q_{2,\tau}(u) = -\phi(u)(\ln u)^\tau$ and

$$Q_{3,\tau}(u) = \widetilde{C}\,\frac{(\ln u)^\tau}{uf(u)} + \frac{\int_B^u f(s)\phi(s)\,\mathrm{d}s}{(\rho+2)uf(u)\phi(u)}\phi(u)(\ln u)^\tau,$$

where $\widetilde{C} \in \mathbf{R}$ is a constant. Since either $f\phi \in RV_{\rho+\eta+1}$ or $-f\phi \in RV_{\rho+\eta+1}$, by Proposition 2.4,

$$\lim_{u\to\infty} \frac{uf(u)\phi(u)}{\int_B^u f(x)\phi(x)\,\mathrm{d}x} = \rho+\eta+2.$$

If (i) holds, then $\lim_{u\to\infty} Q_{2,\tau}(u) = -\ell^\star$ and $\lim_{u\to\infty} Q_{3,\tau}(u) = \ell^\star(\rho+2)^{-2}$. Thus,

$$\lim_{u\to\infty} T_{1,\tau}(u) = \lim_{u\to\infty} Q_{1,\tau}(u) = -\ell^\star/(\rho+2)^2.$$

If (ii) holds, then by Proposition 2.3, we have $\lim_{u\to\infty}(\ln u)^\tau\phi(u) = 0$. It follows that

$$\lim_{u\to\infty} Q_{2,\tau}(u) = \lim_{u\to\infty} Q_{3,\tau}(u) = 0$$

which yields $\lim_{u\to\infty} T_{1,\tau}(u) = 0$. Note that the proof is finished if $\xi_0 = 1$, since $T_{2,\tau}(u) = 0$ for each $u > 0$.

Arguing by contradiction, let us suppose that $\xi_0 \neq 1$. Then, by (1.11),

$$T_{2,\tau}(u) = \xi_0^\rho\left[\exp\left\{\int_u^{\xi_0 u} \frac{\phi(t)}{t}\,\mathrm{d}t\right\} - 1\right](\ln u)^\tau, \quad \forall u > B/\xi_0.$$

But, $\lim_{u\to\infty} \phi(us)/s = 0$, uniformly with respect to $s \in [\xi_0, 1]$. So

$$\lim_{u\to\infty}\int_u^{\xi_0 u} \frac{\phi(t)}{t}\,\mathrm{d}t = \lim_{u\to\infty}\int_1^{\xi_0} \frac{\phi(su)}{s}\,\mathrm{d}s = 0$$

which leads to

$$\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \lim_{u\to\infty}\left(\int_u^{\xi_0 u} \frac{\phi(t)}{t}\,\mathrm{d}t\right)(\ln u)^\tau.$$

If (i) occurs, then by Proposition 2.1, we have

$$\lim_{u\to\infty} T_{2,\tau}(u) = \xi_0^\rho \lim_{u\to\infty}(\ln u)^\tau\phi(u)\int_1^{\xi_0} \frac{\phi(tu)}{\phi(u)}\,\frac{\mathrm{d}t}{t} = \xi_0^\rho\ell^\star\ln\xi_0.$$

If (ii) occurs, then by Proposition 2.3, we infer that

$$\lim_{u\to\infty} T_{2,\tau}(u) = \frac{-\xi_0^\rho}{\tau} \lim_{u\to\infty} \big[\phi(\xi_0 u) - \phi(u)\big](\ln u)^{\tau+1} = 0.$$

The proof of Lemma 2.2 is now complete. □

Lemma 2.3. *If* $k \in \mathcal{K}_{(01],\tau}$ *and* f *satisfies either* (i) *or* (ii) *of Theorem* 1.3*, then*

$$\mathcal{H}(t) := (-\ln t)^\tau \left(1 - \frac{k^2(t) f(\xi_0 h(t))}{\xi_0 h''(t)}\right) \to \rho\widetilde{\chi} \quad \textit{as } t \searrow 0, \tag{2.14}$$

where $\widetilde{\chi}$ *is defined by* (1.14).

Proof. Using (2.8), we write $\mathcal{H}(t) = \frac{k^2(t) f(h(t))}{h''(t)} \sum_{i=1}^3 \mathcal{H}_i(t)$, for $t > 0$ small, where

$$\begin{cases} \mathcal{H}_1(t) := 2\Xi\big(h(t)\big)(-\ln t)^\tau \left[\left(\dfrac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)' - \ell_1\right], \\ \mathcal{H}_2(t) := 2(1-\ell_1)\left(\dfrac{-\ln t}{\ln h(t)}\right)^\tau T_{1,\tau}\big(h(t)\big) \quad \text{and} \quad \mathcal{H}_3(t) := -\left(\dfrac{-\ln t}{\ln h(t)}\right)^\tau T_{2,\tau}\big(h(t)\big). \end{cases}$$

By Remark 2.3, we find $\lim_{t\searrow 0} \mathcal{H}_1(t) = \rho L_\sharp/(\rho+2)$.

Case (i) (that is, $f \in \mathcal{F}_{\rho\eta}$ with $\eta L_\sharp \neq 0$). By Lemmas 2.1 and 2.2, it turns out that

$$\lim_{t\searrow 0} \mathcal{H}_2(t) = \lim_{t\searrow 0} \mathcal{H}_3(t) = 0 \quad \text{and} \quad \lim_{t\searrow 0} \mathcal{H}(t) = \frac{\rho L_\sharp}{2+\rho\ell_1} =: \rho\widetilde{\chi}.$$

Case (ii) (that is, $f \in \mathcal{F}_{\rho 0,\tau}$ with $[\ell^\star(\ell_1-1)]^2 + L_\sharp^2 \neq 0$). By Lemmas 2.1 and 2.2, we get

$$\lim_{t\searrow 0} \mathcal{H}_2(t) = \frac{-2(1-\ell_1)\ell^\star}{(\rho+2)^2}\left(\frac{\rho\ell_1}{2}\right)^\tau \quad \text{and} \quad \lim_{t\searrow 0} \mathcal{H}_3(t) = \frac{-\ell^\star(2+\rho\ell_1)}{(2+\rho)}\left(\frac{\rho\ell_1}{2}\right)^\tau \ln\xi_0.$$

Thus, we arrive at

$$\lim_{t\searrow 0} \mathcal{H}(t) = \frac{\rho L_\sharp}{2+\rho\ell_1} - \ell^\star\left(\frac{\rho\ell_1}{2}\right)^\tau \left[\frac{2(1-\ell_1)}{(\rho+2)(2+\rho\ell_1)} + \ln\xi_0\right] =: \rho\widetilde{\chi}.$$

This finishes the proof. □

3. Characterization of $\mathcal{K}$ and its subclasses

Definition 2.1 extends to *regular variation at the origin*. We say that Z is regularly varying (on the right) at the origin with index q (and write, $Z \in RV_q(0+)$) if $Z(1/u) \in RV_{-q}$. Moreover, by $Z \in NRV_q(0+)$ we mean that $Z(1/u) \in NRV_{-q}$. The meaning of NRV_q is given by (2.2).

Proposition 3.1. *We have $k \in \mathcal{K}_{(01]}$ if and only if k is non-decreasing near the origin and k belongs to $NRV_\alpha(0+)$ for some $\alpha \geqslant 0$ (where $\alpha = 1/\ell_1 - 1$).*

Proof. If $k \in \mathcal{K}_{(01]}$, then from the definition

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)} \Big/ t = \lim_{t\to 0^+} \left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)' = \ell_1,$$

which implies that

$$\lim_{u\to\infty} \frac{u\frac{\mathrm{d}}{\mathrm{d}u}k(1/u)}{k(1/u)} = \lim_{t\to 0^+} \frac{-tk'(t)}{k(t)} = \frac{\ell_1 - 1}{\ell_1}.$$

Thus $k(1/u)$ belongs to NRV_{1-1/ℓ_1}. Conversely, if k belongs to $NRV_\alpha(0+)$ with $\alpha \geqslant 0$, then k is a positive C^1-function on some interval $(0,\nu)$ and

$$\lim_{t\to 0^+} \frac{tk'(t)}{k(t)} = \alpha. \tag{3.1}$$

By Proposition 2.4, we deduce

$$\lim_{t\to 0^+} \frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)} = \lim_{u\to\infty} \frac{\int_u^\infty x^{-2}k(1/x)\,\mathrm{d}x}{u^{-1}k(1/u)} = \frac{1}{1+\alpha}. \tag{3.2}$$

Combining (3.1) and (3.2), we get $\lim_{t\to 0^+}(\int_0^t k(s)\,\mathrm{d}s/k(t))' = 1/(1+\alpha)$. If, in addition, k is non-decreasing near 0, then $k \in \mathcal{K}$ with $\ell_1 = 1/(1+\alpha)$. Note that by (3.1), k is increasing near the origin if $\alpha > 0$; however, when k is slowly varying at 0, then we cannot draw any conclusion about the monotonicity of k near the origin (see Remark 2.2). □

Remark 3.1. By Propositions 3.1 and 2.1, we deduce $k \in \mathcal{K}_{(01]}$ if and only k is of the form

$$k(t) = c_0 t^\alpha \exp\left\{\int_t^{c_1} \frac{E(y)}{y}\,\mathrm{d}y\right\} \quad (0 < t < c_1),\ \text{for some } 0 \leqslant \alpha\ (= 1/\ell_1 - 1), \tag{3.3}$$

where $c_0, c_1 > 0$ are constants, $E \in C[0,c_1)$ with $E(0) = 0$ and (only for $\ell_1 = 1$) $E(t) \leqslant \alpha$.

Proposition 3.2. *We have $k \in \mathcal{K}_{(01],\tau}$ if and only if k is of the form* (3.3) *where, in addition,*

$$\lim_{t\searrow 0}(-\ln t)^\tau E(t) = \ell_\sharp \in \mathbf{R} \quad \text{with } \ell_\sharp = (1+\alpha)^2 L_\sharp. \tag{3.4}$$

Proof. Suppose k satisfies (3.3) and (3.4). A simple calculation leads to

$$\lim_{t\searrow 0}(-\ln t)^\tau \left[\frac{1-\ell_1}{\ell_1} - \frac{tk'(t)}{k(t)}\right] = \lim_{t\searrow 0}(-\ln t)^\tau E(t) = \ell_\sharp. \tag{3.5}$$

By L'Hospital's rule, we find

$$\lim_{t\searrow 0}(-\ln t)^\tau\left[\ell_1-\frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)}\right]=\lim_{t\searrow 0}\frac{(\ell_1-1)+\ell_1 tk'(t)/k(t)}{(-\ln t)^{-\tau}\left[1+\frac{tk'(t)}{k(t)}-\frac{\tau}{\ln t}\right]}$$
$$=-\ell_1^2\lim_{t\searrow 0}(-\ln t)^\tau\left[\frac{1-\ell_1}{\ell_1}-\frac{tk'(t)}{k(t)}\right]=\frac{-\ell_\sharp}{(\alpha+1)^2}. \tag{3.6}$$

We see that, for each $t>0$ small,

$$\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)'-\ell_1=\frac{tk'(t)}{k(t)}\left[\ell_1-\frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)}\right]+\ell_1\left[\frac{1-\ell_1}{\ell_1}-\frac{tk'(t)}{k(t)}\right]. \tag{3.7}$$

By (3.5)–(3.7), we infer that $k\in\mathcal{K}_{(01],\tau}$ with $L_\sharp=\ell_\sharp/(1+\alpha)^2$.

Conversely, if $k\in\mathcal{K}_{(01],\tau}$, then k is of the form (3.3). Moreover, we have

$$\lim_{t\searrow 0}(-\ln t)^\tau\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)}-\ell_1\right)=\lim_{t\searrow 0}\frac{\left(\int_0^t k(s)\,\mathrm{d}s/k(t)\right)'-\ell_1}{(-\ln t)^{-\tau}\left(1-\frac{\tau}{\ln t}\right)}=L_\sharp. \tag{3.8}$$

By (3.7) and (3.8), we deduce

$$L_\sharp=-\alpha L_\sharp+\frac{1}{\alpha+1}\lim_{t\searrow 0}(-\ln t)^\tau E(t).$$

Consequently, $\lim_{t\searrow 0}(-\ln t)^\tau E(t)=(1+\alpha)^2L_\sharp$. Hence, (3.4) holds. □

Proposition 3.3. *We have $k\in\mathcal{K}_0$ if and only if k is of the form*

$$k(t)=d_0\left(\exp\left\{-\int_t^{d_1}\frac{\mathrm{d}x}{x\mathcal{W}(x)}\right\}\right)'\quad(0<t<d_1), \tag{3.9}$$

where $d_0,d_1>0$ are constants and $0<\mathcal{W}\in C^1(0,d_1)$ satisfies $\lim_{t\searrow 0}\mathcal{W}(t)=\lim_{t\searrow 0}t\mathcal{W}'(t)=0$.

Proof. If $k\in\mathcal{K}_0$, then we set

$$\mathcal{W}(t)=\frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)},\quad\text{for } t\in(0,d_1). \tag{3.10}$$

Hence, $\lim_{t\searrow 0}\mathcal{W}(t)=0$ and, for $t>0$ small,

$$t\mathcal{W}'(t)=\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)'-\frac{\int_0^t k(s)\,\mathrm{d}s}{tk(t)}.$$

It follows that $\lim_{t\searrow 0}t\mathcal{W}'(t)=0$. By (3.10), we find

$$\int_t^{d_1}\frac{\mathrm{d}x}{x\mathcal{W}(x)}=\ln\left(\int_0^{d_1}k(s)\,\mathrm{d}s\right)-\ln\left(\int_0^t k(s)\,\mathrm{d}s\right),\quad t\in(0,d_1),$$

so that (3.9) is fulfilled. Conversely, if (3.9) holds, then $\lim_{t\to 0}\int_t^{d_1}\frac{\mathrm{d}x}{x\mathcal{W}(x)}=\infty$ and

$$\int_0^t k(s)\,\mathrm{d}s = d_0\exp\left\{-\int_t^{d_1}\frac{\mathrm{d}x}{x\mathcal{W}(x)}\right\} = tk(t)\mathcal{W}(t),\quad t\in(0,d_1). \tag{3.11}$$

This, together with the properties of $\mathcal{W}$, shows that $k\in\mathcal{K}_0$. □

Proposition 3.4. *We have* $k\in\mathcal{K}_{0,\zeta}$ *if and only if* k *is of the form* (3.9) *where, in addition,*

$$\lim_{t\searrow 0} t^{1-\zeta}\mathcal{W}'(t) = -\ell_\star \quad \textit{with } -\ell_\star = \zeta L_\star/(1+\zeta). \tag{3.12}$$

Proof. If $k\in\mathcal{K}_{0,\zeta}$, then (3.9) and (3.11) are fulfilled. Therefore,

$$L_\star = \lim_{t\searrow 0}\frac{(t\mathcal{W}(t))'}{t^\zeta} = \lim_{t\searrow 0}\frac{\mathcal{W}(t)+t\mathcal{W}'(t)}{t^\zeta} \quad\text{and}\quad \frac{L_\star}{\zeta+1} = \lim_{t\searrow 0}\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)t^{\zeta+1}} = \lim_{t\searrow 0}\frac{\mathcal{W}(t)}{t^\zeta},$$

from which (3.12) follows. Conversely, if (3.9) and (3.12) hold, then $\lim_{t\searrow 0}\mathcal{W}(t)/t^\zeta = -\ell_\star/\zeta$. By (3.11), we infer that

$$\frac{1}{t^\zeta}\left(\frac{\int_0^t k(s)\,\mathrm{d}s}{k(t)}\right)' = \frac{1}{t^\zeta}\big(\mathcal{W}(t)+t\mathcal{W}'(t)\big) \to \frac{-\ell_\star(\zeta+1)}{\zeta} \quad \text{as } t\searrow 0.$$

Thus, $k\in\mathcal{K}_{0,\zeta}$ with $L_\star = -\ell_\star(\zeta+1)/\zeta$. □

Remark 3.2. If $k\in\mathcal{K}_0$ or $k\in\mathcal{K}_{(01],\tau}$ with $(1-\ell_1)^2+L_\sharp^2\neq 0$, then

$$\lim_{t\searrow 0}\frac{k'(t)}{k(t)t^{\theta-1}} = \infty,\quad \text{for every } \theta>0. \tag{3.13}$$

Indeed, if $k\in\mathcal{K}_0$, then $\lim_{t\searrow 0}\frac{tk'(t)}{k(t)}=\infty$. Assuming that $k\in\mathcal{K}_{(01],\tau}$, we deduce (3.13) from (3.1) when $\ell_1\neq 1$, otherwise from (3.4) when $L_\sharp\neq 0$ since

$$\lim_{t\searrow 0}\frac{k'(t)}{k(t)t^{\theta-1}} = \lim_{t\searrow 0} -E(t)t^{-\theta} = -L_\sharp \lim_{t\searrow 0}\frac{t^{-\theta}}{(-\ln t)^\tau} = \infty.$$

Definition 3.1 (see [47]). A non-decreasing function U is Γ-varying at ∞ if U is defined on an interval (A,∞), $\lim_{x\to\infty}U(x)=\infty$ and there is $g:(A,\infty)\to(0,\infty)$ such that

$$\lim_{y\to\infty}\frac{U(y+\lambda g(y))}{U(y)} = \mathrm{e}^\lambda,\quad \forall\lambda\in\mathbf{R}.$$

The function g is called an *auxiliary function* and is unique up to asymptotic equivalence.

Remark 3.3. Under the assumptions of Theorem 1.1, we have

(a) Suppose $\lim_{t\searrow 0}(\int_0^t k(s)\,\mathrm{d}s)^2 r(t) = 1$ and let $\hat{f}(u)$ be such that $\lim_{u\to\infty}\hat{f}(u)/f(u) = 1$ and $j(u) := \hat{f}(u)/u$ is non-decreasing for $u > 0$ large. Then $\lim_{t\searrow 0}\widehat{\varphi}(t)/\varphi(t) = 1$, where $\varphi(t)$ is given by (1.7) and $\widehat{\varphi}(t) = j^{\leftarrow}(r(t))$ for $t > 0$ small.
(b) If $k \in \mathcal{K}$ with $\ell_1 \neq 0$, then $\varphi(1/u) \in RV_{2/(\rho\ell_1)}$.
(c) If $k \in \mathcal{K}_0$, then $\varphi(1/u)$ is Γ-varying at $u = \infty$ with auxiliary function $\frac{\rho u^2\int_0^{1/u}k(s)\,\mathrm{d}s}{2k(1/u)}$.
(d) $\lim_{t\searrow 0}\varphi(t)/h(t) = [2(\rho+2)/\rho^2]^{-1/\rho}$, where $h(t)$ is given by (1.9).

Indeed, by Proposition 2.6 we find $(f(u)/u)^{\leftarrow} \in RV_{1/\rho}$ and $\lim_{u\to\infty}(f(u)/u)^{\leftarrow}/j^{\leftarrow}(u) = 1$. Then, by Proposition 2.1 we deduce (a). We see that (b) follows by Proposition 2.6 since $(\int_0^{1/u}k(s)\,\mathrm{d}s)^{-2} \in RV_{2/\ell_1}$ (cf. Proposition 3.1) and $f(u)/u \in RV_\rho$. If $k \in \mathcal{K}_0$, then by Proposition 3.3 and [47, p. 106], we get $(\int_0^{1/u}k(s)\,\mathrm{d}s)^{-2}$ is Γ-varying at $u = \infty$ with auxiliary function $u\mathcal{W}(1/u)/2$. By [47, p. 36], we conclude (c). Notice that $Y(u) := (1/\int_u^\infty[2F(s)]^{-1/2}\,\mathrm{d}s)^2 \in RV_\rho$ and $Y(h(t)) = (\int_0^t k(s)\,\mathrm{d}s)^{-2}$ for $t > 0$ small. We have $\lim_{u\to\infty} f(u)/[uY(u)] = 2(\rho+2)/\rho^2$ (cf. Remark 2.3). By Proposition 2.6, we achieve (d).

4. Proof of Theorem 1.1

Fix $a \in (-\infty, \lambda_{\infty,1})$. By [12, Theorem 1.1], Eq. (1.1) has at least a large solution.

In what follows, we will prove that (1.6) holds for any large solution. Hence, a standard argument leads to the uniqueness (see, for instance, [25] or [12]).

By virtue of Remark 3.3(d), it is enough to demonstrate (1.8). Let u_a denote an arbitrary large solution of (1.1). Fix $\varepsilon \in (0, 1/2)$ and choose $\delta > 0$ such that

(i) $d(x)$ is a C^2 function on the set $\{x \in \Omega\colon d(x) < \delta\}$;
(ii) k is non-decreasing on $(0,\delta)$;
(iii) $1-\varepsilon < b(x)/k^2(d(x)) < 1+\varepsilon$, $\forall x \in \Omega$ with $0 < d(x) < \delta$ (since (1.5) holds);
(iv) $h'(t) < 0$ and $h''(t) > 0$ for each $t \in (0,\delta)$ (cf. Lemma 2.1).

Define $\xi^{\pm} = [\frac{2+\ell_1\rho}{(1\mp 2\varepsilon)(2+\rho)}]^{1/\rho}$ and $u^{\pm}(x) = \xi^{\pm}h(d(x))$, for any x with $d(x) \in (0,\delta)$.

The proof of (1.8) will be divided into three steps:

Step 1. There exists $\delta_1 \in (0,\delta)$ small such that

$$\begin{cases} \Delta u^+ + au^+ - (1-\varepsilon)k^2(d)f(u^+) \leqslant 0, & \forall x \text{ with } d(x) \in (0,\delta_1),\\ \Delta u^- + au^- - (1+\varepsilon)k^2(d)f(u^-) \geqslant 0, & \forall x \text{ with } d(x) \in (0,\delta_1). \end{cases} \tag{4.1}$$

Indeed, for every $x \in \Omega$ with $0 < d(x) < \delta$, we have

$$\begin{aligned} &\Delta u^{\pm} + au^{\pm} - (1\mp\varepsilon)k^2(d)f(u^{\pm})\\ &\quad = \xi^{\pm}h''(d)\left(1 + a\frac{h(d)}{h''(d)} + \Delta d\,\frac{h'(d)}{h''(d)} - (1\mp\varepsilon)\frac{k^2(d)f(u^{\pm})}{\xi^{\pm}h''(d)}\right) =: \xi^{\pm}h''(d)B^{\pm}(d). \end{aligned} \tag{4.2}$$

By Lemma 2.1, we deduce $\lim_{d\searrow 0}B^{\pm}(d) = \mp\varepsilon/(1\mp 2\varepsilon)$, which proves (4.1).

Step 2. There exists $M^+, \delta^+ > 0$ such that

$$u_a(x) \leqslant u^+(x) + M^+, \quad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

For $x \in \Omega$ with $d(x) \in (0, \delta_1)$, we define $\Psi_x(u) = au - b(x)f(u)$ for each $u > 0$. By Lemma 2.1,

$$\lim_{d(x)\searrow 0} \frac{b(x)f(u^+(x))}{u^+(x)} = \lim_{d\searrow 0} \frac{k^2(d)f(u^+)}{\xi^+ h''(d)} \frac{h''(d)}{h(d)} = \infty. \tag{4.3}$$

From this and (A_1), we infer that there exists $\delta_2 \in (0, \delta_1)$ such that, for any x with $0 < d(x) < \delta_2$,

$$u \mapsto \Psi_x(u) \text{ is decreasing on some interval } (u_x, \infty) \text{ with } 0 < u_x < u^+(x).$$

Hence, for each $M > 0$, we have

$$\Psi_x\big(u^+(x) + M\big) \leqslant \Psi_x\big(u^+(x)\big), \quad \forall x \in \Omega \text{ with } 0 < d(x) < \delta_2. \tag{4.4}$$

Fix $\sigma \in (0, \delta_2/4)$ and set $\mathcal{N}_\sigma := \{x \in \Omega\colon \sigma < d(x) < \delta_2/2\}$.

We define $u^*_\sigma(x) = u^+(d - \sigma, s) + M^+$, where (d, s) are the local coordinates of $x \in \mathcal{N}_\sigma$. We choose $M^+ > 0$ large enough such that

$$u^*_\sigma(\delta_2/2, s) = u^+(\delta_2/2 - \sigma, s) + M^+ \geqslant u_a(\delta_2/2, s), \quad \forall \sigma \in (0, \delta_2/4) \text{ and } \forall s \in \partial\Omega.$$

By (ii), (iii), (4.1) and (4.4), we obtain

$$\begin{aligned}
-\Delta u^*_\sigma(x) &\geqslant au^+(d - \sigma, s) - (1 - \varepsilon)k^2(d - \sigma)f\big(u^+(d - \sigma, s)\big) \\
&\geqslant au^+(d - \sigma, s) - b(x)f\big(u^+(d - \sigma, s)\big) \\
&\geqslant a\big(u^+(d - \sigma, s) + M^+\big) - b(x)f\big(u^+(d - \sigma, s) + M^+\big) \\
&= au^*_\sigma(x) - b(x)f\big(u^*_\sigma(x)\big) \quad \text{in } \mathcal{N}_\sigma.
\end{aligned}$$

So, uniformly with respect to σ, we have

$$\Delta u^*_\sigma(x) + au^*_\sigma(x) \leqslant b(x)f\big(u^*_\sigma(x)\big) \quad \text{in } \mathcal{N}_\sigma. \tag{4.5}$$

Since $u^*_\sigma(x) \to \infty$ as $d \searrow \sigma$, from [12, Lemma 2.1], we get $u_a \leqslant u^*_\sigma$ in $\mathcal{N}_\sigma$, for every $\sigma \in (0, \delta_2/4)$. Letting $\sigma \searrow 0$, we achieve the assertion of Step 2 (with $\delta^+ \in (0, \delta_2/2)$ arbitrarily chosen).

Step 3. There exists $M^-, \delta^- > 0$ such that

$$u_a(x) \geqslant u^-(x) - M^-, \quad \forall x = (d, s) \in \Omega \text{ with } 0 < d < \delta^-. \tag{4.6}$$

For every $r \in (0, \delta)$, define $\Omega_r = \{x \in \Omega\colon 0 < d(x) < r\}$.

Fix $\sigma \in (0, \delta_2/4)$. We define $v^*_\sigma(x) = \lambda u^-(d + \sigma, s)$ for $x = (d, s) \in \Omega_{\delta_2/2}$, where $\lambda \in (0, 1)$ is chosen small enough such that

$$v^*_\sigma(\delta_2/4, s) = \lambda u^-(\delta_2/4 + \sigma, s) \leqslant u_a(\delta_2/4, s), \quad \forall \sigma \in (0, \delta_2/4),\ \forall s \in \partial\Omega. \tag{4.7}$$

Notice that $\limsup_{d\searrow 0}(v_\sigma^* - u_a)(x) = -\infty$. By (ii), (iii), (4.1) and (A_1), we have

$$\begin{aligned}\Delta v_\sigma^*(x) + a v_\sigma^*(x) &= \lambda\big(\Delta u^-(d+\sigma,s) + a u^-(d+\sigma,s)\big)\\ &\geqslant \lambda(1+\varepsilon)k^2(d+\sigma)f\big(u^-(d+\sigma,s)\big) \geqslant (1+\varepsilon)k^2(d)f\big(\lambda u^-(d+\sigma,s)\big)\\ &\geqslant b(x)f\big(v_\sigma^*(x)\big), \quad \forall x=(d,s)\in \Omega_{\delta_2/4}.\end{aligned}$$

Using [12, Lemma 2.1], we derive $v_\sigma^* \leqslant u_a$ in $\Omega_{\delta_2/4}$. Letting $\sigma \searrow 0$, we get

$$\lambda u^-(x) \leqslant u_a(x), \quad \forall x \in \Omega_{\delta_2/4}. \tag{4.8}$$

By Lemma 2.1, $\lim_{d\searrow 0} k^2(d)f(\lambda^2 u^-)/u^- = \infty$. Thus, there exists $\tilde\delta \in (0, \delta_2/4)$ such that

$$k^2(d)f\big(\lambda^2 u^-\big)/u^- \geqslant \lambda^2|a|, \quad \forall x\in\Omega \text{ with } 0<d\leqslant\tilde\delta. \tag{4.9}$$

Choose $\delta_* \in (0,\tilde\delta)$, sufficiently close to $\tilde\delta$, such that

$$h(\delta_*)/h(\tilde\delta) < 1+\lambda. \tag{4.10}$$

For each $\sigma \in (0, \tilde\delta - \delta_*)$, we define $z_\sigma(x) = u^-(d+\sigma,s) - (1-\lambda)u^-(\delta_*,s)$, where $x=(d,s)\in\Omega_{\delta_*}$. We prove that z_σ is positive in Ω_{δ_*} and

$$\Delta z_\sigma + a z_\sigma \geqslant b(x)f(z_\sigma) \quad \text{in } \Omega_{\delta_*}. \tag{4.11}$$

By (iv), $u^-(x)$ decreases with d when $d<\tilde\delta$. This and (4.10) imply that

$$1+\lambda > \frac{u^-(\delta_*,s)}{u^-(\tilde\delta,s)} \geqslant \frac{u^-(\delta_*,s)}{u^-(d+\sigma,s)}, \quad \forall x=(d,s)\in\Omega_{\delta_*}. \tag{4.12}$$

Hence,

$$z_\sigma(x) = u^-(d+\sigma,s)\left(1 - \frac{(1-\lambda)u^-(\delta_*,s)}{u^-(d+\sigma,s)}\right) \geqslant \lambda^2 u^-(d+\sigma,s) > 0, \quad \forall x\in\Omega_{\delta_*}. \tag{4.13}$$

By (4.1), (ii) and (iii), we see that (4.11) follows if

$$(1+\varepsilon)k^2(d+\sigma)\big[f\big(u^-(d+\sigma,s)\big) - f\big(z_\sigma(d,s)\big)\big] \geqslant a(1-\lambda)u^-(\delta_*,s), \quad \forall (d,s)\in\Omega_{\delta_*}. \tag{4.14}$$

The Lagrange mean value theorem and (A_1) show that

$$f\big(u^-(d+\sigma,s)\big) - f\big(z_\sigma(d,s)\big) \geqslant (1-\lambda)u^-(\delta_*,s)f\big(z_\sigma(x)\big)/z_\sigma(x) \tag{4.15}$$

which, combined with (4.9) and (4.13), proves (4.14).

Notice that $\limsup_{d\searrow 0}(z_\sigma - u_a)(x) = -\infty$. By (4.8), we have

$$z_\sigma(x) = u^-(\delta_*+\sigma,s) - (1-\lambda)u^-(\delta_*,s) \leqslant \lambda u^-(\delta_*,s) \leqslant u_a(x), \quad \forall x=(\delta_*,s)\in\Omega.$$

By [12, Lemma 2.1], $z_\sigma \leqslant u_a$ in Ω_{δ_*}, for every $\sigma\in(0,\tilde\delta-\delta_*)$. Letting $\sigma\searrow 0$, we conclude Step 3.

Thus, by Steps 2 and 3, we have

$$\xi^- \leqslant \liminf_{d(x)\searrow 0} \frac{u_a(x)}{h(d(x))} \leqslant \limsup_{d(x)\searrow 0} \frac{u_a(x)}{h(d(x))} \leqslant \xi^+.$$

Taking $\varepsilon \to 0$, we obtain (1.8). This finishes the proof of Theorem 1.1.

5. Proof of Theorem 1.3

Fix $a < \lambda_{\infty,1}$ and denote by u_a the unique large solution of (1.1).

Let $\varepsilon \in (0, 1/2)$ be arbitrary and $\delta > 0$ be such that (i), (ii), (iv) from Section 4 are satisfied.

By (1.10) and Remark 3.2, we can diminish $\delta > 0$ such that

$$\begin{cases} 1 + (\tilde{c} - \varepsilon)d^\theta < b(x)/k^2(d) < 1 + (\tilde{c} + \varepsilon)d^\theta, & \forall x \in \Omega \text{ with } d \in (0, \delta), \\ k^2(t)\big[1 + (\tilde{c} - \varepsilon)t^\theta\big] & \text{is increasing on } (0, \delta). \end{cases} \tag{5.1}$$

Define $u^\pm(x) = \xi_0 h(d)[1 + \chi_\varepsilon^\pm(-\ln d)^{-\tau}]$ for $x \in \Omega$ with $d \in (0, \delta)$, where $\chi_\varepsilon^\pm = \widetilde{\chi} \pm \varepsilon$.

We can assume $u^\pm(x) > 0$ for every $x \in \Omega$ with $d(x) \in (0, \delta)$.

By the Lagrange mean value theorem, we obtain

$$f\big(u^\pm(x)\big) = f\big(\xi_0 h(d)\big) + \xi_0 \chi_\varepsilon^\pm \frac{h(d)}{(-\ln d)^\tau} f'\big(\Psi^\pm(d)\big),$$

where $\Psi^\pm(d) = \xi_0 h(d)[1 + \chi_\varepsilon^\pm \lambda^\pm(d)(-\ln d)^{-\tau}]$, for some $\lambda^\pm(d) \in [0, 1]$.

Since $f(u)/u^{\rho+1}$ is slowly varying, by Proposition 2.1 we find

$$\lim_{d\searrow 0} \frac{f(\Psi^\pm(d))}{f(\xi_0 h(d))} = \lim_{d\searrow 0} \frac{f(u^\pm(d))}{f(\xi_0 h(d))} = 1. \tag{5.2}$$

Step 1. There exists $\delta_1 \in (0, \delta)$ so that

$$\begin{cases} \Delta u^+ + au^+ - k^2(d)\big[1 + (\tilde{c} - \varepsilon)d^\theta\big] f\big(u^+\big) \leqslant 0, & \forall x \in \Omega \text{ with } d < \delta_1, \\ \Delta u^- + au^- - k^2(d)\big[1 + (\tilde{c} + \varepsilon)d^\theta\big] f\big(u^-\big) \geqslant 0, & \forall x \in \Omega \text{ with } d < \delta_1. \end{cases} \tag{5.3}$$

For every $x \in \Omega$ with $d \in (0, \delta)$, we have

$$\Delta u^\pm + au^\pm - k^2(d)\big[1 + (\tilde{c} \mp \varepsilon)d^\theta\big] f\big(u^\pm\big) = \xi_0 \frac{h''(d)}{(-\ln d)^\tau} \mathcal{J}^\pm(d), \tag{5.4}$$

where

$$\begin{aligned} \mathcal{J}^\pm(d) := \bigg[& \chi_\varepsilon^\pm \Delta d \frac{h'(d)}{h''(d)} + \frac{h'(d)}{dh''(d)} \bigg(d(-\ln d)^\tau \Delta d - \frac{2\tau \chi_\varepsilon^\pm}{\ln d} \bigg) + a \frac{h(d)}{h''(d)} \big(\chi_\varepsilon^\pm + (-\ln d)^\tau\big) \\ & + \frac{\tau \chi_\varepsilon^\pm h(d)}{d^2 h''(d) \ln d} \bigg(1 + \frac{\tau + 1}{\ln d} - d\Delta d \bigg) + (-\tilde{c} \pm \varepsilon) d^\theta (-\ln d)^\tau \, \frac{k^2(d) f(\xi_0 h(d))}{\xi_0 h''(d)} \\ & + (-\tilde{c} \pm \varepsilon) \chi_\varepsilon^\pm d^\theta \frac{k^2(d) h(d) f'(\Psi^\pm(d))}{h''(d)} + \mathcal{H}(d) + \mathcal{J}_1^\pm(d) \bigg]. \end{aligned}$$

Here $\mathcal{H}$ is defined by (2.14), while

$$\mathcal{J}_1^{\pm}(d) := \chi_\varepsilon^{\pm}\biggl(1 - \frac{k^2(d)h(d)f'(\Psi^{\pm}(d))}{h''(d)}\biggr).$$

By Lemma 2.1 and (5.2), we infer that

$$\lim_{d\searrow 0} \frac{k^2(d)h(d)f'(\Psi^{\pm}(d))}{h''(d)} = \lim_{d\searrow 0} \frac{\Psi^{\pm}(d)f'(\Psi^{\pm}(d))}{f(\Psi^{\pm}(d))}\frac{k^2(d)f(\xi_0 h(d))}{\xi_0 h''(d)} = \rho + 1.$$

Hence, $\lim_{d\searrow 0} \mathcal{J}_1^{\pm}(d) = -\rho\chi_\varepsilon^{\pm} := -\rho(\widetilde{\chi} \pm \varepsilon)$. Using Lemmas 2.1 and 2.3, we find

$$\lim_{d\searrow 0} \mathcal{J}^{+}(d) = -\rho\varepsilon < 0 \quad \text{and} \quad \lim_{d\searrow 0} \mathcal{J}^{-}(d) = \rho\varepsilon > 0.$$

Therefore, by (5.4) we conclude (5.3).

Step 2. There exists M^+, $\delta^+ > 0$ such that

$$u_a(x) \leqslant u^+(x) + M^+, \quad \forall x \in \Omega \text{ with } 0 < d < \delta^+.$$

We only recover (4.5), the rest being similar to the proof of Step 2 in Theorem 1.1. Indeed, by (5.3), (5.1) and (4.4), we obtain

$$\begin{aligned}
-\Delta u_\sigma^*(x) &\geqslant a u^+(d-\sigma, s) - \bigl[1 + (\tilde{c} - \varepsilon)(d-\sigma)^\theta\bigr]k^2(d-\sigma)f\bigl(u^+(d-\sigma, s)\bigr) \\
&\geqslant a u^+(d-\sigma, s) - \bigl[1 + (\tilde{c} - \varepsilon)d^\theta\bigr]k^2(d)f\bigl(u^+(d-\sigma, s)\bigr) \\
&\geqslant a u^+(d-\sigma, s) - b(x)f\bigl(u^+(d-\sigma, s)\bigr) \\
&\geqslant a\bigl(u^+(d-\sigma, s) + M^+\bigr) - b(x)f\bigl(u^+(d-\sigma, s) + M^+\bigr) \\
&= a u_\sigma^*(x) - b(x)f\bigl(u_\sigma^*(x)\bigr) \quad \text{in } \mathcal{N}_\sigma.
\end{aligned}$$

Step 3. There exists M^-, $\delta^- > 0$ such that

$$u_a(x) \geqslant u^-(x) - M^-, \quad \forall x \in \Omega \text{ with } 0 < d < \delta^-.$$

We proceed in the same way as for proving (4.6). To recover (4.8) (with λ given by (4.7)), we show that $\Delta v_\sigma^* + a v_\sigma^* \geqslant b(x)f(v_\sigma^*)$ in $\Omega_{\delta_2/4}$. Indeed, using (5.1), (5.3) and (A_1), we find

$$\begin{aligned}
\Delta v_\sigma^*(x) + a v_\sigma^*(x) &= \lambda\bigl(\Delta u^-(d+\sigma, s) + a u^-(d+\sigma, s)\bigr) \\
&\geqslant \lambda k^2(d+\sigma)\bigl[1 + (\tilde{c} + \varepsilon)(d+\sigma)^\theta\bigr]f\bigl(u^-(d+\sigma, s)\bigr) \\
&\geqslant k^2(d)\bigl[1 + (\tilde{c} + \varepsilon)d^\theta\bigr]f\bigl(\lambda u^-(d+\sigma, s)\bigr) \\
&\geqslant b(x)f\bigl(v_\sigma^*(x)\bigr), \quad \forall x = (d, s) \in \Omega_{\delta_2/4}.
\end{aligned}$$

Since $\lim_{d\searrow 0} k^2(d)f(\lambda^2 u^-(x))/u^-(x) = \infty$, there exists $\tilde{\delta} \in (0, \delta_2/4)$ such that

$$k^2(d)\bigl[1 + (\tilde{c} + \varepsilon)d^\theta\bigr]f\bigl(\lambda^2 u^-\bigr)/u^- \geqslant \lambda^2|a|, \quad \forall x \in \Omega \text{ with } 0 < d \leqslant \tilde{\delta}. \tag{5.5}$$

By Lemma 2.1, we infer that $u^-(x)$ decreases with d when $d \in (0, \tilde{\delta})$ (if necessary, $\tilde{\delta} > 0$ is diminished). Choose $\delta_* \in (0, \tilde{\delta})$ close enough to $\tilde{\delta}$ such that

$$\frac{h(\delta_*)(1 + \chi_\varepsilon^-(-\ln \delta_*)^{-\tau})}{h(\tilde{\delta})(1 + \chi_\varepsilon^-(-\ln \tilde{\delta})^{-\tau})} < 1 + \lambda. \tag{5.6}$$

Hence, we regain (4.12), (4.13) and (4.15).

By (5.1) and (5.3), we see that (4.11) follows if

$$k^2(d+\sigma)\big[1 + (\tilde{c} + \varepsilon)(d+\sigma)^\theta\big]\big[f\big(u^-(d+\sigma, s)\big) - f\big(z_\sigma(d,s)\big)\big] \geqslant a(1-\lambda)u^-(\delta_*, s) \tag{5.7}$$

for each $(d, s) \in \Omega_{\delta_*}$. Using (4.15), together with (5.5) and (4.13), we arrive at (5.7). From now on, the argument is the same as before. This proves the claim of Step 3.

By Steps 2 and 3, it follows that

$$\begin{cases} \chi_\varepsilon^+ \geqslant \left[-1 + \dfrac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau - \dfrac{M^+(-\ln d)^\tau}{\xi_0 h(d)}, & \forall x \in \Omega \text{ with } d < \delta^+, \\ \chi_\varepsilon^- \leqslant \left[-1 + \dfrac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau + \dfrac{M^-(-\ln d)^\tau}{\xi_0 h(d)}, & \forall x \in \Omega \text{ with } d < \delta^-. \end{cases} \tag{5.8}$$

Using Lemma 2.1, we have

$$\lim_{t \searrow 0} \frac{(-\ln t)^\tau}{h(t)} = \lim_{t \searrow 0} \left(\frac{-\ln t}{\ln h(t)}\right)^\tau \frac{(\ln h(t))^\tau}{h(t)} = \left(\frac{\rho \ell_1}{2}\right)^\tau \lim_{u \to \infty} \frac{(\ln u)^\tau}{u} = 0.$$

Passing to the limit $d \searrow 0$ in (5.8), we obtain

$$\chi_\varepsilon^- \leqslant \liminf_{d \searrow 0} \left[-1 + \frac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau \leqslant \limsup_{d \searrow 0} \left[-1 + \frac{u_a(x)}{\xi_0 h(d)}\right](-\ln d)^\tau \leqslant \chi_\varepsilon^+.$$

By sending ε to 0, the proof of Theorem 1.3 is finished.

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