SIAM J. MATH. ANAL. © 2023 Society for Industrial and Applied Mathematics
Vol. 55, No. 6, pp. 7704–7740

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NORMALIZED SOLUTIONS FOR SCHRÖDINGER EQUATIONS WITH CRITICAL EXPONENTIAL GROWTH IN $\mathbb{R}^2$*

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Abstract. For any $a>0$, we study the existence of normalized solutions and ground state solutions to the following Schrödinger equation with L^2-constraint: $\begin{cases} -\Delta u+\lambda u=b(x)f(u) & x\in\mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2\mathrm{d}x=a, \end{cases}$ where $\lambda\in\mathbb{R}$ is a Lagrange multiplier, the potential $b\in\mathcal{C}(\mathbb{R}^2,(0,\infty))$ satisfies $0<\lim_{|y|\to\infty} b(y)\le \inf_{x\in\mathbb{R}^2} b(x)$ and appears as a converse direction of the Rabinowitz-type trapping potential, and the reaction $f\in\mathcal{C}(\mathbb{R},\mathbb{R})$ enjoys critical exponential growth of Trudinger–Moser type. Under two different kinds of assumptions on f, we prove several new existence results, which, in the context of normalized solutions, can be considered as both counterparts of planar unconstrained critical problems and extensions of unconstrained Schrödinger problems with Rabinowitz-type trapping potential. Especially, in this scenario, we develop some sharp estimates of energy levels and ingenious analysis techniques to restore the compactness which are novel even for $b(x)\equiv$ constant. We believe that these techniques will allow not only treating other L^2-constrained problems in the Trudinger–Moser critical setting but also generalizing previous results to the case of variable potentials.

Key words. Schrödinger equation, normalized solution, critical exponential growth, Trudinger–Moser inequality

MSC codes. 35J20, 35J62, 35Q55

DOI. 10.1137/22M1521675

1. Introduction.

This paper concerns the existence of normalized solutions to the following nonlinear Schrödinger equation with critical exponential growth:

$$\begin{cases} -\Delta u+\lambda u=b(x)f(u), & x\in\mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} u^2\mathrm{d}x=a, \end{cases} \tag{1.1}$$

where $a>0$ is a given constant, $\lambda\in\mathbb{R}$ will arise as a Lagrange multiplier that depends on the solution $u\in H^1(\mathbb{R}^2)$ and is not a priori given, and $b:\mathbb{R}^2\to\mathbb{R}$ and $f:\mathbb{R}\to\mathbb{R}$ satisfy the following basic conditions:

*Received by the editors September 12, 2022; accepted for publication (in revised form) June 15, 2023; published electronically November 9, 2023.
https://doi.org/10.1137/22M1521675
Funding: The work of the authors was partially supported by the National Natural Science Foundation of China grants 11971485 and 12001542 and the Hunan Provincial Natural Science Foundation grants 2022JJ20048 and 2021JJ40703. The work of the second author was supported by the project Nonlinear Differential Systems in Applied Sciences of the Romanian Ministry of Research, Innovation and Digitalization (MCID), project 22, within PNRR-III-C9-2022-I8.

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(B1) $b \in \mathcal{C}(\mathbb{R}^2, (0,\infty))$ and $0 < b_\infty := \lim_{|y|\to\infty} b(y) \le b(x)\ \forall x \in \mathbb{R}^2$;
(F1) $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$, and there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{e^{\alpha t^2}} = \begin{cases} 0 & \forall \alpha > \alpha_0, \\ +\infty & \forall \alpha < \alpha_0; \end{cases} \tag{1.2}$$

(F2) $\lim_{|t|\to 0} f(t)/t^3 = 0$.

As in Adimurthi and Yadava [3] and de Figueiredo, Miyagaki, and Ruf [14], we say that $f(t)$ has critical exponential growth at $t = \pm\infty$ if it satisfies (F1). It was shown by Trudinger [30] and Moser [25] that this kind of nonlinearity has the maximal growth that can be treated variationally in $H^1(\mathbb{R}^2)$, which is motivated by the following Trudinger–Moser inequality.

LEMMA 1.1 [1, 9, 10].
(i) *If* $\alpha > 0$ *and* $u \in H^1(\mathbb{R}^2)$, *then*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x < \infty;$$

(ii) *if* $u \in H^1(\mathbb{R}^2), \|\nabla u\|_2^2 \le 1, \|u\|_2 \le M < \infty$, *and* $\alpha < 4\pi$, *then there exists a constant* $\mathcal{C}(M,\alpha)$, *which depends only on* M *and* α, *such that*

$$\int_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right) \mathrm{d}x \le \mathcal{C}(M,\alpha). \tag{1.3}$$

The main feature of (1.1) is that the desired solutions have a priori prescribed L^2-norms, which are often referred to as normalized solutions in the literature; that is, for given $a > 0$, a couple $(u,\lambda) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ solves (1.1). From the physical point of view, finding normalized solutions seems to be particularly meaningful because the L^2-norms of such solutions are a preserved quantity of the evolution, and their variational characterization can help to analyze the orbital stability or instability; see, for example, [7, 27].

It is well known that normalized solutions to (1.1) can be obtained as critical points of the energy functional $\Phi : H^1(\mathbb{R}^2) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^2} b(x)F(u)\mathrm{d}x \tag{1.4}$$

under the constraint

$$\mathcal{S}_a = \left\{u \in H^1(\mathbb{R}^2) : \|u\|_2^2 = a\right\}. \tag{1.5}$$

It is standard that $\Phi \in \mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$ (see section 2 below), and any critical point u of $\Phi\big|_{\mathcal{S}_a}$ corresponds to a solution to (1.1), with the parameter $\lambda \in \mathbb{R}$ appearing as a Lagrange multiplier. We recall a solution u to be a ground state solution on $\mathcal{S}_a$ if u minimizes the functional Φ among all the solutions to (1.1); that is,

$$\Phi(u) = \inf\left\{\Phi(u) : \|u\|_2^2 = a, \quad \Phi\big|_{\mathcal{S}_a}'(u) = 0\right\}.$$

When f has critical exponential growth, in sharp contrast to the unconstrained problem with fixed $\lambda > 0$:

$$-\Delta u + \lambda u = f(u) \quad \text{in} \quad \mathbb{R}^2, \tag{1.6}$$

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which has been widely studied in the last decades, there seems to be only one recent paper [6] concerning normalized solutions to (1.1) in which the special case that $b(x) \equiv 1$ and $a \in (0,1)$ was considered. In the present paper, we shall focus on the constrained problem (1.1) with critical exponential growth for all $a > 0$. To explain what is at stake, let us first introduce some related results that motivate our research.

Problem (1.1) with $b(x) \equiv 1$ is a special case of the following model

$$\begin{cases} -\Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^N, \ N \geq 2, \\ \displaystyle\int_{\mathbb{R}^N} u^2 \mathrm{d}x = a, \end{cases} \tag{1.7}$$

which has been investigated extensively via the variational methods. There are many existence results on normalized solutions and ground state solutions to (1.7) in these last years, and these existence results depend on the behavior of the nonlinearities at infinity, which determines whether the constrained functional is bounded from below on the L^2-constraint set. From the variational point of view, this behavior gives rise to a new L^2-critical phenomenon, that is, a new L^2-critical exponent $q^* = 2 + 4/N$, which comes from the Gagliardo–Nirenberg inequality (see [11, Theorem 1.3.7]). As we know, if $f(u)$ in (1.7) grows faster than $|u|^{q^*-2}u$ at infinity, then the constrained functional is unbounded from below, and the problem is called L^2-supercritical; otherwise, the constrained functional is bounded from below, and the problem is called L^2-subcritical in the literature. In this sense, (1.1) is clearly L^2-supercritical. Compared with the L^2-subcritical case, more efforts are always needed in the study of the L^2-supercritical case.

1.1. Previous developments and some perspectives. The first contribution to the L^2-supercritical case was made by Jeanjean in a seminal paper [18]. In [18], a radial solution at a mountain pass value to (1.7) was found under the following conditions:

(H0) f is odd;

(H1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$, and there exist $\alpha, \beta \in \mathbb{R}$ satisfying $\frac{2N+4}{N} < \alpha \leq \beta < 2^* = \frac{2N}{N-2}$ such that

$$0 < \alpha F(t) \leq f(t)t \leq \beta F(t) \quad \forall\, t \in \mathbb{R} \setminus \{0\};$$

the existence of ground state solutions was proved if f also satisfies

(H2) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and

$$\tilde{F}'(t)t > \frac{2N+4}{N}\tilde{F}(t) \quad \forall\, t \in \mathbb{R} \setminus \{0\}.$$

Note that the first inequality of (H1) and condition (H2) play analogous roles as the classical Ambrosetti–Rabinowitz condition (AR condition for short) and the Nehari-type condition in the unconstrained context, respectively. Recently, Jeanjean and Lu [21] improved the existence results on ground state solutions obtained in [18] by relaxing (H1) and (H2) to the following conditions:

(H3) $\lim_{t\to 0} f(t)/t^{1+4/N} = 0$ and $\lim_{t\to\infty} F(t)/t^{2+4/N} = +\infty$; moreover,

$$\begin{cases} \lim_{t\to\infty} f(t)/t^{2^*-1} \quad \text{and} \quad f(t)t < \frac{2N}{N-2}F(t) & \text{when} \quad N \geq 3, \\ \lim_{|t|\to\infty} f(t)/e^{\alpha t^2} = 0 \quad \forall\, \alpha > 0 & \text{when} \quad N = 2; \end{cases}$$

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(H4) $[f(t)t - 2F(t)]/|t|^{(N+4)/N}t$ is increasing strictly on $(-\infty, 0)$ and $(0, +\infty)$.
When f satisfies (H1) and a weaker version of (H4),

(H4′) $[f(t)t - 2F(t)]/|t|^{(N+4)/N}t$ is nondecreasing on $(-\infty, 0)$ and $(0, +\infty)$.
Chen and Tang [13] obtained the existence of ground state solutions to (1.7) and also considered the nonautonomous case that $f(u)$ is replaced by $b(x)f(u)$ with $b \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R}^+)$ satisfying additional assumptions. We notice that three different strategies were implemented in [18], [21], and [13]:

(a) constructing a Palais–Smale sequence (PS sequence for short) of the constrained functional satisfying asymptotically the L^2-Pohozaev identity by ingeniously applying the Ekeland's principle to a new auxiliary functional;
(b) constructing a minimizing sequence that is a PS sequence of the constrained functional by subtly adapting the techniques developed by Szulkin and Weth [28, 29];
(c) solving directly the minimization problem of the constrained functional on the L^2-Pohozaev manifold by combining some new inequalities with the deformation lemma.

In the latter two cases, f is not required to be of class $\mathcal{C}^1$ in order to find ground state solutions. We also point out that the proofs for the compactness of PS or minimizing sequences $\{u_n\}$ use the compact embeddings $H_r^1(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ and $H^1_{\text{loc}}(\mathbb{R}^N) \hookrightarrow L^s(\mathbb{R}^N)$ for $2 < s < 2^*$. Both of the embeddings do not work when f has a critical growth in the Sobolev sense if $N \geq 3$ or a critical exponential growth in the Trudinger–Moser sense if $N = 2$, where

$$2^* := \begin{cases} 2N/(N-2) & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2. \end{cases}$$

In 2020, Soave [27] first considered the Schrödinger equation with Sobolev critical growth,

$$\begin{cases} -\Delta u + \lambda u = \mu |u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \ N \geq 3, \\ \displaystyle\int_{\mathbb{R}^N} u^2 \mathrm{d}x = a, \end{cases} \tag{1.8}$$

and proved that, for $\mu a(1-\gamma_q)q < \alpha$, (1.8) has ground state solutions in the L^2-subcritical perturbation case $2 < q < 2 + N/4$ and L^2-supercritical perturbation case $2 + N/4 < q < 2^*$, respectively, where $\alpha = \alpha(N, q)$ is a specific constant depending on N, q and $\gamma_q := N(q-2)/(2q)$. Very recently, Wei and Wu [32] obtained the existence of mountain pass solutions for $2 < q < 2 + N/4$ and proved the existence and nonexistence of ground state solutions for $2 + N/4 < q < 2^*$ with large $\mu > 0$. Moreover, they gave precisely asymptotic behaviors of ground state solutions and mountain pass solutions as μ goes to 0 and its upper bound; see also [19, 20] for $2 + N/4 < q < 2^*$. These results settled several open questions proposed by Soave [27]. For the L^2-supercritical perturbation case $2 + N/4 < q < 2^*$, we also refer to [4, 5, 6], which complement the existence results obtained in [27]. In these works, the compactness was restored successfully by the ingenious combination of the pioneering work of Brezis and Nirenberg [8], the scaling technique introduced by Jeanjean [18], and the concentration-compactness principle due to Lions [22]. It is worth pointing out that these strategies can well solve the obstacles caused by Sobolev critical growth when searching for a solution with a prescribed norm, but it is not available for the planar case that f has critical growth in the Trudinger–Moser sense, that is, the following equation

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$$\begin{cases} -\Delta u + \lambda u = f(u) \quad \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} u^2 \mathrm{d}x = a \end{cases} \tag{1.9}$$

with f satisfying (F1), since additional difficulties arise.

(i) One needs to reselect test functions in $\mathbb{R}^2$ on the L^2-constraint $\mathcal{S}_a$ to control the energy values from above so that PS sequences of the constrained functional are compact at the energy level;

(ii) It is unknown whether the Brezis–Lieb property

$$\int_{\mathbb{R}^2} [f(u_n)u_n - f(\bar{u})\bar{u} - f(u_n - \bar{u})(u_n - \bar{u})] \, \mathrm{d}x = o(1) \tag{1.10}$$

holds if $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^2)$, which plays a crucial role in restoring the compactness to Sobolev critical problems.

Thus, a natural question arises:

- Does (1.9) have solutions or ground state solutions when f is of critical exponential growth?

In 2020, Alves, Ji, and Miyagaki [6] gave partial answers to the above question for $a \in (0,1)$. To state the result, we list the conditions used in [6]:

(F2′) $\lim_{|t|\to 0} |f(t)|/|t|^l = 0$ for some constant $l > 3$;
(F3′) there exists a constant $\mu_0 > 4$ such that $f(t)t \geq \mu_0 F(t) > 0 \ \forall\, t \in \mathbb{R} \setminus \{0\}$;
(F4′) there exist constants $p > 4$ and $\gamma > 0$ such that $F(t) \geq \gamma |t|^p \ \forall\, t \in \mathbb{R}$;
(F5′) the function $\tilde{F}(t) := f(t)t - 2F(t)$ is of class $\mathcal{C}^1$ and satisfies

$$\tilde{F}'(t)t \geq 4\tilde{F}(t) \qquad \forall\, t \in \mathbb{R},$$

where (F3′) is the AR condition in the context of normalized solutions. For all $a > 0$, solutions to (1.9) correspond to critical points of the functional $\Phi^\infty : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\Phi^\infty(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \mathrm{d}x - \int_{\mathbb{R}^2} F(u) \mathrm{d}x \tag{1.11}$$

on the constraint $\mathcal{S}_a$. As a consequence of the Pohozaev identity (see [18, Lemma 2.7]), any solution u of (1.9) lives in the Pohozaev manifold given by

$$\mathcal{M}_a^\infty = \{u \in \mathcal{S}_a : J^\infty(u) = 0\}, \tag{1.12}$$

where J^∞ is called the Pohozaev functional defined by

$$J^\infty(u) = \|\nabla u\|_2^2 - \int_{\mathbb{R}^2} [f(u)u - 2F(u)] \mathrm{d}x \qquad \forall\, u \in H^1(\mathbb{R}^2). \tag{1.13}$$

Let

$$m^\infty(a) := \inf_{u \in \mathcal{M}_a^\infty} \Phi^\infty(u). \tag{1.14}$$

Their result reads as follows in this topic.

THEOREM [AJM] ([6, Theorem 1.2]). *Assume that f satisfies* (F1) *with $\alpha_0 = 4\pi$ and* (F2′), (F3′), *and* (F4′). *If $a \in (0,1)$, then there exists $\gamma^*(a) > 0$ such that* (1.9) *has a radial solution for all $\gamma \geq \gamma^*(a)$, where γ is given by* (F4′)*; moreover, this*

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solution can be chosen as a positive ground state solution if f also satisfies (H0) *and* (F5′).

The proof of Theorem [AJM] is based on the idea introduced by Jeanjean [18]; that is, working on the space $H_r^1(\mathbb{R}^2)$ and for every $a \in (0,1)$, construct a special PS sequence $\{u_n\} \subset \mathcal{S}_a^r := \mathcal{S}_a \cap H_r^1(\mathbb{R}^2)$ such that

$$\Phi^\infty(u_n) \to c_\gamma^\infty(a) > 0, \quad \Phi^\infty|'_{\mathcal{S}_a^r}(u_n) \to 0, \quad \text{and} \quad J^\infty(u_n) \to 0, \tag{1.15}$$

where the mountain pass level $c_\gamma^\infty(a)$ depends on γ in (F4′). To derive the compactness of $\{u_n\}$ in $H_r^1(\mathbb{R}^2)$, with assumptions (F3′) and (F4′), it is obtained that the key approximate evaluation is

$$\limsup_{n\to\infty} \|\nabla u_n\|_2^2 \le \frac{2(\mu_0 - 2)}{\mu_0 - 4} c_\gamma^\infty(a) \to 0 \quad \text{as} \quad \gamma \to \infty, \tag{1.16}$$

rather than a precise upper estimate of energy levels like that in the Sobolev critical case of higher dimensions. In fact, from (1.16), one sees that the energy level can be controlled arbitrarily small just by taking the parameter γ large enough. This perturbative way allows us to avoid the influences of critical exponential growth. Indeed, as long as $\limsup_{n\to\infty} \|\nabla u_n\|_2^2 < 1 - a$, which can be deduced from (1.16) by taking γ large enough, one can prove easily that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} |u_n|^s \left(e^{\alpha u_n^2} - 1\right) \mathrm{d}x = \int_{\mathbb{R}^2} |\bar{u}|^s \left(e^{\alpha \bar{u}^2} - 1\right) \mathrm{d}x \tag{1.17}$$

by using the Trudinger–Moser inequality and the compact embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for $s > 2$, and thus, one can derive the Brezis–Lieb property (1.10) and the convergence

$$\int_{\mathbb{R}^2} [f(u_n)u_n - f(\bar{u})\bar{u}] \, \mathrm{d}x = o(1) \tag{1.18}$$

if $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^2)$ in the same way as those for functions $f(u) \sim |u|^{q-2}u$ with $q > 2$. With (1.10) and (1.18), some powerful tools treating constrained problems with Sobolev critical growth in the higher dimensions are applicable for (1.9). To further obtain a ground state solution, following the arguments of [18, 27], it is established that $c_\gamma^\infty(a) = m^\infty(a)$, where $m^\infty(a)$ is given by (1.14). Note that in this argument, besides the $\mathcal{C}^1$ assumption (F5′), the odd assumption (H0) is required to work with sequences of functions that are Schwartz symmetric.

1.2. Highlights of the paper and main results. Inspired by aforementioned works, especially [6], it is very natural to pose a series of interesting questions, such as the following:

- (Q_1) Alves, Ji, and Miyagaki [6] only considered the case $a \in (0,1)$. A natural question is whether (1.9) has solutions or ground state solutions if $a \ge 1$?
- (Q_2) As we know, for the unconstrained problem (1.6), the energy threshold of the compactness of PS sequences is $2\pi/\alpha_0$. Does the same property hold true for the constrained problem (1.9)?
- (Q_3) If the energy threshold is available for (1.9) in (Q_2), can we find an alternative method to control precisely the energy value from above by the threshold instead of approximate evaluation like (1.16), which is essential in [6]?
- (Q_4) If the energy threshold is available for (1.9) in (Q_2), can we find an explicit lower bound $\gamma^*(a) > 0$ of γ, in place of the implicit expression of existence in [6, Theorem 1.2]?

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The first purpose of this paper is to solve the above questions and establish the existence of normalized solutions and ground state solutions to (1.9) for all $a>0$. In addition, we notice that (F5$'$) used in [6] is the special case of (H2) by taking $N=2$, which has been used and further weakened in the literature, such as (H4) and (H4$'$) mentioned above. It is natural to expect that (F5$'$) could also be weakened accordingly for the search of ground state solutions to (1.9). Our first result will confirm this expectation, as well as not requiring the odd assumption (H0) used in [6]. Before stating this result, besides (F1) and (F2), we introduce the following assumptions:

(F3) $f(t)t \geq 4F(t) := 4\int_0^t f(s)\mathrm{d}s > 0 \ \forall\, t \in \mathbb{R}\setminus\{0\}$;

(F4) $\liminf_{|t|\to\infty} \frac{t^2F(t)}{e^{\alpha_0 t^2}} > 0$;

(F5) there exist constants $M_0>0$ and $\beta_0>0$ such that

$$F(t) \leq M_0|f(t)| \quad \forall\ |t| \geq \beta_0;$$

(F6) $[f(t)t-2F(t)]/|t|^3t$ is nondecreasing on $(-\infty,0)$ and $(0,+\infty)$.

Obviously, (F2) and (F3) are weaker than (F2$'$) and (F3$'$), respectively. Assumptions (F4) and (F5) can date back to the pioneering works of de Figueiredo, Miyagaki, and Ruf [14] and Adimurthi [2], respectively, for the study of the planar unconstrained critical problem (1.6), both of which are reasonable for critical exponential growth functions satisfying (F1) since $f(t)$ behaves like $e^{\alpha_0 t^2}$ at infinity, as pointed out by Figueiredo and Severo [17]. In particular, (F4) is much weaker than the condition

$$\beta_0 := \lim_{|t|\to\infty} \frac{tf(t)}{e^{\alpha_0 t^2}} = +\infty \tag{1.19}$$

introduced in [2], even any relaxed versions of it in the existing literature, such as taking β_0 by a smaller positive constant. This type of condition permits using classical Moser-type functions as test functions to control the energy level by $2\pi/\alpha_0$ in the unconstrained context; see [14, 15]. As pointed out by Masmoudi and Sani [24, Remark 8.2], although (F4) and (F4$'$) both define the behaviors of the nonlinearity at infinity, it still seems to be difficult to compare them because the latter, performing as a global assumption, additionally specifies the growth condition at the origin. Our results in this direction are given in the following two statements.

THEOREM 1.2. *Assume that $a>0$ and f satisfies* (F1), (F2), (F3), (F4), *and* (F5). *Then* (i) (1.9) *has a radial solution;* (ii) (1.9) *has a ground state solution on $\mathcal{S}_a$ if further* (F6) *holds. Moreover, for any solution, the associated Lagrange multiplier λ is positive.*

Set

$$\gamma^*(a) := \frac{\mathcal{C}_p}{a(p-2)}\left[\frac{\alpha_0(p-4)}{4\pi(p-2)}\right]^{(p-4)/2} \ \text{ with } \ \mathcal{C}_p := \inf_{u\in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\|\nabla u\|_2^{p-2}\|u\|_2^2}{\|u\|_p^p}, \tag{1.20}$$

where p is given by (F4$'$).

THEOREM 1.3. *Assume that $a>0$ and that f satisfies* (F1), (F2), (F3), (F4$'$) *with $\gamma>\gamma^*(a)$, and* (F5). *Then, the conclusions of Theorem* 1.2 *hold.*

Remark 1.4.

(i) Theorems 1.2 and 1.3 give complete answers to questions (Q_1), (Q_2), (Q_3), and (Q_4), which may be mutually noninclusive due to assumptions (F4) and (F4$'$).

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(ii) With (F4) and (F4′), we implement two kinds of sharp estimates for energy levels, both of which differ considerably from previous works on normalized solutions. These subtle estimates should be useful for considering other L^2-constrained equations in critical exponential settings.

(iii) Theorem 1.2 can be considered as a counterpart of planar unconstrained critical problem (1.6), originally considered by Adimurthi [2] in the context of normalized solutions, and seems to be the first result on this topic. As in the unconstrained context, estimates of the energy levels depend on the asymptotic behavior of $\frac{t^2F(t)}{e^{\alpha_0 t^2}}$ at infinity. Somehow surprisingly, our hypothesis (F4) seems to be optimal in this respect, which is much weaker than (1.19), even any weakened versions of it in the previous works; however, it remains open whether this optimal hypothesis works in the unconstrained context.

(iv) Theorem 1.3 improves that of [6] in the sense that we not only extend the constrain condition $a \in (0,1)$ used in [6] to $a>0$ but also find an explicit value $\gamma^*(a)>0$ instead of an implicit expression in [6]; besides, our assumptions are weaker than those of [6].

Note that the proof for the existence of a radial solution to (1.9) in Theorems 1.2 and 1.3 can also be applied to the nonautonomous equation (1.1) with radial potential b satisfying

(B_r) $b \in \mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^+)$ is radial and bounded, $\inf_{x\in\mathbb{R}^2} b(x)>0$, and

$$-\infty < \inf_{x\in\mathbb{R}^2} \nabla b(x)\cdot x \le \sup_{x\in\mathbb{R}^2} \nabla b(x)\cdot x \le 0.$$

We have the following corollary.

COROLLARY 1.5. *Assume that $a>0$ and that* (B_r), (F1), (F2), (F3), *and* (F5) *hold. Then,* (1.1) *has a radial solution if f further satisfies either* (F4) *or* (F4′) *with $\gamma>\gamma^*(a)$.*

In the second part of this paper, we shall further study the existence of ground state solutions to nonautonomous form (1.1) with the variable potential $b(x)$ satisfying (B1). This type of variable potential originates from the classic work of Ding and Ni [16] that investigated the positive ground state solutions to the unconstrained Schrödinger equation

$$-\Delta u + u = b(x)f(u), \quad u\in H^1(\mathbb{R}^N), \tag{1.21}$$

where $N\ge 3$, potential b satisfies (B1) replacing $\mathbb{R}^2$ with $\mathbb{R}^N$, and nonlinearity f has Sobolev subcritical growth. Nowadays, it has been widely used in studying various kinds of unconstrained elliptic equations and systems in the subcritical or critical setting, which, as mentioned by Wang and Zeng [31], can be considered as a converse direction of the classical Rabinowitz-type trapping potential introduced by Rabinowitz [26]. In sharp contrast to the above unconstrained case, regarding this type of variable potential, the study of normalized solutions is almost unexplored in the literature. To our knowledge, the first and currently the only paper in this respect is [13], where the authors considered the Sobolev subcritical case in the dimensions $N\ge 3$; nevertheless, nothing is known in the Trudinger–Moser critical setting this paper focuses on. The present paper seems to be the first attempt to generalize the previous results in (1.21) to a new L^2-constrained scenario (1.1) in the Trudinger–Moser critical setting. A novelty of the proof lies in the subtle combination of new strategies, which shall be introduced in the process of treating autonomous form (1.9), and the comparison

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argument in the unconstrained context developed by Rabinowitz [26] to restore the compactness of PS sequence $\{u_n\}$, which is more delicate because one has to deal with the lack of both translation invariance of Φ and the noncompactness of the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^s(\mathbb{R}^2)$ for $s>2$, as well as whether $\bar{u} \in \mathcal{S}_a$ is unknown if $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^2)$. To state our results in this direction, on (1.1), besides (B1), (F1), (F2), (F3), and (F4) (or (F4′)), we also introduce the following conditions:

(F7) There exists a constant $\theta \in [2,4]$ such that $[f(t)t-\theta F(t)]/|t|^3 t$ is nondecreasing on $(-\infty,0)$ and $(0,+\infty)$;

(B2) $b \in \mathcal{C}^1(\mathbb{R}^2,\mathbb{R})$, and $t \mapsto (\theta-2)b(tx) - \nabla b(tx)\cdot(tx)$ is nonincreasing on $(0,\infty)$ for every $x \in \mathbb{R}^2$;

(B3) $2b(x) + \nabla b(x)\cdot x \geq 0 \ \forall\ x \in \mathbb{R}^2$, and the inequality strictly holds for some $\Lambda \subset \mathbb{R}^2$.

Without loss of generality, we may assume that $b_\infty = 1$. Our results are as follows.

THEOREM 1.6. *Assume that $a>0$ and that* (B1), (B2), (B3), (F1), (F2), (F3), (F4), *and* (F7) *hold. Then,* (1.1) *has a ground state solution on $\mathcal{S}_a$, and the associated Lagrange multiplier λ is positive.*

THEOREM 1.7. *Assume that $a>0$ and that* (B1), (B2), (B3), (F1), (F2), (F3), (F4′) *with $\gamma > \gamma^*(a)$, and* (F7) *hold. Then, the conclusions of Theorem* 1.6 *hold.*

We recall that any solution of (1.1) lives in the L^2-Pohozaev manifold given by

$$\mathcal{M}_a = \left\{ u \in \mathcal{S}_a : J(u) := \frac{\mathrm{d}}{\mathrm{d}t}\Phi(tu_t)\Big|_{t=1} = 0 \right\}. \tag{1.22}$$

For given $a>0$, we identify the suspected ground state solution energy

$$m_a := \inf_{u \in \mathcal{M}_a} \Phi(u). \tag{1.23}$$

If a solution u_a of (1.1) satisfies $\Phi(u_a) = m_a$, then it is a ground state solution.

Remark 1.8. As will be seen, to restore the compactness when searching for ground state solutions, we need to obtain the monotonicity of $a \mapsto m^\infty(a)$ and $a \mapsto m(a)$ (see Lemmas 3.6 and 4.5), which turns out to be the key ingredient. As a by-product of this study, we can further establish the following asymptotic behaviors of the ground state energy (see Lemma 4.6 and Corollary 4.7):

$$\lim_{a\to 0^+} m^\infty(a) = +\infty, \lim_{a\to+\infty} m^\infty(a) = 0, \lim_{a\to 0^+} m(a) = +\infty, \quad \text{and} \quad \lim_{a\to+\infty} m(a) = 0.$$

Remark 1.9. Theorems 1.6 and 1.7, to some extent, can be viewed as the extension of existence results on ground state solutions to unconstrained Schrödinger problems with the Rabinowitz-type trapping potential in the context of normalized solutions. To our knowledge, there have not been any similar results in the literature in this respect.

1.3. Sketch of the proofs. First, we treat the autonomous problem (1.9) and give ideas of the proofs of Theorems 1.2 and 1.3. To obtain a radial solution of (1.9) for any $a>0$, following the argument developed in Jeanjean [18] and working in the space $H_r^1(\mathbb{R}^2)$, we first construct a special PS sequence $\{u_n\} \subset \mathcal{S}_a^r := \mathcal{S}_a \cap H_r^1(\mathbb{R}^2)$ satisfying

$$\Phi^\infty(u_n) \to c_r^\infty(a) > 0, \quad \Phi^\infty|'_{\mathcal{S}_a^r}(u_n) \to 0, \quad \text{and} \quad J^\infty(u_n) \to 0 \tag{1.24}$$

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for every $a>0$; see section 3.1. Since the Nehari manifold cannot contribute here due to the presence of unknown Lagrange multipliers, it is the additional property related to the Pohozaev identity $J^\infty(u_n)\to 0$ that allows us to obtain the boundedness of $\{\|u_n\|\}$, whose proof does not require AR condition (F3$'$) used in [6]. As noted before, a major difficulty lies in the analysis of the convergence of $\{u_n\}$. Indeed, due to the simultaneous appearance of the L^2-constraint and the nonlinear term with critical exponential growth, several classical tools are not available anymore, and this forces the implementation of new strategies and techniques to restore the compactness of $\{u_n\}$, which are summarized as follows.

(I) Obtain a sharp upper estimate $c_r^\infty(a)<2\pi/\alpha_0$ instead of the approximate evaluation like (1.16) in [6].
With (F4) and (F4$'$), we shall propose two different strategies to get this fine estimate. The new strategies will conquer the aforementioned difficulty (i), which read as follows.
- The case (F4) holds.
Note that the Moser-type functions used in the unconstrained context do not work anymore since both testing functions and testing paths must be restricted on the set $\mathcal{S}_a$. To settle this issue, we construct a finer path with a new sequence of testing functions in an ingenious way and successfully control energy to make sure that it is less than the threshold $2\pi/\alpha_0$ by means of (F4); see Lemmas 2.8 and 2.9.
- The case (F4$'$) holds.
Different from that of [6], we select skillfully a sequence of functions related to the Gagliardo–Nirenberg inequality as testing functions. It is this special choice that permits us to find an explicit lower bound $\gamma^*(a)>0$ of γ to control precisely energy with (F4$'$); see Lemma 2.10.

(II) Prove that $u_n\to\bar u$ in $H_r^1(\mathbb{R}^2)$ for some $\bar u\in H_r^1(\mathbb{R}^2)\setminus\{0\}$, up to a subsequence, which conquer the aforementioned difficulty (ii).
Having at hand the above conclusion (I), we first prove that up to a subsequence, $\{u_n\}$ has a nontrivial weak limit $\bar u\in H^1(\mathbb{R}^2)$ with $\|\bar u\|_2^2\in(0,a]$ provided $c_r^\infty(a)<2\pi/\alpha_0$. However, it is not sufficient to justify that $\bar u$ is a normalized solution of (1.9) because whether $\|\bar u\|_2^2=a$ is unclear. To verify it, we prove that there exists a Lagrange multiplier $\bar\lambda>0$ such that $-\Delta\bar u+\bar\lambda u=f(\bar u)$ and conclude the compactness of $\{u_n\}$ from $\bar\lambda>0$. This strategy effectively addresses the lack of the compactness on f in $L^1(\mathbb{R}^2)$ and the Brezis–Lieb property in $H_r^1(\mathbb{R}^2)$.

The proof for the existence of ground state solutions to (1.9) consists of the following steps.

Step 1. Working directly in $H^1(\mathbb{R}^2)$ in place of $H_r^1(\mathbb{R}^2)$ used above, we construct a certain PS sequence $\{u_n\}\subset\mathcal{S}_a$ of Φ^∞ with the additional property $J^\infty(u_n)\to 0$ at the level $c^\infty(a)$ (see (3.28) below), which is bounded in $H^1(\mathbb{R}^2)$.

Step 2. By adapting the argument of the Nehari manifold in the unconstrained context and combining some new inequalities related with $\Phi^\infty(u)$ and $J^\infty(u)$, we prove that $c^\infty(a)=m^\infty(a)$; see Lemma 3.5.

Step 3. By developing more robust arguments than before, we prove the compactness of $\{u_n\}$, up to translations.

Note that Step 3 turns out to be the most challenging and requires a deeper analysis and more delicate techniques. Indeed, the proof of the compactness in the aforementioned (II) uses essentially the fact that the embedding $H_r^1(\mathbb{R}^2)\hookrightarrow L^s(\mathbb{R}^2)$ is compact for any $s>2$; it fails in $H^1(\mathbb{R}^2)$ since there is no odd condition (H0), and we can-

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not work with sequences of functions that are Schwartz symmetric, like that in the almost all previous related works. To address this issue, we show that the ground state energy map $a \mapsto m^\infty(a)$ is nonincreasing on $(0, \infty)$, with the help of (F6) as noted in Remark 1.8, and, by making full use of this monotonicity, we prove that, up to a subsequence and up to translations, there is a weak limit $\bar{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$ of $\{u_n\}$ such that $\|u_n - \bar{u}\|_2 \to 0$, where $\bar{u} \neq 0$ can be obtained by controlling the energy level as in the above (I). It permits reducing the problem of strong convergence to the one of showing the positivity of the Lagrange multiplier like that in the last part of the above (II). This strategy was initially proposed by Bellazzini, Jeanjean, and Luo [7], who studied Schrödinger–Poisson systems, and was further developed by Jeanjean and Lu [21] for the study of (1.9) with Sobolev subcritical growth. One should, however, note that the argument in [21] used essentially the convergence of $f(u_n)u_n$ in $L^1(\mathbb{R}^2)$, and so it is unavailable in the critical exponential setting. It is worth pointing out that by developing new arguments, we confirm that the above strategy is also effective for (1.9) with critical exponential growth without (H0). Our arguments are more robust and involved than before in the sense that they require neither the convergence of $f(u_n)u_n$ in $L^1(\mathbb{R}^2)$ nor the Brezis–Lieb property (1.10) and should be adapted to treat other L^2-constrained problems with critical exponential growth.

Finally, we deal with the more complex nonautonomous problem (1.1) and present crucial ingredients for the proofs of Theorems 1.6 and 1.7. Our proofs are based on the line of proofs for the search of ground state solutions to (1.9). However, special care and extra effort are always needed due to the aforementioned unpleasant facts in the presence of the variable potential $b(x)$, summarized as follows.

- Derive that the obtained mountain pass level equals the ground state energy like those of Step 1 and Step 2 in (1.1).
  For this, we have to carefully analyze the behaviors of both $t \mapsto b(tx)$ and $t \mapsto \nabla b(tx) \cdot tx$ for any $x \in \mathbb{R}^2$ and develop more technical arguments so that key inequalities on the energy functional and the Pohozaev functional of (1.9) can be extended to the nonautonomous case (1.1), where a slightly stronger monotonicity condition (F6) compared to (F5) is required.
- Prove the nontriviality of the weak limit of the obtained PS sequence $\{u_n\}$, up to a subsequence.
  Note that for this, the argument treating (1.9) before is invalid because the corresponding energy no longer has translation invariance. To overcome this lack, we first compare ground state energies between the nonautonomous problem (1.1) and its "limit problem" (1.9) and conclude that $m(a) < m^\infty(a)$, then prove that either a weak limit of $\{u_n\}$ is nontrivial or that $m(a) \geq m^\infty(a)$; thus, this contradiction ends the proof. Although this idea comes from Rabinowitz [26] and has been used in many other contexts, the related proofs explored in the previous works fail to work because they require the concentration-compactness principle due to Lions [23] and the Brezis–Lieb property like (1.10). This forces us to develop more robust arguments to complete the proof.

1.4. Organization of the paper. In section 2, we present some preliminary results for (1.1), which will be used in the rest of the paper. In section 3, we study the existence of radial solutions and ground state solutions for autonomous equation (1.9) and complete the proof of Theorem 1.2. Section 4 is devoted to the study of the existence of ground state solutions for (1.1) and finishing the proof of Theorem 1.6, where the proofs of Theorems 1.3 and 1.6 are given in Remark 4.10.

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Throughout the paper, we make use of the following notations:

• $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space equipped with the inner product and norm

$$(u,v)=\int_{\mathbb{R}^2}(\nabla u\cdot\nabla v+uv)\mathrm{d}x,\quad \|u\|=(u,u)^{1/2}\qquad \forall\ u,v\in H^1(\mathbb{R}^2);$$

• $H_r^1(\mathbb{R}^2)$ denotes the space of spherically symmetric functions belonging to $H^1(\mathbb{R}^2)$:

$$H_r^1(\mathbb{R}^2):=\{u\in H^1(\mathbb{R}^2)\ \big|\ u(x)=u(|x|)\text{ a.e. in }\mathbb{R}^2\};$$

• $L^s(\mathbb{R}^2)(1\le s<\infty)$ denotes the Lebesgue space with the norm $\|u\|_s=(\int_{\mathbb{R}^2}|u|^s\mathrm{d}x)^{1/s}$;

• For any $u\in H^1(\mathbb{R}^2)\setminus\{0\}$, $u_t(x):=u(tx)$ for $t>0$;

• For any $x\in\Omega$ and $r>0$, $B_r(x):=\{y\in\Omega:|y-x|<r\}$ and $B_r=B_r(0)$;

• $C_1,C_2,\dots$ denote positive constants possibly different in different places, which are dependent on $a>0$.

2. Preliminary results. Under assumptions (F1) and (F2), fixing $\alpha>\alpha_0$, we know that, for any $\varepsilon>0$ and any $q\ge 1$, there exists $C_{\alpha,\varepsilon,q}>0$ such that

$$|f(t)|\le\varepsilon|t|^3+C_{\alpha,\varepsilon,q}e^{\alpha t^2}|t|^q\qquad\forall\ t\in\mathbb{R};\tag{2.1}$$

moreover, using (2.1), we deduce that for any $\varepsilon>0$, there exists $C_{\alpha,\varepsilon}>0$ such that

$$|F(t)|\le\varepsilon|t|^4+C_{\alpha,\varepsilon}e^{\alpha t^2}|t|^4\qquad\forall\ t\in\mathbb{R}.\tag{2.2}$$

In this section, we always assume $b\in\mathcal{C}^1(\mathbb{R}^2,\mathbb{R}^+)\cap L^\infty(\mathbb{R}^2,\mathbb{R}^+)$ with $\inf_{x\in\mathbb{R}^2}b(x)\ge 1$ and $-\infty<\inf_{x\in\mathbb{R}^2}\nabla b(x)\cdot x\le\sup_{x\in\mathbb{R}^2}\nabla b(x)\cdot x\le 0$. By Lemma 1.1 and (2.2), we have $\Phi\in\mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$.

Noting that (F1) and (F2) imply that Φ is no longer bounded from below on $\mathcal{S}_a$, we shall look for a critical point satisfying a minimax characterization. For this, we give the following definition.

DEFINITION 2.1. *For given $a>0$, we say that Φ possesses a mountain pass geometry on $\mathcal{S}_a$ if there exists $\rho_a>0$ such that*

$$c(a):=\inf_{g\in\Gamma_a}\max_{\tau\in[0,1]}\Phi(g(\tau))>\max_{g\in\Gamma_a}\max\{\Phi(g(0)),\Phi(g(1))\},\tag{2.3}$$

where $\Gamma_a:=\{g\in\mathcal{C}([0,1],\mathcal{S}_a):\|\nabla g(0)\|_2^2\le\rho_a,\Phi(g(1))<0\}$.

We want to prove that for any $a>0$, there exist PS sequences for Φ restricted to $\mathcal{S}_a$ at the level $c(a)$. To derive the boundedness of PS sequences, we manage to look for more information related to the L^2-Pohozaev identity, which will be obtained in the following subsection. From now on, we always let $a>0$.

2.1. A special PS sequence with the extra property. In this subsection, we shall find a special PS sequence $\{u_n\}$ of Φ restricted on $\mathcal{S}_a$ with the extra property $J(u_n)\to 0$, where

$$\Phi(tu_t)=\frac{t^2}{2}\|\nabla u\|_2^2-\frac{1}{t^2}\int_{\mathbb{R}^2}b(x/t)F(tu)\mathrm{d}x\qquad\forall\ t>0,\ u\in H^1(\mathbb{R}^2)\tag{2.4}$$

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and

$$J(u) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(tu_t)\Big|_{t=1} = \|\nabla u\|_2^2 + \int_{\mathbb{R}^2}[2b(x)+\nabla b(x)\cdot x]F(u)\mathrm{d}x - \int_{\mathbb{R}^2} b(x)f(u)u\mathrm{d}x \quad \forall\ u\in H^1(\mathbb{R}^2). \tag{2.5}$$

To this end, we first prove that Φ has a mountain pass geometry on the constraint $\mathcal{S}_a$; this reads as follows.

Lemma 2.2. *Assume that* (F1), (F2), *and* (F3) *hold. Then,*

(i) *there exists $K(a)>0$ small enough that $\Phi(u)>0$ and $J(u)>0$ if $u\in A_{2K}$ and*

$$0<\sup_{u\in A_K}\Phi(u)<\inf\left\{\Phi(u): u\in\mathcal{S}_a, \|\nabla u\|_2^2 = 2K(a)\right\}, \tag{2.6}$$

where

$$A_K = \left\{u\in\mathcal{S}_a : \|\nabla u\|_2^2\le K(a)\right\} \quad \textit{and} \quad A_{2K} = \left\{u\in\mathcal{S}_a : \|\nabla u\|_2^2 \le 2K(a)\right\}; \tag{2.7}$$

(ii) $\Gamma_a = \{g\in\mathcal{C}([0,1],\mathcal{S}_a) : \|\nabla g(0)\|_2^2\le K(a), \Phi(g(1))<0\}\ne\emptyset$ *and*

$$\begin{aligned} c(a) = \inf_{g\in\Gamma_a}\max_{t\in[0,1]}\Phi(g(t)) &\ge \inf\left\{\Phi(u): u\in\mathcal{S}_a, \|\nabla u\|_2^2 = 2K(a)\right\}\\ &> \max_{g\in\Gamma_a}\max\{\Phi(g(0)),\Phi(g(1))\}. \end{aligned} \tag{2.8}$$

Proof. (i) By the Gagliardo–Nirenberg inequality, we have

$$\|u\|_s^s \le \mathcal{C}_s^{-1}\|u\|_2^2\|\nabla u\|_2^{s-2} \quad \text{for} \quad s>2, \tag{2.9}$$

where $\mathcal{C}_s>0$ is a constant determined by s. Using (ii) of Lemma 1.1, we have

$$\begin{aligned} \int_{\mathbb{R}^2}\left(e^{2\alpha u^2}-1\right)\mathrm{d}x &= \int_{\mathbb{R}^2}\left(e^{2\alpha\|\nabla u\|_2^2(u/\|\nabla u\|_2)^2}-1\right)\mathrm{d}x\\ &\le C_1 \quad \forall\ u\in\mathcal{S}_a, \|\nabla u\|_2\le\sqrt{\pi/\alpha}. \end{aligned} \tag{2.10}$$

By (2.1), (2.2), (2.9), and (2.10), we have

$$\begin{aligned} &\int_{\mathbb{R}^2} b(x)F(u)\mathrm{d}x + \int_{\mathbb{R}^2} b(x)[f(u)u-2F(u)]\mathrm{d}x + \int_{\mathbb{R}^2}|\nabla b(x)\cdot x|F(u)\mathrm{d}x\\ &\quad\le \varepsilon\|u\|_4^4 + C_\varepsilon\int_{\mathbb{R}^2}\left(e^{\alpha u^2}-1\right)|u|^4\mathrm{d}x\\ &\quad\le \varepsilon\mathcal{C}_4^{-1}a\|\nabla u\|_2^2 + C_\varepsilon\left[\int_{\mathbb{R}^2}\left(e^{2\alpha u^2}-1\right)\mathrm{d}x\right]^{1/2}\|u\|_8^4\\ &\quad\le \varepsilon\mathcal{C}_4^{-1}a\|\nabla u\|_2^2 + C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}\|\nabla u\|_2^3 \quad \forall\ u\in\mathcal{S}_a, \|\nabla u\|_2\le\sqrt{\pi/\alpha}. \end{aligned} \tag{2.11}$$

Now, let $0<K<\pi/\alpha$ be arbitrary but fixed, and suppose that $u,u_0,v,v_0\in\mathcal{S}_a$ such that $\|\nabla u\|_2^2\le K$, $\|\nabla v\|_2^2\le 2K$, $\|\nabla u_0\|_2^2 = K/2$, and $\|\nabla v_0\|_2^2 = 2K$. From (1.4), (2.5), and (2.11), by setting $\varepsilon = \mathcal{C}_4/(8a)$ in (2.11), we deduce that, for small enough $K>0$,

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$$\begin{aligned}\Phi(v) &= \frac{1}{2}\|\nabla v\|_2^2 - \int_{\mathbb{R}^2} b(x)F(v)\mathrm{d}x \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - 2a\varepsilon\mathcal{C}_4^{-1}\|\nabla v\|_2^2 - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(\|\nabla v\|_2^2)^{3/2} \\ &= \frac{1}{4}\|\nabla v\|_2^2 - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(\|\nabla v\|_2^2)^{3/2} > 0, \qquad (2.12)\end{aligned}$$

$$\begin{aligned}J(v) &= \|\nabla v\|_2^2 + \int_{\mathbb{R}^2}[2b(x)+\nabla b(x)\cdot x]F(v)\mathrm{d}x - \int_{\mathbb{R}^2} b(x)f(v)v\mathrm{d}x \\ &\geq \frac{1}{2}\|\nabla v\|_2^2 - 2a\varepsilon\mathcal{C}_4^{-1}\|\nabla v\|_2^2 - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(\|\nabla v\|_2^2)^{3/2} \\ &= \frac{1}{4}\|\nabla v\|_2^2 - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(\|\nabla v\|_2^2)^{3/2} > 0, \qquad (2.13)\end{aligned}$$

$$\begin{aligned}\Phi(v_0)-\Phi(u) &= \frac{1}{2}\|\nabla v_0\|_2^2 - \int_{\mathbb{R}^2} b(x)F(v_0)\mathrm{d}x - \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^2} b(x)F(u)\mathrm{d}x \\ &\geq \frac{1}{2}K - 2a\varepsilon\mathcal{C}_4^{-1}K - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(2K)^{3/2} \\ &= \frac{1}{4}K - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(2K)^{3/2} \geq \frac{1}{8}K, \qquad (2.14)\end{aligned}$$

and

$$\begin{aligned}\Phi(u_0) &= \frac{1}{2}\|\nabla u_0\|_2^2 - \int_{\mathbb{R}^2} b(x)F(u_0)\mathrm{d}x \\ &\geq \frac{1}{4}K - a\varepsilon\mathcal{C}_4^{-1}K/2 - C_\varepsilon C_1^{1/2}\mathcal{C}_8^{-1/2}\sqrt{a}(K/2)^{3/2} \geq \frac{1}{8}K. \qquad (2.15)\end{aligned}$$

Using (2.12), (2.13), (2.14), and (2.15), we know that there exists $K = K(a) > 0$ sufficiently small that $\Phi(u) > 0$ and $J(u) > 0$ if $u \in A_{2K}$, and (2.6) holds.

(ii) We first prove that $\Gamma_a \neq \emptyset$. Using (F1) and (F3), it is easy to see that

$$\lim_{|t|\to+\infty} \frac{F(t)}{e^{\alpha_0 t^2/2}} = +\infty. \qquad (2.16)$$

For any given $w \in \mathcal{S}_a$, we have $\|tw_t\|_2 = \|w\|_2$, and so, $tw_t \in \mathcal{S}_a$ for every $t > 0$. Then, (2.4) and (2.16) yield that

$$\Phi(tw_t) \to -\infty \quad \text{as} \quad t \to +\infty. \qquad (2.17)$$

Thus, we can deduce that there exist $t_1 > 0$ small enough and $t_2 > 0$ large enough such that

$$\begin{aligned}&\|\nabla(t_1 w_{t_1})\|_2^2 = t_1^2\|\nabla w\|_2^2 \leq K(a), \\ &\|\nabla(t_2 w_{t_2})\|_2^2 = t_2^2\|\nabla w\|_2^2 > 2K(a), \text{ and } \Phi(t_2 w_{t_2}) < 0.\end{aligned} \qquad (2.18)$$

Let $g_0(t) := (t_1 + (t_2 - t_1)t)w_{t_1+(t_2-t_1)t}$. Then, $g_0 \in \Gamma_a$, and so, $\Gamma_a \neq \emptyset$. Now, using the intermediate value theorem, for any $g \in \Gamma_a$, there exists $t_0 \in (0,1)$, depending on g, such that $\|\nabla g(t_0)\|_2^2 = 2K(a)$ and

$$\max_{t\in[0,1]} \Phi(g(t)) \geq \Phi(g(t_0)) \geq \inf\left\{\Phi(u) : u \in \mathcal{S}_a, \|\nabla u\|_2^2 = 2K(a)\right\},$$

which, together with the arbitrariness of $g \in \Gamma_a$, implies that

$$c(a) = \inf_{g\in\Gamma_a}\max_{t\in[0,1]} \Phi(g(t)) \geq \inf\left\{\Phi(u) : u \in \mathcal{S}_a, \|\nabla u\|_2^2 = 2K(a)\right\}. \qquad (2.19)$$

Hence, (2.8) follows directly from (2.6) and (2.19), and the proof is completed. ∎

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Remark 2.3. From (2.13), we can deduce that, for any $a>0$, there exists a constant $\rho(a)>0$ just depending on $a>0$ such that $\|\nabla u\|_2 \geq \rho(a)$ for all $u \in \mathcal{M}_a$.

Let us define a continuous map $\beta : H := H^1(\mathbb{R}^2)\times\mathbb{R} \to H^1(\mathbb{R}^2)$ by

$$\beta(v,t)(x) = e^t v(e^t x) \quad \text{for } v \in H^1(\mathbb{R}^2),\ t\in\mathbb{R}, \text{ and } x \in \mathbb{R}^2 \tag{2.20}$$

and consider the following auxiliary functional:

$$\tilde{\Phi}(v,t) = \Phi(\beta(v,t)) = \frac{e^{2t}}{2}\|\nabla v\|_2^2 - \frac{1}{e^{2t}}\int_{\mathbb{R}^2} b(e^{-t}x)F(e^t v)\mathrm{d}x, \tag{2.21}$$

where H is a Banach space equipped with the scalar product

$$((v_1,s_1),(v_2,s_2))_H = (v_1,v_2) + s_1 s_2 \qquad \forall\ (v_i,s_i)\in H,\ i=1,2$$

and corresponding norm $\|(v,t)\|_H := \left(\|v\|^2 + |t|^2\right)^{1/2}$ for all $(v,s)\in H$. We see that $\tilde{\Phi}'(v,t)$ is of class $\mathcal{C}^1$, and for any $(w,s)\in H$,

$$\begin{aligned}
\left\langle \tilde{\Phi}'(v,t),(w,s)\right\rangle &= e^{2t}\int_{\mathbb{R}^2}\nabla v\cdot\nabla w\mathrm{d}x + e^{2t}s\|\nabla v\|_2^2 - \frac{1}{e^{2t}}\int_{\mathbb{R}^2} b(e^{-t}x)f(e^t v)e^t w\mathrm{d}x\\
&\quad + \frac{s}{e^{2t}}\int_{\mathbb{R}^2}\left[2b(e^{-t}x) + \nabla b(e^{-t}x)\cdot e^{-t}x\right]F(e^t v)\mathrm{d}x\\
&\quad - \frac{s}{e^{2t}}\int_{\mathbb{R}^2} b(e^{-t}x)f(e^t v)e^t v\mathrm{d}x\\
&= \langle \Phi'(\beta(v,t)),\beta(w,t)\rangle + sJ(\beta(v,t)).
\end{aligned}\tag{2.22}$$

We shall prove that $\tilde{\Phi}$ also possesses a kind of mountain pass geometrical structure on $\mathcal{S}_a\times\mathbb{R}$.

LEMMA 2.4. *Assume that* (F1), (F2), *and* (F3) *hold. Let $v\in\mathcal{S}_a$ be arbitrary but fixed. Then, we have the following:*

(i) $\|\nabla\beta(v,t)\|_2 \to 0$ *and* $\Phi(\beta(v,t))\to 0$ *as* $t\to-\infty$;

(ii) $\|\nabla\beta(v,t)\|_2 \to +\infty$ *and* $\Phi(\beta(v,t))\to -\infty$ *as* $t\to+\infty$;

(iii) *There exist $s_1<0$ and $s_2>0$, depending on a and v, such that the functions $\tilde{v}_1=\beta(v,s_1)$ and $\tilde{v}_2=\beta(v,s_2)$ satisfy*

$$\|\nabla\tilde{v}_1\|_2^2 \leq K(a), \quad \|\nabla\tilde{v}_2\|_2^2 > 2K(a), \quad \textit{and} \quad \Phi(\tilde{v}_2)<0.$$

Proof. (i) A straightforward calculation shows that

$$\|\beta(v,t)\|_2^2 = a \quad\text{and}\quad \|\nabla\beta(v,t)\|_2^2 = \frac{e^{2t}}{2}\|\nabla v\|_2^2 \qquad \forall\ t\in\mathbb{R}. \tag{2.23}$$

Since $e^{2t}\to 0$ as $t\to-\infty$, using (2.11), (2.21), and (2.23), we can deduce that (i) holds, arguing as in the proof of (i) of Lemma 2.2.

(ii) From (2.21) and the fact that $e^{2t}\to+\infty$ as $t\to+\infty$, item (ii) follows as in the proof of (2.17).

(iii) Item (iii) follows from (2.23) and the above items (i) and (ii). ∎

LEMMA 2.5. *Assume that* (F1), (F2), *and* (F3) *hold. Let $v\in\mathcal{S}_a$ be arbitrary but fixed. Then,*

$$c(a) = \tilde{c}(a) := \inf_{\tilde{g}\in\tilde{\Gamma}_a}\max_{\tau\in[0,1]}\tilde{\Phi}(\tilde{g}(\tau)) > \max_{\tilde{g}\in\tilde{\Gamma}_a}\max\left\{\tilde{\Phi}(\tilde{g}(0)),\tilde{\Phi}(\tilde{g}(1))\right\}, \tag{2.24}$$

where

$$\tilde{\Gamma}_a := \{\tilde{g}\in\mathcal{C}([0,1],\mathcal{S}_a\times\mathbb{R}) : \tilde{g}(0) = (\tilde{g}_1(0),0), \|\nabla\tilde{g}_1(0)\|_2^2\leq K(a), \tilde{\Phi}(\tilde{g}(1))<0\}.$$

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Proof. Note that $\tilde{\Gamma}_a \neq \emptyset$ due to (iii) of Lemma 2.4. Since $\Gamma_a = \{\beta \circ \tilde{g} : \tilde{g} \in \tilde{\Gamma}_a\}$, i.e., $c(a) = \tilde{c}(a)$, (2.8) and (2.21) lead to

$$\tilde{c}(a) = c(a) > \max_{g \in \Gamma_a} \max\{\Phi(g(0)), \Phi(g(1))\} = \max_{\tilde{g} \in \tilde{\Gamma}_a} \max\{\tilde{\Phi}(\tilde{g}(0)), \tilde{\Phi}(\tilde{g}(1))\}.$$

This completes the proof. ∎

Following [33], we know that, for any $a > 0$, $\mathcal{S}_a$ is a submanifold of $H^1(\mathbb{R}^2)$ with codimension 1, and the tangent space at $\mathcal{S}_a$ is defined as

$$T_u = \left\{ v \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} uv \mathrm{d}x = 0 \right\}. \tag{2.25}$$

The norm of the $\mathcal{C}^1$ restriction functional $\Phi|'_{\mathcal{S}_a}(u)$ is defined by

$$\|\Phi|'_{\mathcal{S}_a}(u)\| = \sup_{v \in T_u, \|v\|=1} \langle \Phi'(u), v \rangle . \tag{2.26}$$

As in Jeanjean [18], for every $(u,t) \in \mathcal{S}_a \times \mathbb{R}$, we define the following linear space

$$\tilde{T}_{u,t} = \left\{ (v,s) \in H : \int_{\mathbb{R}^2} uv \mathrm{d}x = 0 \right\} \tag{2.27}$$

and the norm of the derivative of the $\mathcal{C}^1$ restriction functional $\tilde{\Phi}|_{\mathcal{S}_a \times \mathbb{R}}$ by

$$\|\tilde{\Phi}|'_{\mathcal{S}_a \times \mathbb{R}}(u,t)\| = \sup_{(v,s) \in \tilde{T}_{u,t}, \|(v,s)\|_H = 1} \left\langle \tilde{\Phi}|'_{\mathcal{S}_a \times \mathbb{R}}(u,t), (v,s) \right\rangle . \tag{2.28}$$

In the same way as [18, Proposition 2.2], we have the following proposition.

PROPOSITION 2.6. *Assume that $\tilde{\Phi}$ has a mountain pass geometry on the constraint $\mathcal{S}_a \times \mathbb{R}$. Let $a > 0$ and $\{\tilde{g}_n\} \subset \tilde{\Gamma}_a$ be such that*

$$\max_{\tau \in [0,1]} \tilde{\Phi}(\tilde{g}_n(\tau)) \leq \tilde{c}(a) + \frac{1}{n} \quad \forall\, n \in \mathbb{N}. \tag{2.29}$$

Then, there exists a sequence $\{(v_n, t_n)\} \subset \mathcal{S}_a \times \mathbb{R}$ such that

(i) $\tilde{\Phi}(v_n, t_n) \in \left[\tilde{c}(a) - \frac{1}{n}, \tilde{c}(a) + \frac{1}{n}\right]$;

(ii) $\min_{\tau \in [0,1]} \|(v_n, t_n) - \tilde{g}_n(\tau)\|_H \leq \frac{1}{\sqrt{n}}$;

(iii) $\|\tilde{\Phi}|'_{\mathcal{S}_a \times \mathbb{R}}(v_n, t_n)\| \leq \frac{2}{\sqrt{n}}$; *i.e.,*

$$|\langle \tilde{\Phi}'(v_n, t_n), (v,s) \rangle| \leq \frac{2}{\sqrt{n}} \|(v,s)\|_H \quad \forall\, (v,s) \in \tilde{T}_{v_n, t_n}.$$

Note that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \tilde{\Phi}(v,t) &= \left\langle \tilde{\Phi}'(v,t), (0,1) \right\rangle \\
&= e^{2t} \|\nabla v\|_2^2 + \frac{1}{e^{2t}} \int_{\mathbb{R}^2} \left[2b(e^{-t}x) + \nabla b(e^{-t}x) \cdot e^{-t}x \right] F(e^t v) \mathrm{d}x \\
&\quad - \frac{1}{e^{2t}} \int_{\mathbb{R}^2} b(e^{-t}x) f(e^t v) e^t v \mathrm{d}x \\
&= J(\beta(v,t)) \quad \forall\, (v,t) \in H.
\end{aligned} \tag{2.30}$$

With the aforementioned lemmas, we can get the desired sequence as follows.

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LEMMA 2.7. *Assume that* (F1), (F2), *and* (F3) *hold. Then, there exists a bounded sequence* $\{u_n\} \subset \mathcal{S}_a$ *such that*

$$\Phi(u_n) \to c(a) > 0, \quad \Phi|'_{\mathcal{S}_a}(u_n) \to 0, \quad \text{and} \quad J(u_n) \to 0. \tag{2.31}$$

Proof. Let

$$u_n = \beta(v_n, t_n) \quad \text{and} \quad g_n(\tau) = \beta(\tilde{g}_n(\tau)) \text{ for } \tau \in [0,1], \tag{2.32}$$

where β is defined by (2.20); v_n, t_n, and $\tilde{g}_n$ are given in Proposition 2.6. Then, $u_n \in \mathcal{S}_a$ and $g_n \in \Gamma_a$ by (2.23) and (ii) of Lemma 2.2. Moreover, by (2.21), (2.30), Lemma 2.5, and Proposition 2.6, we have

$$\Phi(u_n) = \tilde{\Phi}(v_n, t_n) \in \left[c(a) - \frac{1}{n}, c(a) + \frac{1}{n}\right] \tag{2.33}$$

and

$$J(u_n) = \left\langle \tilde{\Phi}'(v_n, t_n), (0,1) \right\rangle \to 0. \tag{2.34}$$

Using (F3) and (F5), we know that, for any $\delta > 0$, there exists $R_\delta > 0$ such that

$$f(t)t \geq \delta F(t) > 0 \qquad \forall\ |t| \geq R_\delta. \tag{2.35}$$

Noting that $b(x) \geq 1$ and $\nabla b(x) \cdot x \leq 0\ \forall\ x \in \mathbb{R}^2$, it follows from (2.33), (2.34), and (2.35) with $\delta = 8$ that

$$\begin{aligned} c(a) + o(1) &= \Phi(u_n) - \frac{1}{4}J(u_n) \\ &= \frac{1}{4}\|\nabla u_n\|_2^2 + \frac{1}{4}\int_{|u_n|<R_8} b(x)\left[f(u_n)u_n - 6F(u_n)\right]\mathrm{d}x \\ &\quad + \frac{1}{4}\int_{|u_n|\geq R_8} b(x)\left[f(u_n)u_n - 6F(u_n)\right]\mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^2} \nabla b(x)\cdot x F(u_n)\mathrm{d}x \\ &\geq \frac{1}{4}\|\nabla u_n\|_2^2 + \frac{3}{16}\int_{|u_n|\geq R_8} b(x) f(u_n)u_n \mathrm{d}x - C_2\|u_n\|_2^2, \end{aligned} \tag{2.36}$$

which implies that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. To finish the proof, it remains to prove that $\Phi|'_{\mathcal{S}_a}(u_n) \to 0$; i.e., $\langle \Phi'(u_n), w\rangle \to 0$ for all $w \in T_{u_n}$. For this, we just need to show that $\{(\beta(w, -t_n), 0)\} \subset T_{v_n, t_n}$ and $\{(\beta(w, -t_n), 0)\}$ is bounded in H since

$$\langle \Phi'(u_n), w\rangle = \left\langle \tilde{\Phi}'(v_n, t_n), (\beta(w, -t_n), 0)\right\rangle \leq \frac{2}{\sqrt{n}}\|(\beta(w, -t_n), 0)\|_H \qquad \forall\ w \in T_{u_n}. \tag{2.37}$$

Indeed, for any $w \in T_{u_n}$, i.e.,

$$\int_{\mathbb{R}^2} u_n w \mathrm{d}x = \int_{\mathbb{R}^2} e^{t_n} v_n(e^{t_n}x) w(x)\mathrm{d}x = 0,$$

we have

$$\int_{\mathbb{R}^2} v_n(x)\beta(w, -t_n)(x)\mathrm{d}x = \int_{\mathbb{R}^2} v_n(x) e^{-t_n} w(e^{-t_n}x)\mathrm{d}x = \int_{\mathbb{R}^2} e^{t_n} v_n(e^{t_n}x) w(x)\mathrm{d}x = 0,$$

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which implies that

$$(\beta(w,-t_n),0)\in T_{v_n,t_n}. \tag{2.38}$$

Moreover, by (ii) of Proposition 2.6, we have

$$|t_n|\le \min_{\tau\in[0,1]}\|(v_n,t_n)-\tilde{g}_n(\tau)\|_H\le 1 \text{ for large } n\in\mathbb{N},$$

which leads to

$$\|(\beta(w,-t_n),0)\|_H^2=\|\beta(w,-t_n)\|^2=e^{-2t_n}\|\nabla w\|_2^2+\|w\|_2^2\le\ e^2\|w\|^2 \text{ for large } n\in\mathbb{N}. \tag{2.39}$$

This shows that $\{(\beta(w,-t_n),0)\}\subset T_{v_n,t_n}$ is bounded in H. Jointly with (2.38), we get $\Phi|'_{\mathcal{S}_a}(u_n)\to 0$. From this, (2.33), and (2.34), we conclude that $\{u_n\}$, defined by (2.32), is bounded and satisfies (2.31). The proof is completed. ∎

To overcome the difficulties caused by the critical exponential growth in the context of normalized solutions, we need to establish some crucial energy estimates for the minimax level $c(a)$ given by (2.8), which are given in the next subsection.

2.2. Energy estimates for minimax level. In this subsection, we give a precise estimation for the energy level $c(a)$ given by (2.8), which helps us to restore the compactness in the critical exponential case.

Let $\kappa:=\liminf_{|t|\to\infty}\frac{t^2F(t)}{e^{\alpha_0 t^2}}$. By (F4), we know that $\kappa>0$. Then, we can choose $d>0$ such that $\kappa>\frac{4\pi}{ed^2\alpha_0^3}$. For large $n\in\mathbb{N}$, let $R_n\ge d$ be such that

$$\begin{aligned}a=&\frac{d^2}{16\log n}\left(1+2\log 2+2\log^2 2-\frac{4}{n^2}-\frac{8}{n^2}\log n\right)\\&+\frac{\log^2 2}{48(2R_n-d)\log n}\left(8R_n^3+4R_n^2d-10R_nd^2+3d^3\right).\end{aligned} \tag{2.40}$$

Then, one has

$$\lim_{n\to\infty}\frac{R_n^2}{\log n}=\frac{12a}{\log^2 2}. \tag{2.41}$$

Now, we define the following new Moser-type functions $w_n(x)$ supported in $B_{R_n}:=B_{R_n}(0)$:

$$w_n(x)=\frac{1}{\sqrt{2\pi}}\begin{cases}\sqrt{\log n}, & 0\le|x|\le d/n,\\ \frac{\log(d/|x|)}{\sqrt{\log n}}, & d/n\le|x|\le d/2,\\ \frac{2(R_n-|x|)\log 2}{(2R_n-d)\sqrt{\log n}}, & d/2\le|x|\le R_n,\\ 0, & |x|\ge R_n.\end{cases} \tag{2.42}$$

Computing directly, we get that, for large $n\in\mathbb{N}$,

$$\|\nabla w_n\|_2^2=\int_{\mathbb{R}^2}|\nabla w_n|^2\mathrm{d}x=1-\frac{\log 2}{\log n}+\frac{(2R_n+d)\log^2 2}{2(2R_n-d)\log n}\le 1 \tag{2.43}$$

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and

$$
\begin{aligned}
\|w_n\|_2^2 &= \int_{\mathbb{R}^2} |w_n|^2 \mathrm{d}x \\
&= \int_0^{d/n} (\log n) r\mathrm{d}r + \int_{d/n}^{d/2} \frac{\log^2(d/r)}{\log n} r\mathrm{d}r + \int_{d/2}^{R_n} \frac{4(R_n-r)^2 \log^2 2}{(2R_n-d)^2 \log n} r\mathrm{d}r \\
&= \frac{d^2}{16 \log n}\left(1 + 2\log 2 + 2\log^2 2 - \frac{4}{n^2} - \frac{8}{n^2}\log n\right) \\
&\quad + \frac{\log^2 2}{48(2R_n-d)\log n}\left(8R_n^3 + 4R_n^2 d - 10R_n d^2 + 3d^3\right) \\
&= a. \qquad (2.44)
\end{aligned}
$$

LEMMA 2.8. *Assume that* (F1), (F2), (F3), *and* (F4) *hold. Then, there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$
\sup_{t>0} \Phi(t(w_{\bar{n}})_t) < \frac{2\pi}{\alpha_0}. \qquad (2.45)
$$

Proof. Using the fact that $\kappa > 4\pi/(ed^2\alpha_0^3)$ and (F4), we may choose small $\varepsilon > 0$ and large $t_\varepsilon > 0$ such that

$$
1 + \log \frac{(2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3}{8\pi(1+\varepsilon)^{5/2}} > 0 \qquad (2.46)
$$

and

$$
t^2 F(t) \geq (\kappa - \varepsilon) e^{\alpha_0 t^2} \quad \forall\ |t| \geq t_\varepsilon. \qquad (2.47)
$$

Using (2.4) and (2.43), we have

$$
\begin{aligned}
\Phi(t(w_n)_t) &= \frac{t^2}{2}\|\nabla w_n\|_2^2 - \frac{1}{t^2}\int_{\mathbb{R}^2} b(x/t) F(tw_n)\mathrm{d}x \\
&\leq \frac{t^2}{2} - \frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x \quad \forall\ t > 0 \text{ for large } n \in \mathbb{N}. \qquad (2.48)
\end{aligned}
$$

There are four cases to distinguish. In the following, we agree that all inequalities hold for large $n \in \mathbb{N}$ without mentioning.

Case (i) $t \in [0, \sqrt{\frac{2\pi}{\alpha_0}}]$. Then, by $F(t) \geq 0$, $\forall\ t \in \mathbb{R}$ and (2.48), we have

$$
\Phi(t(w_n)_t) \leq \frac{t^2}{2} - \frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x \leq \frac{t^2}{2} \leq \frac{\pi}{\alpha_0}, \qquad (2.49)
$$

which yields the existence of $\bar{n} \in \mathbb{N}$, satisfying (2.45).

Case (ii) $t \in [\sqrt{\frac{2\pi}{\alpha_0}}, \sqrt{\frac{4\pi}{\alpha_0}}]$. In this case, $tw_n(x) \geq t_\varepsilon$ for $x \in B_{d/\sqrt{n}}$ and large $n \in \mathbb{N}$. Then, it follows from (2.42) and (2.47) that

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$$
\begin{aligned}
\frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x &\geq \frac{1}{t^2}\int_{B_{d/\sqrt{n}}} F(tw_n)\mathrm{d}x \geq \int_{B_{d/\sqrt{n}}} \frac{(\kappa-\varepsilon)e^{\alpha_0 t^2 w_n^2}}{t^4 w_n^2}\mathrm{d}x \\
&\geq \frac{(\kappa-\varepsilon)\alpha_0^2}{8\pi\log n}\int_{B_{d/\sqrt{n}}} e^{\alpha_0 t^2 w_n^2}\mathrm{d}x \\
&= \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{8n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2n^2\log n\int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s^2 - 2s}\mathrm{d}s\right] \\
&\geq \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{8n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2n^2\log n\int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s - 2}\mathrm{d}s\right] \\
&\geq \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{8n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + \frac{4\pi}{\alpha_0 t^2}\left(n^{(2\pi)^{-1}\alpha_0 t^2} - n^{(4\pi)^{-1}\alpha_0 t^2}\right)\right] \\
&\geq \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{4n^2\log n}e^{(2\pi)^{-1}\alpha_0 t^2\log n} - O\left(\frac{1}{n\log n}\right). \qquad (2.50)
\end{aligned}
$$

Using (2.48) and (2.50), we are led to

$$
\begin{aligned}
\Phi(t(w_n)_t) &\leq \frac{t^2}{2} - \frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x \\
&\leq \frac{t^2}{2} - \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{4n^2\log n}e^{(2\pi)^{-1}\alpha_0 t^2\log n} + O\left(\frac{1}{n\log n}\right) \\
&:= \varphi_n(t) + O\left(\frac{1}{n\log n}\right). \qquad (2.51)
\end{aligned}
$$

Choosing $t_n > 0$ such that $\varphi_n'(t_n) = 0$, then we have

$$
1 = \frac{(\kappa-\varepsilon)d^2\alpha_0^3}{4\pi n^2}e^{(2\pi)^{-1}\alpha_0 t_n^2\log n}. \qquad (2.52)
$$

It follows that

$$
\begin{aligned}
t_n^2 &= \frac{4\pi}{\alpha_0}\left[1 + \frac{\log 4\pi - \log((\kappa-\varepsilon)d^2\alpha_0^3)}{2\log n}\right] \\
&= \frac{4\pi}{\alpha_0} - \frac{2\pi}{\alpha_0\log n}\log\frac{(\kappa-\varepsilon)d^2\alpha_0^3}{4\pi} \qquad (2.53)
\end{aligned}
$$

and

$$
\varphi_n(t) \leq \varphi_n(t_n) = \frac{t_n^2}{2} - \frac{\pi}{\alpha_0\log n} \qquad \forall\ t \geq 0. \qquad (2.54)
$$

Using (2.53) and (2.54), we are led to

$$
\begin{aligned}
\varphi_n(t) &\leq \frac{t_n^2}{2} - \frac{\pi}{\alpha_0\log n} \\
&= \frac{2\pi}{\alpha_0} - \frac{\pi}{\alpha_0\log n}\log\frac{(\kappa-\varepsilon)d^2\alpha_0^3}{4\pi} - \frac{\pi}{\alpha_0\log n} \\
&= \frac{2\pi}{\alpha_0} - \frac{\pi}{\alpha_0\log n}\log\frac{e(\kappa-\varepsilon)d^2\alpha_0^3}{4\pi},
\end{aligned}
$$

which, together with (2.51), yields

$$
\Phi(t(w_n)_t) \leq \frac{2\pi}{\alpha_0} - \frac{\pi}{\alpha_0\log n}\log\frac{e(\kappa-\varepsilon)d^2\alpha_0^3}{4\pi} + O\left(\frac{1}{\log^2 n}\right).
$$

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Then, we deduce from (2.46) that (2.45) holds for some $\bar{n} \in \mathbb{N}$.

Case (iii) $t \in [\sqrt{\frac{4\pi}{\alpha_0}}, \sqrt{\frac{4\pi}{\alpha_0}(1+\varepsilon)}]$. In this case, $tw_n(x) \geq t_\varepsilon$ for $x \in B_{d/\sqrt{n}}$ and large $n \in \mathbb{N}$; then (2.42) and (2.47) yield

$$
\begin{aligned}
&\frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x \geq \frac{1}{t^2}\int_{B_{d/\sqrt{n}}} F(tw_n)\mathrm{d}x \geq \int_{B_{d/\sqrt{n}}} \frac{(\kappa-\varepsilon)e^{\alpha_0 t^2 w_n^2}}{t^4 w_n^2}\mathrm{d}x \\
&\geq \frac{(\kappa-\varepsilon)\alpha_0^2}{8\pi(1+\varepsilon)^2\log n}\int_{B_{d/\sqrt{n}}} e^{\alpha_0 t^2 w_n^2}\mathrm{d}x \\
&= \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{8(1+\varepsilon)^2 n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2n^2\log n\int_{1/2}^{1} n^{(2\pi)^{-1}\alpha_0 t^2 s^2 - 2s}\mathrm{d}s\right] \\
&\geq \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{8(1+\varepsilon)^2 n^2\log n}\left[e^{(2\pi)^{-1}\alpha_0 t^2\log n} + 2\log n\int_{1-\varepsilon}^{1} n^{[(1-\varepsilon)(2\pi)^{-1}\alpha_0 t^2 + 2\varepsilon]s}\mathrm{d}s\right] \\
&\geq \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{8(1+\varepsilon)^2 n^2\log n}\left\{e^{(2\pi)^{-1}\alpha_0 t^2\log n} + \frac{1}{1+\varepsilon}e^{[(1-\varepsilon)(2\pi)^{-1}\alpha_0 t^2 + 2\varepsilon]\log n}\right\} \\
&\quad - O\left(\frac{1}{n^{2\varepsilon^2}\log n}\right) \\
&\geq \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{4(1+\varepsilon)^{5/2} n^{2-\varepsilon}\log n}e^{(2-\varepsilon)(4\pi)^{-1}\alpha_0 t^2\log n} - O\left(\frac{1}{n^{2\varepsilon^2}\log n}\right).
\end{aligned} \tag{2.55}
$$

Using (2.48) and (2.55), we are led to

$$
\begin{aligned}
\Phi(t(w_n)_t) &\leq \frac{t^2}{2} - \frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x \\
&\leq \frac{t^2}{2} - \frac{(\kappa-\varepsilon)d^2\alpha_0^2}{4(1+\varepsilon)^{5/2} n^{2-\varepsilon}\log n}e^{(2-\varepsilon)(4\pi)^{-1}\alpha_0 t^2\log n} + O\left(\frac{1}{n^{2\varepsilon^2}\log n}\right) \\
&:= \psi_n(t) + O\left(\frac{1}{n^{2\varepsilon^2}\log n}\right).
\end{aligned} \tag{2.56}
$$

Choosing $\hat{t}_n > 0$ satisfying $\psi_n'(\hat{t}_n) = 0$, then we have

$$
1 = \frac{(2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3}{8\pi(1+\varepsilon)^{5/2} n^{2-\varepsilon}}e^{(2-\varepsilon)(4\pi)^{-1}\alpha_0 \hat{t}_n^2\log n}, \tag{2.57}
$$

which implies that

$$
\begin{aligned}
\hat{t}_n^2 &= \frac{4\pi}{\alpha_0}\left[1 + \frac{\log\left(8\pi(1+\varepsilon)^{5/2}\right) - \log\left((2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3\right)}{(2-\varepsilon)\log n}\right] \\
&= \frac{4\pi}{\alpha_0} + \frac{4\pi}{(2-\varepsilon)\alpha_0\log n}\log\frac{8\pi(1+\varepsilon)^{5/2}}{(2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3}.
\end{aligned} \tag{2.58}
$$

From (2.56) and (2.58), we deduce that

$$
\begin{aligned}
\psi_n(t) \leq \psi_n(\hat{t}_n) &= \frac{\hat{t}_n^2}{2} - \frac{2\pi}{(2-\varepsilon)\alpha_0\log n} \\
&= \frac{2\pi}{\alpha_0} + \frac{2\pi}{(2-\varepsilon)\alpha_0\log n}\log\frac{8\pi(1+\varepsilon)^{5/2}}{(2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3} - \frac{2\pi}{(2-\varepsilon)\alpha_0\log n} \\
&= \frac{2\pi}{\alpha_0} - \frac{2\pi}{(2-\varepsilon)\alpha_0\log n}\left[1 + \log\frac{(2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3}{8\pi(1+\varepsilon)^{5/2}}\right],
\end{aligned}
$$

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which, together with (2.56), yields that

$$\Phi(t(w_n)_t) \leq \frac{2\pi}{\alpha_0} - \frac{\pi}{(2-\varepsilon)\alpha_0 \log n}\left[1 + \log\frac{(2-\varepsilon)(\kappa-\varepsilon)d^2\alpha_0^3}{8\pi(1+\varepsilon)^{5/2}}\right] + O\left(\frac{1}{\log^2 n}\right).$$

Then, it follows from (2.46) that (2.45) holds for some $\bar{n} \in \mathbb{N}$.

Case (iv) $t \in (\sqrt{\frac{4\pi}{\alpha_0}(1+\varepsilon)}, +\infty)$. Since $tw_n(x) \geq t_\varepsilon$ for $x \in B_{d/\sqrt{n}}$ and large $n \in \mathbb{N}$, we deduce from (2.42) and (2.48) that

$$\begin{aligned}\Phi(t(w_n)_t) &\leq \frac{t^2}{2} - \frac{1}{t^2}\int_{\mathbb{R}^2} F(tw_n)\mathrm{d}x \\ &\leq \frac{t^2}{2} - \frac{2\pi^2(\kappa-\varepsilon)d^2}{n^2 t^4 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n} \\ &\leq \frac{2\pi(1+\varepsilon)}{\alpha_0} - \frac{\alpha_0^2(\kappa-\varepsilon)d^2}{8(1+\varepsilon)^2\log n} e^{2\varepsilon \log n} \leq \frac{4\pi}{3\alpha_0}, \end{aligned} \tag{2.59}$$

where we have used the fact that the function

$$\phi_n(t) := \frac{t^2}{2} - \frac{2\pi^2(\kappa-\varepsilon)d^2}{n^2 t^4 \log n} e^{(2\pi)^{-1}\alpha_0 t^2 \log n}$$

is decreasing on $t \in \left(\sqrt{\frac{4\pi}{\alpha_0}(1+\varepsilon)}, +\infty\right)$ for large n. In fact,

$$\phi_n'(t) = t - \frac{2\pi^2(\kappa-\varepsilon)d^2}{n^2 t^5 \log n}\left(\frac{\alpha_0 t^2 \log n}{\pi} - 4\right) e^{(2\pi)^{-1}\alpha_0 t^2 \log n}.$$

Assume that $s_n \geq \sqrt{\frac{4\pi}{\alpha_0}(1+\varepsilon)}$ such that $\phi_n'(s_n) = 0$ for large n. Then,

$$s_n^6 = \frac{2\pi(\kappa-\varepsilon)d^2}{n^2}\left(\alpha_0 s_n^2 - \frac{4\pi}{\log n}\right) e^{(2\pi)^{-1}\alpha_0 s_n^2 \log n}, \tag{2.60}$$

which yields

$$\begin{aligned} s_n^2 &= \frac{4\pi}{\alpha_0}\left[1 + \frac{\log s_n^6 - \log\left(2\pi(\kappa-\varepsilon)d^2\left(\alpha_0 s_n^2 - \frac{4\pi}{\log n}\right)\right)}{2\log n}\right] \\ &= \frac{4\pi}{\alpha_0} + \frac{2\pi}{\alpha_0 \log n}\log\frac{s_n^6}{2\pi(\kappa-\varepsilon)d^2\left(\alpha_0 s_n^2 - \frac{4\pi}{\log n}\right)}. \end{aligned} \tag{2.61}$$

This implies that $\lim_{n\to\infty} s_n^2 = \frac{4\pi}{\alpha_0}$, which is a contradiction. So, $\phi_n(t)$ is decreasing on

$$t \in \left(\sqrt{\frac{4\pi}{\alpha_0}(1+\varepsilon)}, +\infty\right)$$

for large n. Thus, (2.45) holds for some $\bar{n} \in \mathbb{N}$.

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The above four cases show that (2.45) holds for some $\bar{n} \in \mathbb{N}$, and the proof is completed. ∎

LEMMA 2.9. *Assume that* (F1), (F2), (F3), *and* (F4) *hold. Then,* $c(a) < 2\pi/\alpha_0$.

Proof. Let $w_{\bar{n}}$ be given by Lemma 2.8. Since $\|\nabla t(w_{\bar{n}})_t\|_2^2 = t^2\|\nabla w_{\bar{n}}\|_2^2$, we know that there exist $t_w > 0$ small enough and $T_w > 0$ large enough that $\|\nabla t_w(w_{\bar{n}})_{t_w}\|_2^2 \leq K(a)$ and $\Phi(T_w(w_{\bar{n}})_{T_w}) < 0$ by (2.17). Set

$$g_0(\tau) = [(1-\tau)t_w + \tau T_w](w_{\bar{n}})_{(1-\tau)t_w+\tau T_w} \quad \forall\ \tau \in [0,1].$$

Then, $g_0 \in \Gamma_a$. Jointly with the definition of $c(a)$, we have $c(a) < 2\pi/\alpha_0$ for any $a > 0$. ∎

LEMMA 2.10. *Assume that f satisfies* (F1), (F2), (F3), *and* (F4′) *with* $\gamma > \gamma^*(a)$. *Then,* $c(a) < 2\pi/\alpha_0$, *where* $\gamma^*(a)$ *is given by* (1.20).

Proof. Since

$$\mathcal{C}_p = \inf_{u \in H^1(\mathbb{R}^2)\setminus\{0\}} \frac{\|\nabla u\|_2^{p-2}\|u\|_2^2}{\|u\|_p^p}, \tag{2.62}$$

we can choose $v_n \in \mathcal{S}_a$ such that

$$\mathcal{C}_p \leq \frac{\|\nabla v_n\|_2^{p-2}\|v_n\|_2^2}{\|v_n\|_p^p} = \frac{\|\nabla v_n\|_2^{p-2}a}{\|v_n\|_p^p} < \mathcal{C}_p + \frac{1}{n} \quad \forall\ n \in \mathbb{N}. \tag{2.63}$$

Note that

$$\begin{aligned} \Phi(t(v_n)_t) &= \frac{t^2}{2}\|\nabla v_n\|_2^2 - \frac{1}{t^2}\int_{\mathbb{R}^2} b(x/t)F(tv_n)\mathrm{d}x \\ &\leq \frac{t^2}{2}\|\nabla v_n\|_2^2 - \gamma t^{p-2}\|v_n\|_p^p := g_n(t) \quad \forall\ t>0,\ n \in \mathbb{N}. \end{aligned} \tag{2.64}$$

Let

$$t_n^{p-4} = \frac{\|\nabla v_n\|_2^2}{\gamma(p-2)\|v_n\|_p^p} \tag{2.65}$$

such that $g_n'(t_n) = 0$. It is easy to see that $g_n(t) \leq g_n(t_n)$ for all $t > 0$. Then, it follows from (2.63) and (2.64) that

$$\begin{aligned} \Phi(t(v_n)_t) \leq g_n(t_n) &= \frac{p-4}{2(p-2)}\frac{1}{[\gamma(p-2)]^{2/(p-4)}}\left(\frac{\|\nabla v_n\|_2^{p-2}}{\|v_n\|_p^p}\right)^{2/(p-4)} \\ &\leq \frac{p-4}{2(p-2)}\frac{1}{[\gamma(p-2)]^{2/(p-4)}}\left(\frac{\mathcal{C}_p+\frac{1}{n}}{a}\right)^{2/(p-4)} \quad \forall\ t>0,\ n \in \mathbb{N}. \end{aligned} \tag{2.66}$$

Since $p > 4$ and $\gamma > \gamma^*(a)$, then there exists $\epsilon_0 > 0$ such that

$$\gamma = \gamma^*(a)(1-\epsilon_0)^{(4-p)/2} = \frac{\mathcal{C}_p}{a(p-2)}\left[\frac{\alpha_0(p-4)}{4\pi(p-2)(1-\epsilon_0)}\right]^{(p-4)/2}. \tag{2.67}$$

By (2.66) and (2.67), we have

$$\Phi(t(v_n)_t) \leq \left(\frac{\mathcal{C}_p+\frac{1}{n}}{\mathcal{C}_p}\right)^{2/(p-4)}\frac{2\pi(1-\epsilon_0)}{\alpha_0} \quad \forall\ t>0,\ n \in \mathbb{N},$$

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which implies that there exists $\bar{n} \in \mathbb{N}$ large enough that

$$\max_{t>0} \Phi(t(v_{\bar{n}})_t) < \frac{2\pi}{\alpha_0}. \tag{2.68}$$

Replacing $w_{\bar{n}}$ by $v_{\bar{n}}$ in the proof of Lemma 2.9, we can get $c(a) \leq \max_{t>0} \Phi(t(v_{\bar{n}})_t)$ for any $\gamma > \gamma^*(a)$. From this and (2.68), we derived the desired conclusion, and so, the proof is completed. ∎

To guarantee that the obtained bounded PS sequence is strongly convergent up to a subsequence, we give some crucial statements.

2.3. To restore the compactness.

LEMMA 2.11. *Assume that* (F1) *and* (F2) *hold. Let* $u_n \rightharpoonup \bar{u}$ *in* $H^1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2} |f(u_n)u_n| \mathrm{d}x \leq K_0$ *for some constant* $K_0 > 0$.

(i) *Then,* $\lim_{n\to\infty} \int_{\mathbb{R}^2} f(u_n)\phi \mathrm{d}x = \int_{\mathbb{R}^2} f(\bar{u})\phi \mathrm{d}x$ *for any* $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$.

(ii) *Suppose that* $\Omega \subset \mathbb{R}^2$ *and* $u_n \to \bar{u}$ *in* $L^q(\Omega)$ *for some* $q \geq 2$. *If further* (F5) *holds, then* $\lim_{n\to\infty} \int_\Omega F(u_n)\mathrm{d}x = \int_\Omega F(\bar{u})\mathrm{d}x$.

Proof. Item (i) follows directly from [14, Lemma 2.1]. Arguing as in Assertion 2 of [12, Proof of Theorem 1.4], we can conclude item (ii). ∎

PROPOSITION 2.12 [7, Proposition 4.1]. *Let* $\{u_n\} \subset \mathcal{S}_a$ *be a bounded PS sequence satisfying* (2.31). *Then, there is a sequence* $\{\lambda_n\} \subset \mathbb{R}$ *such that, up to a subsequence,*

(1) $u_n \rightharpoonup \bar{u}$ *in* $H^1(\mathbb{R}^2)$ *and* $\lambda_n \to \bar{\lambda}$ *in* $\mathbb{R}$;

(2) $-\Delta u_n - \lambda_n u_n - b(x) f(u_n) \to 0$ *in* $(H^1(\mathbb{R}^2))^*$;

(3) $-\Delta u_n - \bar{\lambda} u_n - b(x) f(u_n) \to 0$ *in* $(H^1(\mathbb{R}^2))^*$.

In addition, assume that (F1) *and* (F2) *hold and that* $\int_{\mathbb{R}^2} |f(u_n)u_n| \mathrm{d}x \leq K_0$ *for some constant* $K_0 > 0$ *as required in Lemma* 2.11; *then*

(4) $-\Delta \bar{u} - \bar{\lambda}\bar{u} - b(x) f(\bar{u}) = 0$ *in* $(H^1(\mathbb{R}^2))^*$.

LEMMA 2.13. *Assume that* (F1), (F2), *and* (F3) *hold. If there exist* $u \in H^1(\mathbb{R}^2)$ *and* $\lambda \in \mathbb{R}$ *such that*

$$-\Delta u - \lambda u = b(x) f(u), \qquad x \in \mathbb{R}^2, \tag{2.69}$$

then $J(u) = 0$, *where* J *is defined by* (2.5).

Proof. By a standard argument, we can derive the following Pohozaev identity:

$$P(u) := \lambda \|u\|_2^2 + \int_{\mathbb{R}^2} [2b(x) + \nabla b(x) \cdot x] F(u) = 0. \tag{2.70}$$

Multiplying (2.69) by u and integrating, we get

$$\|\nabla u\|_2^2 - \lambda \|u\|_2^2 - \int_{\mathbb{R}^2} b(x) f(u) u \mathrm{d}x = 0. \tag{2.71}$$

By (2.70) plus (2.71), we conclude $J(u) = 0$ as desired. ∎

3. Normalized solutions for autonomous equation (1.9).

3.1. Radial solutions for (1.9). In this subsection, we shall establish the existence of radial solutions for (1.9) and give the proof of (i) of Theorem 1.2. For this, we work in the space $H_r^1(\mathbb{R}^2)$ because this space embeds compactly in $L^s(\mathbb{R}^2)$ for all $s > 2$, which helps to restore the compactness. Moreover, by Palais' principle of symmetric

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criticality [33], it is well known that the solutions in $H_r^1(\mathbb{R}^2)$ are in fact solutions in whole $H^1(\mathbb{R}^2)$. In this subsection, we shall consider the problem in $\mathcal{S}_a^r = \mathcal{S}_a \cap H_r^1(\mathbb{R}^2)$.

Proof of (i) *of Theorem* 1.2. In the same way as Lemmas 2.7 and 2.9, we can deduce that, for any $a>0$, there exists a bounded sequence $\{u_n\} \subset \mathcal{S}_a^r$ such that

$$\Phi^\infty(u_n) \to c_r^\infty(a) \in (0, 2\pi/\alpha_0), \quad \Phi^\infty|'_{\mathcal{S}_a^r}(u_n) \to 0, \quad \text{and} \quad J^\infty(u_n) \to 0 \tag{3.1}$$

with

$$c_r^\infty(a) = \inf_{g\in\Gamma_{r,a}^\infty} \max_{t\in[0,1]} \Phi^\infty(g(t)) > \max_{g\in\Gamma_{r,a}^\infty} \max\{\Phi^\infty(g(0)), \Phi^\infty(g(1))\}, \tag{3.2}$$

where $\Gamma_{r,a}^\infty = \{g \in \mathcal{C}([0,1], \mathcal{S}_a^r) : \|\nabla g(0)\|_2^2 \le K(a), \Phi^\infty(g(1)) < 0\}$ and $K(a)$ is given in Lemma 2.2. Then, there exists $\bar{u} \in H_r^1(\mathbb{R}^2)$ such that, passing to a subsequence,

$$u_n \rightharpoonup \bar{u} \ \text{ in } H_r^1(\mathbb{R}^2), \quad u_n \to \bar{u} \ \text{ in } L^s(\mathbb{R}^2) \text{ for } s>2, \quad u_n \to \bar{u} \text{ a.e. in } \mathbb{R}^2. \tag{3.3}$$

Since $f(t)t \ge 0$ for any $t \in \mathbb{R}$, arguing as in the proof of (2.36), we have that $\{f(u_n)u_n\}$ is bounded in $L^1(\mathbb{R}^2)$. From Proposition 2.12, we know that there is a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that, up to a subsequence,

$$\lambda_n \to \bar{\lambda} \in \mathbb{R}; \tag{3.4}$$

moreover, we have that

$$-\Delta u_n - \lambda_n u_n - f(u_n) \to 0, \qquad -\Delta u_n - \bar{\lambda} u_n - f(u_n) \to 0 \tag{3.5}$$

in $(H_r^1(\mathbb{R}^2))^*$ and

$$-\Delta \bar{u} - \bar{\lambda}\bar{u} - f(\bar{u}) = 0 \tag{3.6}$$

by (i) of Lemma 2.11. Noting that $u_n \to \bar{u}$ in $L^s(\mathbb{R}^2)$ for $s>2$, it follows from (ii) of Lemma 2.11 that

$$\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = \int_{\mathbb{R}^2} F(\bar{u})\mathrm{d}x + o(1). \tag{3.7}$$

To prove that $\bar{u}$ is a radial solution to (1.9), it suffices to show that $\|\bar{u}\|_2^2 = a$ by (3.6). For this, we prove below three claims in turn.

Claim 1. $\bar{u} \ne 0$.

Otherwise, we suppose that $u_n \rightharpoonup 0$ in $H_r^1(\mathbb{R}^2)$. Then, (1.11), (3.1), and (3.7) give

$$\|\nabla u_n\|^2 = 2\Phi^\infty(u_n) + 2\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = 2c_r^\infty(a) + o(1) := \frac{4\pi}{\alpha_0}(1-3\bar{\varepsilon}) + o(1) \tag{3.8}$$

for some constant $\bar{\varepsilon} > 0$. Choosing $q \in (1,2)$ such that

$$\frac{(1+\bar{\varepsilon})(1-3\bar{\varepsilon})q}{1-\bar{\varepsilon}} < 1, \tag{3.9}$$

using (F1), we get

$$|f(t)|^q \le C_1 \left[e^{\alpha_0(1+\bar{\varepsilon})qt^2} - 1\right] \qquad \forall\ |t| \ge 1. \tag{3.10}$$

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By (3.8), (3.9), (3.10), and (ii) of Lemma 1.1, we have

$$\begin{aligned}\int_{|u_n|\geq 1}|f(u_n)|^q\mathrm{d}x &\leq C_1\int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar\varepsilon)qu_n^2}-1\right]\mathrm{d}x\\ &= C_1\int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar\varepsilon)q\|u_n\|^2(u_n/\|u_n\|)^2}-1\right]\mathrm{d}x \leq C_2.\end{aligned}\tag{3.11}$$

Noting that $q/(q-1)>2$ and $u_n\to 0$ in $L^s(\mathbb{R}^2)$ for $s>2$, by (3.3), (3.11), and the Hölder inequality, we have

$$\int_{|u_n|\geq 1}f(u_n)u_n\mathrm{d}x \leq \left[\int_{|u_n|\geq 1}|f(u_n)|^q\mathrm{d}x\right]^{1/q}\|u_n\|_{q/(q-1)} = o(1).\tag{3.12}$$

Moreover, by (F2), we have

$$\int_{|u_n|<1}f(u_n)u_n\mathrm{d}x \leq C_3\|u_n\|_4^4 = o(1).\tag{3.13}$$

Then, it follows from (1.11), (1.13), (3.1), (3.7), (3.12), and (3.13) that

$$c_r^\infty(a)+o(1)=\Phi^\infty(u_n)-\frac{1}{2}J^\infty(u_n)=\frac{1}{2}\int_{\mathbb{R}^2}\left[f(u_n)u_n-4F(u_n)\right]\mathrm{d}x=o(1),\tag{3.14}$$

which is a contradiction due to $c_r^\infty(a)>0$ for any $a>0$. This shows that $\bar u\neq 0$ as claimed.

Claim 2. $\int_{\mathbb{R}^2}f(u_n)(u_n-\bar u)\mathrm{d}x=o(1)$.

Using (3.6) and arguing as in the proof of Lemma 2.13, we have $J^\infty(\bar u)=0$. This, jointly with (F3), (1.11), and (1.13), implies that

$$\Phi^\infty(\bar u)=\Phi^\infty(\bar u)-\frac{1}{2}J^\infty(\bar u)=\frac{1}{2}\int_{\mathbb{R}^2}\left[f(\bar u)\bar u-4F(\bar u)\right]\mathrm{d}x\geq 0.\tag{3.15}$$

By (1.11), (3.1), (3.7), and (3.15), we have

$$\begin{aligned}c_r^\infty(a)+o(1)=\Phi^\infty(u_n)&=\frac{1}{2}\|\nabla u_n\|_2^2-\int_{\mathbb{R}^2}F(u_n)\mathrm{d}x\\ &=\frac{1}{2}\left(\|\nabla(u_n-\bar u)\|_2^2+\|\nabla\bar u\|_2^2\right)-\int_{\mathbb{R}^2}F(\bar u)\mathrm{d}x+o(1)\\ &=\frac{1}{2}\|\nabla(u_n-\bar u)\|_2^2+\Phi^\infty(\bar u)+o(1)\\ &\geq\frac{1}{2}\|\nabla(u_n-\bar u)\|_2^2+o(1).\end{aligned}\tag{3.16}$$

Since $0<c_r^\infty(a)<2\pi/\alpha_0$ for any $a>0$, similarly as in (3.8), it follows from (3.16) that there exists $\bar\varepsilon>0$ such that

$$\|\nabla(u_n-\bar u)\|_2^2\leq\frac{(1-3\bar\varepsilon)4\pi}{\alpha_0}<\frac{4\pi}{\alpha_0}\quad\text{for large } n\in\mathbb{N}.\tag{3.17}$$

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By (3.9), (3.10), (3.17), Young's inequality, and Lemma 1.1, we have

$$
\begin{aligned}
\int_{|u_n|\geq 1}|f(u_n)|^q\mathrm{d}x &\leq C_1\int_{|u_n|\geq 1}\left[e^{\alpha_0(1+\bar{\varepsilon})qu_n^2}-1\right]\mathrm{d}x\\
&\leq C_1\int_{|u_n|\geq 1}\left[e^{\alpha_0(1+\bar{\varepsilon})^2\bar{\varepsilon}^{-1}q\bar{u}^2}e^{\alpha_0(1+\bar{\varepsilon})^2q(u_n-\bar{u})^2}-1\right]\mathrm{d}x\\
&\leq \frac{(q-1)C_1}{q}\int_{|u_n|\geq 1}\left[e^{\alpha_0(1+\bar{\varepsilon})^2\bar{\varepsilon}^{-1}q^2(q-1)^{-1}\bar{u}^2}-1\right]\mathrm{d}x\\
&\quad +\frac{C_1}{q}\int_{|u_n|\geq 1}\left[e^{\alpha_0(1+\bar{\varepsilon})^2q^2(u_n-\bar{u})^2}-1\right]\mathrm{d}x\\
&\leq \frac{(q-1)C_1}{q}\int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})^2\bar{\varepsilon}^{-1}q^2(q-1)^{-1}\bar{u}^2}-1\right]\mathrm{d}x\\
&\quad +\frac{C_1}{q}\int_{\mathbb{R}^2}\left[e^{\alpha_0(1+\bar{\varepsilon})^2q^2(u_n-\bar{u})^2}-1\right]\mathrm{d}x\leq C_4.
\end{aligned}
\tag{3.18}
$$

Noting that $q/(q-1)>2$, by (3.3), (3.18), and the Hölder inequality, we have

$$
\int_{|u_n|\geq 1}f(u_n)(u_n-\bar{u})\mathrm{d}x\leq\left[\int_{|u_n|\geq 1}|f(u_n)|^q\mathrm{d}x\right]^{1/q}\|u_n-\bar{u}\|_{q/(q-1)}=o(1).
\tag{3.19}
$$

Moreover, by (F1) and (F2), we have

$$
\int_{|u_n|<1}f(u_n)(u_n-\bar{u})\mathrm{d}x\leq C_5\|u_n\|_{9/2}^3\|u_n-\bar{u}\|_3=o(1).
\tag{3.20}
$$

Hence, Claim 2 follows from directly (3.19) and (3.20).

Claim 3. $u_n\to\bar{u}$ in $H_r^1(\mathbb{R}^2)$.

Note that (3.5) yields

$$
\|\nabla u_n\|_2^2+\lambda_n\|u_n\|_2^2-\int_{\mathbb{R}^2}f(u_n)u_n\mathrm{d}x\to 0
\tag{3.21}
$$

and

$$
\int_{\mathbb{R}^2}(\nabla u_n\cdot\nabla\bar{u}+\lambda_nu_n\bar{u})\,\mathrm{d}x-\int_{\mathbb{R}^2}f(u_n)\bar{u}\mathrm{d}x\to 0.
\tag{3.22}
$$

By (3.21) minus $J^\infty(u_n)\to 0$ and using (3.4) and (3.7), we have

$$
\bar{\lambda}a+o(1)=\lambda_n\|u_n\|_2^2=2\int_{\mathbb{R}^2}F(u_n)\mathrm{d}x+o(1)=2\int_{\mathbb{R}^2}F(\bar{u})\mathrm{d}x+o(1),
$$

which, together with $F(t)>0$ for $t\neq 0$, yields $\bar{\lambda}>0$. By (3.21) minus (3.22), using the above Claim 2, we have

$$
\int_{\mathbb{R}^2}[\nabla u_n\cdot\nabla(u_n-\bar{u})+\lambda_nu_n(u_n-\bar{u})]\,\mathrm{d}x=o(1),
\tag{3.23}
$$

which, together with $u_n\rightharpoonup\bar{u}$ in $H_r^1(\mathbb{R}^2)$ and $\lambda\to\bar{\lambda}>0$, implies that $u_n\to\bar{u}$ in $H_r^1(\mathbb{R}^2)$. The proof of Claim 3 is completed.

Next, using Palais' principle of symmetric criticality [33], the above function $\bar{u}\in H_r^1(\mathbb{R}^2)\setminus\{0\}$ is in fact a radial solution of (1.9) in $H^1(\mathbb{R}^2)$, and so, the proof of (i) of Theorem 1.2 is completed. ∎

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3.2. Ground state solutions for (1.9). In this subsection, we shall obtain the existence of ground state solutions for (1.9) and finish the proof of (ii) of Theorem 1.2. For this, we shall directly work on the space $H^1(\mathbb{R}^2)$ instead of $H^1_r(\mathbb{R}^2)$ used in the last subsection. First, we establish some important inequalities.

LEMMA 3.1. *Assume that* $f \in \mathcal{C}(\mathbb{R},\mathbb{R})$ *satisfies* (F2) *and* (F6). *Then,*

$$t^{-2}F(tu) - F(u) + \frac{1-t^2}{2}[f(u)u - 2F(u)] \geq 0 \quad \forall\, u \in \mathbb{R}, \quad t > 0 \tag{3.24}$$

and

$$f(u)u - 4F(u) \geq 0 \quad \forall\, u \in \mathbb{R}. \tag{3.25}$$

Proof. Using (F6), a standard argument shows that (3.24) holds. Moreover, (3.25) follows by letting $t \to 0$ in (3.24). ∎

LEMMA 3.2. *Assume that* (F1), (F2), *and* (F6) *hold. Then,*

$$\Phi^\infty(u) \geq \Phi^\infty(tu_t) + \frac{1-t^2}{2}J^\infty(u) \quad \forall\, u \in \mathcal{S}_a, \quad t > 0. \tag{3.26}$$

Proof. By (1.11), (1.13), and (3.24), we have

$$\begin{aligned}\Phi^\infty(u) &= \Phi^\infty(tu_t) + \frac{1-t^2}{2}J^\infty(u) \\ &\quad + \int_{\mathbb{R}^2}\left\{t^{-2}F(tu) - F(u) + \frac{1-t^2}{2}[f(u)u - 2F(u)]\right\}\mathrm{d}x \\ &\geq \Phi^\infty(u_t) + \frac{1-t^2}{2}J^\infty(u) \quad \forall\, u \in \mathcal{S}_a, \quad t > 0.\end{aligned}$$

∎

COROLLARY 3.3. *Assume that* (F1), (F2), *and* (F6) *hold. Then,*

$$\Phi^\infty(u) = \max_{t>0}\Phi^\infty(tu_t) \quad \forall\, u \in \mathcal{M}_a^\infty. \tag{3.27}$$

By a standard argument, we can get the following lemma.

LEMMA 3.4. *Assume that* (F1), (F2), *and* (F6) *hold. Then, for any* $u \in \mathcal{S}_a$, *there exists* $t_u > 0$ *such that* $t_u u_{t_u} \in \mathcal{M}_a^\infty$.

By Lemma 2.2, we have

$$c^\infty(a) := \inf_{g\in\Gamma_a^\infty}\max_{t\in[0,1]}\Phi^\infty(g(t)) > \max_{g\in\Gamma_a^\infty}\max\{\Phi^\infty(g(0)), \Phi^\infty(g(1))\}. \tag{3.28}$$

$\Gamma_a^\infty = \{g \in \mathcal{C}([0,1],\mathcal{S}_a) : \|\nabla g(0)\|_2^2 \leq K(a), \Phi^\infty(g(1)) < 0\}$, and $K(a)$ is given in Lemma 2.2.

LEMMA 3.5. *Assume that* (F1), (F2), (F3), *and* (F6) *hold. Then,*

$$c^\infty(a) = m^\infty(a) = \inf_{u\in\mathcal{M}_a^\infty}\Phi^\infty(u) = \inf_{u\in\mathcal{S}_a}\max_{t>0}\Phi^\infty(tu_t).$$

Proof. Using Corollary 3.3 and Lemma 3.4, it is easy to see that

$$m^\infty(a) = \inf_{u\in\mathcal{M}_a^\infty}\Phi^\infty(u) = \inf_{u\in\mathcal{S}_a}\max_{t>0}\Phi^\infty(tu_t).$$

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To finish the proof, it remains to show that $c^\infty(a) = m^\infty(a)$. For this, we first prove that $c^\infty(a) \le m^\infty(a)$. As in the proof of Lemma 2.4, we know that for any $u \in \mathcal{M}_a^\infty$, there exist $t_1 > 0$ small enough and $t_2 > 1$ large enough that $\|\nabla(t_1 u_{t_1})\|_2^2 \le K(a)$ and $\Phi^\infty(t_2(u_{t_2})) < 0$. Letting

$$g^\infty(\tau) = [(1-\tau)t_1 + \tau t_2]u_{(1-\tau)t_1+\tau t_2} \quad \forall\ \tau \in [0,1],$$

jointly with the definition of $c^\infty(a)$, then we have $g^\infty \in \Gamma_a^\infty$. By (3.27), we have

$$c^\infty(a) \le \max_{\tau\in[0,1]} \Phi^\infty(g^\infty(\tau)) = \Phi^\infty(u),$$

and so, $c^\infty(a) \le m^\infty(a) = \inf_{u\in\mathcal{M}_a^\infty} \Phi^\infty(u)$ for any $a > 0$.

On the other hand, by (3.26) with $t \to 0$, we have

$$J^\infty(u) \le 2\Phi^\infty(u) \quad \forall\ u \in \mathcal{S}_a,$$

which implies that

$$J^\infty(g(1)) \le 2\Phi^\infty(g(1)) < 0 \quad \forall\ g \in \Gamma_a^\infty.$$

Since $\|g(0)\|_2^2 \le K(a)$, by (i) of Lemma 2.2, we have $J^\infty(g(0)) > 0$. Hence, any path in Γ_a^∞ has to cross $\mathcal{M}_a^\infty$. This shows that

$$\max_{\tau\in[0,1]} \Phi^\infty(g(\tau)) \ge \inf_{u\in\mathcal{M}_a^\infty} \Phi^\infty(u) = m^\infty(a) \quad \forall\ g \in \Gamma_a^\infty,$$

and so, $c^\infty(a) \ge m^\infty(a)$ due to the arbitrariness of g. Therefore, $c^\infty(a) = m^\infty(a)$ for any $a > 0$, and the proof is completed. □

Besides the above characterization of the mountain pass type, we give further the behavior of $m^\infty(a)$ for $a > 0$, which is crucial to recover the compactness of PS sequences. In fact, we establish the same conclusion on the behavior of $m(a)$ as $m^\infty(a)$ for $a > 0$ in Lemma 4.5 below. So, to avoid repetition, we omit the proof of Lemma 3.6 here, which can be deduced obviously from Lemma 4.5.

LEMMA 3.6. *Assume that* (F1), (F2), (F3), (F5), *and* (F6) *hold. The function* $a \mapsto m^\infty(a)$ *is continuous and nonincreasing on* $(0,\infty)$. *Moreover, if* $m^\infty(a)$ *is achieved, then* $m^\infty(a) > m^\infty(a')$ *for any* $a' > a$.

Proof of (ii) *of Theorem* 1.2. In the same way as Lemmas 2.7 and 2.9, we can deduce that for any $a > 0$, there exists a bounded sequence $\{u_n\} \subset \mathcal{S}_a$ such that

$$\Phi^\infty(u_n) \to c^\infty(a) \in (0, 2\pi/\alpha_0), \quad \Phi^\infty|'_{\mathcal{S}_a}(u_n) \to 0, \quad \text{and} \quad J^\infty(u_n) \to 0, \tag{3.29}$$

where $c^\infty(a)$ is given by (3.28). To obtain the existence of ground state solutions for (1.9), we split the proof into several claims.

Claim 1. $\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_2(y)} |u_n|^2 \mathrm{d}x > 0$.

Otherwise, if $\delta = 0$, then by Lions' concentration compactness principle [23] or [33, Lemma 1.21], we have $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s > 2$. Arguing as in the proof of (2.36), we have

$$\int_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x \le C_1. \tag{3.30}$$

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For any given $\varepsilon \in (0, M_0C_1/\beta_0)$, we choose $M_\varepsilon > M_0C_1/\varepsilon > \beta_0$; then it follows from (F5) and (3.30) that

$$\begin{aligned}\int_{|u_n|\geq M_\varepsilon} F(u_n)\mathrm{d}x &\leq M_0 \int_{|u_n|\geq M_\varepsilon} |f(u_n)|\mathrm{d}x \\ &\leq \frac{M_0}{M_\varepsilon}\int_{|u_n|\geq M_\varepsilon} f(u_n)u_n\mathrm{d}x < \varepsilon. \end{aligned} \tag{3.31}$$

By (F1) and (F2), we obtain

$$\int_{|u_n|\leq M_\varepsilon} F(u_n)\mathrm{d}x \leq C_\varepsilon \|u_n\|_4^4 = o(1) \tag{3.32}$$

and

$$\int_{|u_n|\leq 1} f(u_n)u_n\mathrm{d}x \leq C_\varepsilon \|u_n\|_4^4 = o(1), \tag{3.33}$$

where $C_\varepsilon > 0$ is a constant depending on ε. Due to the arbitrariness of $\varepsilon > 0$, we derive from (3.31) and (3.32) that

$$\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = o(1). \tag{3.34}$$

Since $0 < c^\infty(a) < 2\pi/\alpha_0$ for any $a > 0$, similarly as in (3.8), it follows from (3.16) that there exists $\bar{\varepsilon} > 0$ such that (3.17) holds. Hence, it follows from (1.11), (3.29), and (3.34) that there exists a constant $\bar{\varepsilon} > 0$ such that

$$\|\nabla u_n\|^2 = 2\Phi^\infty(u_n) + 2\int_{\mathbb{R}^2} F(u_n)\mathrm{d}x = 2c^\infty(a) + o(1) := \frac{4\pi}{\alpha_0}(1-3\bar{\varepsilon}) + o(1). \tag{3.35}$$

By replacing $c_r^\infty(a)$ in (3.14) by $c^\infty(a) > 0$, we then get a contradiction with the fact $c^\infty(a) > 0$. This shows that $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $y_n \in \mathbb{R}^2$ such that

$$\int_{B_1(y_n)} |u_n|^2\mathrm{d}x > \frac{\delta}{2}. \tag{3.36}$$

Let $\tilde{u}_n(x) = u_n(x + y_n)$. Then,

$$\int_{B_1(0)} |\tilde{u}_n|^2\mathrm{d}x > \frac{\delta}{2}, \tag{3.37}$$

and so, there exists $\tilde{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that, passing to a subsequence,

$$\tilde{u}_n \rightharpoonup \tilde{u} \text{ in } H^1(\mathbb{R}^2), \quad \tilde{u}_n \to \tilde{u} \text{ in } L^s_{\mathrm{loc}}(\mathbb{R}^2) \text{ for } s > 2, \quad \tilde{u}_n \to \tilde{u} \text{ a.e. in } \mathbb{R}^2. \tag{3.38}$$

Moreover, (3.29) gives

$$\Phi^\infty(\tilde{u}_n) \to c^\infty(a) \in (0, 2\pi/\alpha_0) \text{ and } J^\infty(\tilde{u}_n) \to 0, \quad n \to \infty. \tag{3.39}$$

As in Proposition 2.12, we know that there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that (3.4) holds; moreover, (3.5) also holds in $(H^1(\mathbb{R}^2))^*$. Thus, we have

$$-\Delta\tilde{u}_n + \lambda_n\tilde{u}_n - f(\tilde{u}_n) \to 0 \text{ and } -\Delta\tilde{u}_n + \bar{\lambda}\tilde{u}_n - f(\tilde{u}_n) \to 0 \text{ in } (H^1(\mathbb{R}^2))^*, \tag{3.40}$$

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which, together with (3.38) and (i) of Lemma 2.11, yields

$$-\Delta\tilde{u}+\bar{\lambda}\tilde{u}-f(\tilde{u})=0 \text{ in } (H^1(\mathbb{R}^2))^*. \tag{3.41}$$

Claim 2. $\tilde{u}_n\to\tilde{u}$ in $L^2(\mathbb{R}^2)$.

Using (3.41) and arguing as in the proof of Lemma 2.13, we have $J^\infty(\tilde{u})=0$. Letting $\|\tilde{u}\|_2^2:=\tilde{a}\in(0,a]$, in view of Lemma 3.4, there exists $\tilde{t}>0$ such that $\tilde{t}\tilde{u}_{\tilde{t}}\in\mathcal{M}^\infty_{\tilde{a}}$, and so, $\Phi^\infty\left(\tilde{t}\tilde{u}_{\tilde{t}}\right)\geq m^\infty(\tilde{a})$. From (1.11), (1.13), (3.39), Fatou's lemma, and Lemmas 3.2, 3.5, and 3.6, we have

$$\begin{aligned}
m^\infty(a)=c^\infty(a)&=\lim_{n\to\infty}\left[\Phi^\infty(\tilde{u}_n)-\frac{1}{2}J^\infty(\tilde{u}_n)\right]\\
&=\frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^2}[f(\tilde{u}_n)\tilde{u}_n-4F(\tilde{u}_n)]\,\mathrm{d}x\\
&\geq\frac{1}{2}\int_{\mathbb{R}^2}[f(\tilde{u})\tilde{u}-4F(\tilde{u})]\,\mathrm{d}x\\
&=\Phi^\infty(\tilde{u})-\frac{1}{2}J^\infty(\tilde{u})\geq\Phi^\infty\left(\tilde{t}\tilde{u}_{\tilde{t}}\right)-\frac{t^2}{2}J^\infty(\tilde{u})\\
&\geq m^\infty(\tilde{a})\geq m^\infty(a),
\end{aligned}$$

which implies that

$$\Phi^\infty(\tilde{u})=m^\infty(\tilde{a})=m^\infty(a),\qquad \|\tilde{u}\|_2^2=\tilde{a}. \tag{3.42}$$

This shows that $m^\infty(\tilde{a})$ is achieved. Then, it follows from the last conclusion of Lemma 3.6 that $\|\tilde{u}\|_2^2=\tilde{a}=a=\|\tilde{u}_n\|_2^2$. This shows that $\tilde{u}_n\to\tilde{u}$ in $L^2(\mathbb{R}^2)$ as claimed.

Claim 3. $\|\nabla(\tilde{u}_n-\tilde{u})\|_2^2\to 0$.

Using the above Claim 2, it is easy to see that $\tilde{u}_n\to\tilde{u}$ in $L^s(\mathbb{R}^2)$ for all $s\geq 2$. From (ii) of Lemma 2.11, we have $\int_{\mathbb{R}^2}F(\tilde{u}_n)\mathrm{d}x=\int_{\mathbb{R}^2}F(\tilde{u})\mathrm{d}x+o(1)$. In the same way as the proof of Claim 2 in the proof of (i) of Theorem 1.2, we can get $\int_{\mathbb{R}^2}f(\tilde{u}_n)(\tilde{u}_n-\tilde{u})\mathrm{d}x=o(1)$ by replacing $H^1_r(\mathbb{R}^2)$ with $H^1(\mathbb{R}^2)$. Arguing as in the proof of Claim 3 in the proof of (i) of Theorem 1.2 and using (3.4), (3.40), and (3.41), we can deduce that $\lambda_n\to\bar{\lambda}>0$ and

$$\int_{\mathbb{R}^2}\left[\nabla\tilde{u}_n\cdot\nabla(\tilde{u}_n-\tilde{u})+\bar{\lambda}\tilde{u}_n(\tilde{u}_n-\tilde{u})\right]\mathrm{d}x=o(1).$$

Jointly with $\tilde{u}_n\rightharpoonup\tilde{u}$ in $H^1(\mathbb{R}^2)$, we conclude that $\|\nabla(\tilde{u}_n-\tilde{u})\|_2^2\to 0$ as claimed.

From (3.39), (3.41), Lemma 3.5, and the above Claim 2 and Claim 3, we conclude that, for any $a>0$, $(\tilde{u},\bar{\lambda})$ solves (1.9), and $\Phi^\infty(\bar{u})=m^\infty(a)$. This shows that $\tilde{u}$ is a ground state solution for (1.9), and so, Theorem 1.2 is proved. ∎

4. Ground state solutions for nonautonomous equation (1.1). In this section, we establish the existence of ground state solutions to (1.1) and complete the proof of Theorem 1.6.

First, similarly as [13, Lemmas 2.1–2.3], we have the following two lemmas.

Lemma 4.1. *Assume that* (B1), (B2), *and* (B3) *hold. Then,*

$$-t^{2-\theta}[b(x)-b(tx)]+\frac{1}{\theta-2}\left(t^{2-\theta}-1\right)\nabla b(x)\cdot x\geq 0\quad\forall\, x\in\mathbb{R}^2,\quad t>0 \tag{4.1}$$

and

$$\nabla b(x)\cdot x\leq 0\quad\forall\, x\in\mathbb{R}^2,\qquad \nabla b(x)\cdot x\to 0 \text{ as } |x|\to\infty. \tag{4.2}$$

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If (B2) *holds, then* $t^2b(tx)$ *is nondecreasing on* $t\in(0,\infty)$ *for every* $x\in\mathbb{R}^2$*, and there exist* $t_1,t_2>0$ *and* $\Lambda_0\subset\mathbb{R}^2$ *such that* $t^2b(tx)$ *is strictly increasing on* $t\in[t_1,t_2]$ *for every* $x\in\Lambda_0$.

LEMMA 4.2. *Assume that* (B1), (B2), (F1), (F2), *and* (F7) *hold. Then,*

$$\begin{aligned}&t^{-2}b(t^{-1}x)F(tu)-b(x)F(u)+\frac{1-t^2}{2}b(x)[f(u)u-2F(u)]\\&-\frac{1-t^2}{2}\nabla b(x)\cdot xF(u)\geq 0\quad\forall\ x\in\mathbb{R}^2,\ t>0,\ u\in\mathbb{R}.\end{aligned}\tag{4.3}$$

Similarly as Lemma 3.2, we get the following inequality.

LEMMA 4.3. *Assume that* (B1), (B2), (F1), (F2), *and* (F7) *hold. Then,*

$$\Phi(u)\geq\Phi(tu_t)+\frac{1-t^2}{2}J(u)\quad\forall\ u\in\mathcal{S}_a,\quad t>0.\tag{4.4}$$

As in the proof of Lemma 3.5, we have the following result.

LEMMA 4.4. *Assume that* (B1), (B2), (B3), (F1), (F2), (F3), *and* (F7) *hold. Then*

$$c(a)=m(a)=\inf_{u\in\mathcal{M}_a}\Phi(u)=\inf_{u\in\mathcal{S}_a}\max_{t>0}\Phi(tu_t)>0,$$

where $c(a)$ *is given in Lemma* 2.2.

LEMMA 4.5. *Assume that* (B1), (B2), (B3), (F1), (F2), (F3), *and* (F7) *hold. Then the function* $a\mapsto m(a)$ *is continuous and nonincreasing on* $(0,\infty)$*. In particular, if* $m(a)$ *is achieved, then* $m(a)>m(a')$ *for any* $a'>a$.

Proof. By a standard argument, we can derive the continuity of $m(a)$. Now, we prove the monotonicity of $m(a)$. For any $a_2>a_1>0$, it follows that there exists $\{u_n\}\subset\mathcal{M}_{a_1}$ such that

$$m(a_1)\leq\Phi(u_n)<m(a_1)+\frac{1}{n}.\tag{4.5}$$

Let $\xi:=\sqrt{a_2/a_1}\in(1,\infty)$ and $v_n(x):=u_n(\xi^{-1}x)$. Then $\|v_n\|_2^2=a_2$ and $\|\nabla v_n\|_2=\|\nabla u_n\|_2$. As in Lemma 3.4, there exists $t_n>0$ such that $t_n(v_n)_{t_n}\in\mathcal{M}_{a_2}$. Then, it follows from (B2), (1.4), (4.5), and Lemmas 4.1 and 4.3 that

$$\begin{aligned}m(a_2)&\leq\Phi\left(t_n(v_n)_{t_n}\right)\\&=\Phi\left(t_n(u_n)_{t_n}\right)+t_n^{-2}\int_{\mathbb{R}^2}\left[b(t_n^{-1}x)F(t_nu_n)-\xi^2b(\xi t_n^{-1}x)F(t_nu_n)\right]\mathrm{d}x\\&\leq\Phi(u_n)<m(a_1)+\frac{1}{n},\end{aligned}$$

which shows that $m(a_2)\leq m(a_1)$ by letting $n\to\infty$.

Next, we assume that $m(a)$ is achieved; i.e., there exists $\tilde{u}\in\mathcal{M}_a$ such that $\Phi(\tilde{u})=m(a)$. For any given $a'>a$, let $\tilde{\xi}=a'/a\in(1,\infty)$, and let $\tilde{v}(x):=\tilde{u}(\tilde{\xi}^{-1}x)$. Then, $\|\tilde{v}\|_2^2=a'$ and $\|\nabla\tilde{v}\|_2=\|\nabla\tilde{u}\|_2$. As in Lemma 3.4, there exists $t_0>0$ such that $t_0\tilde{v}_{t_0}\in\mathcal{M}_{a'}$. From (B2), (1.4), and Lemmas 4.1 and 4.3, we then deduce that

$$\begin{aligned}m(a')&\leq\Phi\left(t_0\tilde{v}_{t_0}\right)\\&=\Phi\left(t_0\tilde{u}_{t_0}\right)+t_0^{-2}\int_{\mathbb{R}^2}\left[b(t_0^{-1}x)F(t_0\tilde{u})-\tilde{\xi}^2b(\tilde{\xi}t_0^{-1}x)F(t_0\tilde{u})\right]\mathrm{d}x\\&<\Phi\left(t_0\tilde{u}_{t_0}\right)\leq\Phi(\tilde{u})=m(a),\end{aligned}$$

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where the strict inequality follows from the last conclusion of Lemma 4.1 and the fact that $\tilde{\xi} > 1$ and $F(t) > 0$ for all $t \neq 0$. This shows that $m(a') < m(a)$, and the proof is completed. ∎

LEMMA 4.6. *Assume that* (B1), (B2), (B3), (F1), (F2), (F3), *and* (F7) *hold. Then*

$$\lim_{a\to 0^+} m(a) = +\infty \text{ and } \lim_{a\to+\infty} m(a) = 0.$$

Proof. We first prove that $m(a) \to +\infty$ as $a \to 0^+$. Arguing by contradiction, using Lemma 4.5, we may assume that there exist a sequence $\{u_n\} \subset H^1(\mathbb{R}^2) \setminus \{0\}$ and a constant $M_1 > 0$ such that

$$\|u_n\|_2 \to 0, \quad J(u_n) = 0 \text{ and } \Phi(u_n) \le M_1 \quad \forall\, n \in \mathbb{N}. \tag{4.6}$$

Set

$$t_n = 2\sqrt{M_1}/\|\nabla u_n\|_2 \text{ and } v_n = t_n(u_n)_{t_n}. \tag{4.7}$$

Noting that $\|v_n\|_2 = \|u_n\|_2 \to 0$ by (4.6) and (4.7), it follows from (ii) of Lemma 2.11 that

$$\int_{\mathbb{R}^2} b(x)F(v_n)\mathrm{d}x \to 0. \tag{4.8}$$

From (1.4), (4.6), (4.7), (4.8), and Lemma 4.3, we derive

$$M_1 \ge \Phi(u_n) \ge \Phi(v_n) = \frac{1}{2}\|\nabla v_n\|_2^2 - \int_{\mathbb{R}^2} b(x)F(v_n)\mathrm{d}x = \frac{t_n^2}{2}\|\nabla u_n\|_2^2 + o(1) = 2M_1 + o(1).$$

This contradiction shows that $\lim_{a\to 0^+} m(a) = +\infty$.

We next prove that $m(a) \to 0$ as $a \to +\infty$. Fix $u \in \mathcal{S}_1 \cap L^\infty(\mathbb{R}^2)$, and set

$$u_a = \sqrt{a}u \in \mathcal{S}_a \quad \forall\, a \ge 1. \tag{4.9}$$

As in Lemma 3.4, there exists $t_a > 0$ such that $t_a(u_a)_{t_a} \in \mathcal{M}_a$. Then, it follows from Lemma 4.4 that

$$0 < m(a) \le \Phi(t_a(u_a)_{t_a}) \le \frac{at_a^2}{2}\|\nabla u\|_2^2, \quad \forall\, a \ge 1. \tag{4.10}$$

To complete the proof, using Lemma 4.5, it suffices to show that $at_a^2 \to 0$ as $a \to +\infty$. Since $J(t_a(u_a)_{t_a}) = 0$ for all $a \ge 1$, by (B1) with $b_\infty = 1$, (F3), (4.2), and (4.9), we have

$$\begin{aligned} a\|\nabla u\|_2^2 &= \|\nabla u_a\|_2^2 \\ &= \frac{1}{t_a^4}\int_{\mathbb{R}^2} b(x/t_a)f(\sqrt{a}t_a u)\sqrt{a}t_a u\mathrm{d}x \\ &\quad - \frac{1}{t_a^4}\int_{\mathbb{R}^2} [2b(x/t_a) + \nabla b(x/t_a)\cdot(x/t_a)]\, F(\sqrt{a}t_a u)\mathrm{d}x \\ &\ge \frac{2}{t_a^4}\int_{\mathbb{R}^2} F(\sqrt{a}t_a u)\mathrm{d}x = 2\int_{\mathbb{R}^2} \frac{F(\sqrt{a}t_a u)}{(\sqrt{a}t_a u)^4} a^2u^4\mathrm{d}x \quad \forall\, a \ge 1, \end{aligned}$$

which gives

$$\frac{1}{a}\|\nabla u\|_2^2 \ge 2\int_{\mathbb{R}^2} \frac{F(\sqrt{a}t_a u)}{(\sqrt{a}t_a u)^4} u^4\mathrm{d}x \ge 0 \quad \forall\, a \ge 1. \tag{4.11}$$

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Noting that $F(t)/t^4$ is nondecreasing on $t\in(-\infty,0)\cup(0,+\infty)$ by (F3), it follows from (F2) and (4.11) with $a\to+\infty$ that $at_a^2\to 0$ as $a\to+\infty$. The proof is completed. ◻

Arguing as in the above proof, we can also get the similar asymptotic behavior of $m^\infty(a)$ as follows.

COROLLARY 4.7. *Assume that* (F1), (F2), (F3), *and* (F6) *hold. Then*

$$\lim_{a\to 0^+} m^\infty(a)=+\infty \ \textit{and} \ \lim_{a\to+\infty} m^\infty(a)=0.$$

LEMMA 4.8. *Assume that* (B1), (B2), (B3), (F1), (F2), (F3), *and* (F7) *hold. Then* $m(a)<m^\infty(a)$.

Proof. In view of (ii) of Theorem 1.2, there exists $\bar u\in\mathcal{M}_a^\infty$ such that

$$(\Phi^\infty\big|_{\mathcal{S}_a})'(\bar u)=0 \quad\text{and}\quad \Phi^\infty(\bar u)=m^\infty(a). \tag{4.12}$$

As in Lemma 3.4, there exists $\bar t>0$ such that $\bar t\bar u_{\bar t}\in\mathcal{M}_a$. Since $b(x)\ge(\not\equiv)1\ \forall\ x\in\mathbb{R}^2$ and $F(\bar t\bar u)>0$, then it follows from (1.4), (1.11), (4.12) and Corollary 3.3 that

$$\begin{aligned} m^\infty(a)=\Phi^\infty(\bar u)\ge\Phi^\infty(\bar t\bar u_{\bar t})&=\frac{\bar t^2}{2}\|\nabla\bar u\|_2^2-\frac{1}{\bar t^2}\int_{\mathbb{R}^2}F(\bar t\bar u)\mathrm{d}x\\ &>\frac{\bar t^2}{2}\|\nabla\bar u\|_2^2-\frac{1}{\bar t^2}\int_{\mathbb{R}^2}b(x/\bar t)F(\bar t\bar u)\mathrm{d}x=\Phi(\bar t\bar u_{\bar t})\ge m(a).\end{aligned}$$

This shows that $m(a)<m^\infty(a)$ for any $a>0$. ◻

LEMMA 4.9. *Assume that* (F1) *and* (F2) *hold. Let* $\{u_n\}\subset H^1(\mathbb{R}^2)$ *be a sequence satisfying* $u_n\rightharpoonup 0$ *in* $H^1(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2}F(u_n)\mathrm{d}x\le C_0$ *for some constant* $C_0>0$. *Then,*

$$\int_{\mathbb{R}^2}[b(x)-1]F(u_n)\mathrm{d}x=o(1). \tag{4.13}$$

Proof. Since $0<1=\lim_{|y|\to\infty}b(y)\le b(x)$, using (2.36), we can deduce that (3.30) holds; moreover, for any given $\varepsilon>0$, there exists $R_\varepsilon>0$ such that

$$1\le b(x)\le 1+\varepsilon \qquad \forall\ |x|\ge R_\varepsilon,$$

which yields

$$\int_{|x|\ge R_\varepsilon}[b(x)-1]F(u_n)\mathrm{d}x\le\varepsilon\int_{\mathbb{R}^2}F(u_n)\mathrm{d}x\le\varepsilon C_0. \tag{4.14}$$

Noting that $u_n\to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^2)$ for $s\ge 2$, then (ii) of Lemma 2.11 gives

$$\int_{|x|\le R_\varepsilon}F(u_n)\mathrm{d}x=o(1). \tag{4.15}$$

Hence, (4.13) follows from (4.14) and (4.15) since $\varepsilon>0$ is arbitrary. ◻

Proof of Theorem 1.6. From Lemmas 2.7 and 2.9, we know that, for any $a>0$, there exists a bounded sequence $\{u_n\}\subset\mathcal{S}_a$ such that

$$\Phi(u_n)\to c(a)\in(0,2\pi/\alpha_0),\quad \Phi|'_{\mathcal{S}_a}(u_n)\to 0,\quad\text{and}\quad J(u_n)\to 0, \tag{4.16}$$

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where $c(a)$ is given by (2.8). Then, there exists $\bar{u} \in H^1(\mathbb{R}^2)$ such that, passing to a subsequence,

$$u_n \rightharpoonup \bar{u} \ \text{ in } H^1(\mathbb{R}^2), \quad u_n \to \bar{u} \ \text{ in } L^s_{\text{loc}}(\mathbb{R}^2) \text{ for } s > 2, \quad u_n \to \bar{u} \text{ a.e. in } \mathbb{R}^2. \tag{4.17}$$

Now, we claim that $\bar{u} \neq 0$.

Arguing by contradiction, suppose that $\bar{u} = 0$. Then, $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$. We expect to derive a contradiction with $m(a) < m^\infty(a)$ obtained in Lemma 4.8, and so, we need to obtain some information related to $\Phi^\infty(u_n)$. Since $\|u_n\|_2^2 = a > 0$, using Lemma 3.4, for every $n \in \mathbb{N}$, there exists $t_n > 0$ such that $t_n(u_n)_{t_n} \in \mathcal{M}_a^\infty$, i.e., $J^\infty(t_n(u_n)_{t_n}) = 0$, and so $\Phi^\infty(t_n(u_n)_{t_n}) \geq m^\infty(a)$. We next prove that both $\{t_n\}$ and $\{1/t_n\}$ are bounded. Using (4.16) and arguing as in Claim 1 for the proof of (ii) of Theorem 1.2, we can derive that

$$\delta = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_2(y)} |u_n|^2 \mathrm{d}x > 0.$$

Going if necessary to a subsequence, we may assume the existence of $y_n \in \mathbb{R}^2$ such that (3.36) holds. Let $\tilde{u}_n(x) = u_n(x + y_n)$. Then, there exists $\tilde{u} \in H^1(\mathbb{R}^2) \setminus \{0\}$ such that, passing to a subsequence, (3.38) holds. Since $J^\infty(t_n(u_n)_{t_n}) = 0$ for all $n \in \mathbb{N}$, then (F3) and (1.13) give

$$\begin{aligned}\|\nabla u_n\|_2^2 &= \frac{1}{t_n^4} \int_{\mathbb{R}^2} [f(t_n u_n) t_n u_n - 2F(t_n u_n)]\,\mathrm{d}x \\ &\geq \frac{2}{t_n^4} \int_{\mathbb{R}^2} F(t_n u_n)\mathrm{d}x = \frac{2}{t_n^4} \int_{\mathbb{R}^2} F(t_n \tilde{u}_n)\mathrm{d}x.\end{aligned} \tag{4.18}$$

Since $\tilde{u}_n \to \tilde{u}$ a.e. in $\mathbb{R}^2$, from (F1), Fatou's lemma, and (4.18), we can deduce easily that $\{t_n\}$ is bounded. This, jointly with (4.18), yields that $\int_{\mathbb{R}^2} F(t_n u_n)\mathrm{d}x$ is bounded. Using the fact that $t_n u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$ and Lemma 4.9, we have

$$\int_{\mathbb{R}^2} [1 - b(x)] F(t_n u_n)\mathrm{d}x = o(1). \tag{4.19}$$

Noting that $t_n(u_n)_{t_n} \in \mathcal{M}_a^\infty$ for every $n \in \mathbb{N}$, in view of Remark 2.3, we know that there exists a constant $\rho(a) > 0$ such that $\|\nabla(t_n(u_n)_{t_n})\|_2^2 = t_n^2 \|\nabla u_n\|_2^2 \geq \rho(a)$, which, together with the boundednss of $\{\|\nabla u_n\|_2\}$, implies that $\{1/t_n\}$ is bounded as well. Then, it follows from (1.4), (1.11), (1.14), (4.16), (4.19), and Lemmas 4.3 and 4.4 that

$$\begin{aligned} m(a) = c(a) &= \Phi(u_n) \\ &\geq \Phi(t_n(u_n)_{t_n}) + \frac{1 - t_n^2}{2} J(u_n) = \Phi(t_n(u_n)_{t_n}) + o(1) \\ &= \Phi^\infty(t_n(u_n)_{t_n}) + \frac{1}{t_n^2} \int_{\mathbb{R}^2} [1 - b(x)] F(t_n u_n)\mathrm{d}x + o(1) \\ &\geq m^\infty(a) + o(1), \end{aligned}$$

which gives $m(a) \geq m^\infty(a)$. This contradicts Lemma 4.8. Therefore, the claim that $\bar{u} \neq 0$ is proved.

By adapting the proofs of Claims 2 and 3 for the proof of (ii) of Theorem 1.2, we can deduce that $u_n \to \bar{u}$ in $H^1(\mathbb{R}^2)$, and so, $\Phi(\bar{u}) = m(a)$. Hence, for any $a > 0$, $(\bar{u}, \bar{\lambda})$ solves (1.1) by Proposition 2.12, and $\bar{u}$ is a ground state solution of (1.1) for any $a > 0$. From Lemmas 4.5 and 4.6, we have that the function $a \mapsto m(a)$ is strictly decreasing on $a \in (0, +\infty)$, $\lim_{a\to 0^+} m(a) = +\infty$, and $\lim_{a\to+\infty} m(a) = 0$. ∎

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Remark 4.10. The ideas of proofs of Theorems 1.3 and 1.7 are almost the same as those of Theorems 1.2 and 1.6, respectively. In fact, to conclude Theorems 1.3 and 1.7, we just need to replace Lemma 2.9 used in the proof of Theorems 1.2 and 1.6 by Lemma 2.10.

Acknowledgments. We would like to thank the associate editor and the reviewers for their helpful remarks.

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