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Fractional Choquard logarithmic equations with Stein-Weiss potential

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A B S T R A C T

In the present paper, we are concerned with the following fractional p-Laplacian Choquard logarithmic equation

$$(-\Delta)_p^s u+V(x)|u|^{p-2}u+(\ln|\cdot|*|u|^p)|u|^{p-2}u=\left(\int_{\mathbb{R}^N}\frac{F(y,u)}{|y|^\beta|x-y|^\mu}dy\right)\frac{f(x,u)}{|x|^\beta}\ \text{ in } \mathbb{R}^N,$$

where $N=sp\geq 2$, $s\in(0,1)$, $0<\mu<N$, $\beta\geq 0$, $2\beta+\mu\leq N$ and $(-\Delta)_p^s$ denotes the fractional p-Laplace operator, the potential $V\in\mathcal{C}(\mathbb{R}^N,[0,\infty))$, and $f:\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is continuous. Under mild conditions and combining variational and topological methods, we obtain the existence of axially symmetric solutions both in the exponential subcritical case and in the exponential critical case. We point out that we take advantage of some refined analysis techniques to get over the difficulty carried by the competition of the Choquard logarithmic term and the Stein-Weiss nonlinearity. Moreover, in the exponential critical case, we extend the nonlinearities to more general cases compared with the existing results.

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1. Introduction

1.1. Overview

In this paper, we are concerned with the fractional p-Laplacian Choquard logarithmic problems with exponential critical or subcritical Stein-Weiss type nonlinearity. The features of this paper are the following:

- (i) the appearance of the Stein-Weiss convolution term generates the lack of translation invariance;
- (ii) the presence of the Choquard logarithmic term with the peculiarity that is unbounded and sign-changing, generates an unconventional workspace;
- (iii) the certain energy level creates a bridge between exponential critical growth case and the compactness of the fractional p-Laplacian Choquard logarithmic problems;
- (iv) the analysis developed in this paper is concerned with the combined effects of the Choquard logarithmic term and the exponential-subcritical/critical nonlinearity;

Since the contents of the paper are closely concerned with the weighted nonlocal Stein-Weiss problem, we briefly recall the related background and some pioneering contributions in this field, and we start with the weighted L^p estimates for the fractional integral

$$(T_\mu\phi)(x) = \int\limits_{\mathbb{R}^N} \frac{\phi(y)}{|x-y|^\mu}\mathrm{d}y, \quad 0<\mu<N,$$

which is a fundamental problem in the field of harmonic analysis and also plays a crucial role in the analysis developed in our paper. Such weighted L^p estimates are generated from quite natural phenomena, which can be summarized as that the appearance of some suitable symmetry hypotheses, notably radial symmetry, contribute to improving the classical estimates and some embedding properties of function spaces. For example, the classical radial estimate by Strauss [43] establishes that all radial functions $u \in H^1(\mathbb{R}^N)(N \geq 2)$ satisfy

$$|x|^{(N-1)/2}|u(x)| \leq C\|\nabla u\|_{L^2}, \quad |x| \geq 1,$$

which implies that the information of gradient in $H^1(\mathbb{R}^2)$ can give a pointwise bound and also reveal the decay of u. However, this is false in the general case. So such weighted L^p estimates have practical significance in the large wide of mathematical fields.

A series of studies have been done on the weighted L^p estimates for the fractional integral T_μ. Historically, Hardy & Littlewood [23] first considered the weighted L^p estimates for the one-dimensional fractional integral operator T_μ, then Sobolev [41] extended it to the N-dimensional case. Later, Stein & Weiss [42] obtained the following two-weight extension of the Hardy-Littlewood-Sobolev inequality, which is known as the Stein-Weiss inequality.

Proposition 1. ***(Doubly weighted Hardy-Littlewood-Sobolev inequality)*** *Let $t,s>1$ and $0<\mu<N$ with $\vartheta+\beta>0$, $\frac{1}{t}+\frac{\mu+\vartheta+\beta}{N}+\frac{1}{s}=2$, $\vartheta<\frac{N}{t'}$, $\beta<\frac{N}{s'}$, $g_1 \in L^t(\mathbb{R}^N)$ and $g_2 \in L^s(\mathbb{R}^N)$, where t' and s' denote the Hölder conjugate of t and s, respectively. Then there exists a constant $C(N,\mu,\vartheta,\beta,t,s)$, independent of g_1, g_2 such that*

$$\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{g_1(x)g_2(y)}{|x-y|^\mu|y|^\vartheta|x|^\beta}\mathrm{d}x\mathrm{d}y \leq C(N,\mu,\vartheta,\beta,t,s)\|g_1\|_t\|g_2\|_s. \tag{1}$$

For $\vartheta = \beta = 0$, it is reduced to the Hartree type (also called the Choquard type) nonlinearity, which is driven by the classical Hardy-Littlewood-Sobolev inequality (See [30]).

The Stein-Weiss inequality provides quantitative information to characterize integrability for integral operators and is intrinsically determined by their dilation character. The study and understanding of the Stein-Weiss inequality have aroused an increasing interest among many scholars due to its importance in applications to problems in harmonic analysis and partial differential equations. Now, we take a look into the related applications concerning with Stein-Weiss term. Giacomoni et al. [8] studied the polyharmonic Kirchhoff equations involving the critical Choquard type exponential nonlinearity with singular weights. It is worth mentioning the beautiful work of Du et al. [18], where they investigated the following equation,

$$-\Delta u = \frac{1}{|x|^\alpha}\left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^{2^*_{\alpha,\mu}}}{|x-y|^\mu |y|^\alpha}\mathrm{d}y\right)|u(x)|^{2^*_{\alpha,\mu}-2}u, \quad x \in \mathbb{R}^N,$$

where $2^*_{\alpha,\mu} = (2N - 2\alpha - \mu)/(N-2)$. The authors developed a nonlocal version of the concentration-compactness principle to investigate the existence of solutions and study the regularity, and the symmetry of positive solutions by moving plane arguments under the critical case, as well as the results under the subcritical case. By using the moving plane arguments in integral form, Yang et al. [49] obtained the symmetry, regularity, and asymptotic properties of the weighted nonlocal system with critical exponents related to the Stein-Weiss inequality. Regarding other related results, we refer to Alves & Shen [7], Biswas et al. [9], Yang & Zhou [50], Zhang & Tang [53], Zhang, Tang & Rădulescu [54], and the references therein.

Another typical feature of our problem is the appearance of the exponential critical nonlinearities in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$. We start a short description of the development of this research with the Trudinger-Moser inequality. There are several results about the Trudinger-Moser inequality in the Sobolev-Slobodeckij spaces [1,3,20,33,34]. Concerning the N-dimensional fractional p-Laplacian equation, based on the result [38] and by applying a slightly modified version of the Trudinger-Moser sequence, Parini & Ruf [37] established the following local fractional Trudinger-Moser inequality.

Lemma 2. *Let Ω be a bounded, open domain of $\mathbb{R}^N (N \geq 2)$ with Lipschitz boundary, and let $s \in (0,1)$, $sp = N$. Then there exists an exponent α of the fractional Trudinger-Moser inequality such that*

$$\sup_{u \in \tilde{W}^{s,p}_0(\Omega), [u]_{s,p} \leq 1} \int\limits_\Omega \exp(\alpha |u|^{\frac{N}{N-s}}) < +\infty.$$

Set

$$\alpha_* = \alpha_*(s,\Omega) := \sup\left\{\alpha : \sup_{u \in \tilde{W}^{s,p}_0(\Omega), [u]_{s,p} \leq 1} \int\limits_\Omega \exp(\alpha |u|^{\frac{N}{N-s}}) < +\infty\right\}.$$

Moreover, $\alpha_* \leq \alpha^*_{s,N}$, where

$$\alpha^*_{s,N} := N\left(\frac{2(N\omega_N)^2\Gamma(p+1)}{N!}\sum_{k=0}^{+\infty}\frac{(N+k-1)!}{k!}\frac{1}{(N+2k)^p}\right)^{\frac{s}{N-s}},$$

and for $s \in (0,1)$ the Sobolev-Slobodeckij space $\tilde{W}^{s,p}_0(\Omega)$ is defined by the completion of $\mathcal{C}^\infty_0(\Omega)$ with respect of the norm

$$u \mapsto \|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + [u]^p_{s,p}\right)^{\frac{1}{p}},$$

where $[u]_{s,p}$ *is the Gagliardo seminorm*

$$[u]_{s,p} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.$$

Then Zhang [51] generalized the local fractional Trudinger-Moser inequality [37] to the whole space as follows.

Lemma 3. *Let* $s \in (0,1)$ *and* $sp = N$*. Then for every* $0 \le \alpha < \alpha_*$*, the following inequality*

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{\frac{N}{N-s}})\mathrm{d}x < +\infty \tag{2}$$

holds, where $\Phi_{N,s}(t) = e^t - \sum_{i=0}^{j_p-2} \frac{t^j}{j!}$ *and* $j_p := \min\{j \in \mathbb{N} : j \ge p\}$*. Moreover, for* $\alpha \ge \alpha^*_{s,N}$*,*

$$\sup_{u \in W^{s,p}(\mathbb{R}^N), \|u\|_{W^{s,p}(\mathbb{R}^N)} \le 1} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha |u|^{\frac{N}{N-s}})\mathrm{d}x = +\infty,$$

where $s \in (0,1)$ *and* $p > 2$*, the Sobolev space* $W^{s,p}(\mathbb{R}^N)$ *is defined by*

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]^p_{s,p} < +\infty\}.$$

Remark 1. As explored by Zhang [51, Remark 1.2], $\alpha^*_{s,N}$ is just an upper bound of α_*, but they did not give the precise value of α_*.

Based on the Lemma 3, we say that $f(x,t)$ has subcritical exponential growth at $t = +\infty$ if it satisfies

(F1) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\sup_{x \in \mathbb{R}^N, |s| \le |t|} |f(x,s)| < +\infty$ for every $t \in \mathbb{R}$, and

$$\lim_{t \to +\infty} f(x,t)\exp(-\alpha t^{\frac{N}{N-s}}) = 0 \quad \text{uniformaly on } x \in \mathbb{R}^N \text{ for all } \alpha > 0.$$

We say that $f(x,t)$ has the critical exponential growth at $t = +\infty$ if it satisfies

(F1′) $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $\sup_{x \in \mathbb{R}^N, |s| \le |t|} |f(x,s)| < +\infty$ for every $t \in \mathbb{R}$, and there exists $\alpha_0 > 0$ such that

$$\lim_{t \to +\infty} f(x,t)\exp(-\alpha t^{\frac{N}{N-s}}) = 0 \;\; \text{uniformly on } x \in \mathbb{R}^N \text{ for all } \alpha > \alpha_0$$

and

$$\lim_{t \to +\infty} f(x,t)\exp(-\alpha t^{\frac{N}{N-s}}) = +\infty \;\; \text{uniformly on } x \in \mathbb{R}^N \text{ for all } \alpha < \alpha_0.$$

Based on the Trudinger-Moser inequalities, many authors considered the existence of weak solutions for the N-dimensional nonlinear equations as an application. On the bounded domain Ω, we refer to de Freitas [22] for quasilinear problems involving the N-Laplacian operator; Lam & Lu [27] for polyharmonic equations; Lam & Lu [28] for elliptic equations and systems. As for the whole space, we refer to Lam & Lu [26] for N-Laplacian equations and Alves & Figueiredo [5] for quasilinear problems.

Among the investigations into the nonlinear partial differential equation with Stein-Weiss term in the exponential critical case, to the best of our knowledge, there is the only work Biswas et al. [9]. The authors considered a kind of quasilinear Schrödinger equations,

$$-\Delta_N u - \Delta_N(u^2)u + V(x)|u|^{N-2}u = \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u)}{|y|^\beta |x-y|^\mu} dy\right) \frac{f(x,u)}{|x|^\beta} \quad \text{in} \quad \mathbb{R}^N, \tag{3}$$

where $N \geq 2$, $0 < \mu < N$, $\beta \geq 0$ and $2\beta + \mu \leq N$. The potential $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function satisfying $0 < V_0 \leq V(x)$ for all $x \in \mathbb{R}^N$ and $\lim_{|x|\to+\infty} V(x) = V_\infty$, and the nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function with critical exponential growth in the sense of the Trudinger-Moser inequality. Different from the condition (f) in [10], they applied the following assumption

(f') assume that

$$\lim_{s\to+\infty} \frac{sf(x,s)F(x,s)}{\exp\left(2|s|^{\frac{2N}{N-1}}\right)} = \infty, \quad \text{uniformly in } x \in \mathbb{R}^N,$$

to exclude the vanishing case of the Cerami sequence. Then compared with the energy level of the limit equation, which is the equation (3) with $V(x) = V_\infty$, they could verify that equation (3) has a non-trivial positive weak solution.

For related results on the fractional p-Laplacian equations in $\mathbb{R}^N$ with the nonlinearity satisfying exponential critical growth, we refer to [10,35,39]. Pei [39] investigated the existence of nontrivial solutions to a class of quasilinear fractional p-Laplacian problems without Ambrosetti-Rabinowitz (AR) condition. Nguyen [35] investigated the singular Schrödinger systems involving the fractional p-Laplacian and exponential critical nonlinearities in the sense of the Trudinger-Moser inequality. Recently, Böer & Miyagaki [10] considered the fractional p-Laplacian Choquard logarithmic equation involving a nonlinearity with the exponential critical and subcritical growth. In particular, they studied the existence and multiplicity of the equation

$$(-\Delta)_p^s u + |u|^{p-2}u + (\ln|\cdot| * |u|^p)|u|^{p-2}u = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{4}$$

where $N = sp$, $s \in (0,1)$, $p > 2$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous, with primitive $F(t) = \int_0^t f(\tau)\mathrm{d}\tau$, and $(-\Delta)_p^s$ denotes the fractional p-Laplace operator. To guarantee that the Cerami sequence and the minimizing sequence of the ground state energy level satisfy the energy estimation, they relied on following hypothesis

(f) there exist $q > 2p$ and $C_q > \frac{[2(q-p)]^{\frac{q-p}{p}}}{q^{\frac{q}{p}}} \frac{S_q^q}{\rho_0^{q-p}}$ such that $F(t) \geq C_q|t|^q$ for all $t \in \mathbb{R}$,

where the parameters $S_q, \rho_0 > 0$ defined in [10, Lemma 3.7] involve the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. After excluding the vanishing case of the Cerami sequence $\{u_n\}$, they verified carefully the properties that

$$\phi'(\tilde{u}_n)(\tilde{u}_n - u) \to 0 \quad \text{and} \quad \int\limits_{\mathbb{R}^N} f(\tilde{u}_n)(\tilde{u}_n - u)\mathrm{d}x \to 0 \quad \text{as} \quad n \to +\infty,$$

where u is the weak limit of the translated sequence $\{\tilde{u}_n\}$, and ϕ is the energy functional of equation (4). These properties contributed to obtaining the convergence $\tilde{u}_n \to u$ in X', where X' is the subspace of

$W^{s,p}(\mathbb{R}^N)$. Moreover, they verified that there exists a constant C such that $\|v(x+y_n)\|_{X'} \leq C\|u_n\|$, where v is an arbitrary element in X'. Then it is easy to check that u is a nontrivial critical point of ϕ in X'. In the second part of [10], they used the genus theory to prove the multiplicity result. Here, we also refer the reader to [37,51] and the references therein for the relative progress of such problems.

Finally, it is worth mentioning that there has been increasing interests into the study of fractional p-Laplacian and non-local operators of elliptic type in recent years. Such type of operators arise in a quite natural way in many different applications, such as optimization, finance, phase transitions, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws and water waves. See [21,24,31,47,48] and the references therein for more detailed introductions and applications on the fractional p-Laplacian. We also refer to the work by Molica Bisci, Rădulescu & Servadei [32] for a comprehensive analysis of nonlocal fractional problems.

1.2. Main goal and difficulties

In this paper, we are concerned with the following fractional p-Laplacian Choquard logarithmic problems with exponential critical or subcritical Stein-Weiss type nonlinearity,

$$(-\Delta)_p^s u + V(x)|u|^{p-2}u + (\ln|\cdot| * |u|^p)|u|^{p-2}u = \left(\int_{\mathbb{R}^N} \frac{F(y,u)}{|y|^\beta |x-y|^\mu} dy\right) \frac{f(x,u)}{|x|^\beta} \quad \text{in} \quad \mathbb{R}^N, \tag{5}$$

where $N = sp \geq 2$, $s \in (0,1)$, $0 < \mu < N$, $\beta \geq 0$, $2\beta + \mu \leq N$ and $(-\Delta)_p^s$ denotes the fractional p-Laplacian operator, which, up to normalization factors, can be defined as

$$(-\Delta)_p^s u(x) = C(N,s) \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} \mathrm{d}y$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. Throughout this paper we omit the normalizing constant to simplify the expressions. The potential $V \in \mathcal{C}(\mathbb{R}^N, [0,\infty))$, and $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous nonlinearity with exponential critical and subcritical growth, and with primitive $F(x,t) = \int_0^t f(x,\tau)\mathrm{d}\tau$.

Here we point out that the Choquard Logarithmic term together with the Stein-Weiss type nonlinearity with the exponential critical growth brings some difficulties in our analysis, which can be summarized as follows.

i) To overcome the obstacle caused by the sign-changing and unbounded potential

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \ln(|x-y|)|u(x)|^p|u(y)|^p \mathrm{d}x\mathrm{d}y,$$

following the methods in [12,17,44], we shall use the smaller Hilbert space

$$X = \left\{ u \in W^{s,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} [V(x) + \ln(1+|x|)]u^p \mathrm{d}x < \infty \right\}$$

and the norm $\|\cdot\|_*$ that will be introduced in Section 2. However, the norm $\|\cdot\|_*$ lacks translation invariance. Even if we could overcome this difficulty by applying the symmetric bilinear form

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \ln(1+|x-y|)|u(x)|^p|v(y)|^p \mathrm{d}x\mathrm{d}y,$$

this problem still exists due to the appearance of the Stein-Weiss convolution term.

ii) One of the main difficulties comes from dealing with the exponential growth case. We need to verify that if $u_n \rightharpoonup \bar{u}$ in X, then

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x$$

converges to

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))\bar{u}(x)}{|x|^\beta}\mathrm{d}x,$$

which was adopted to find the ground state solution of the equation (5). However, our problem is raised in the exponential growth case, and it does not hold even if $u_n \to \bar{u}$ in $L^q(\mathbb{R}^N)$ for $q \in [p, +\infty)$ since the inequality introduced in Lemma 2 needs the critical exponent α_0 to be less than α_*.

iii) We will be working with an exponential term. In order to guarantee that the Cerami sequence for the ground state satisfies the exponential estimates, we need to give a detailed analysis by using the fractional Trudinger-Moser type inequality in the whole space $\mathbb{R}^N$. However due to the appearance of the Choquard logarithmic term and the Stein-Weiss term, controlling the minimax levels and excluding the case that the Palais-Smale sequence is vanishing become more complicate and difficult.

When $N = 2$, $p = 2$, $s = 1$, and the nonlinearity term is the general nonlinearity $f(x, u)$, problem (5) reduces to the following Schrödinger equation

$$-\Delta u + V(x)u + \left(\int_{\mathbb{R}^2}\ln|x-y|u^2(y)\mathrm{d}y\right)u = f(x,u), \quad x \in \mathbb{R}^2. \tag{6}$$

Similarly, one typical feature of the equation is the appearance of the sign-changing and unbounded logarithmic term, which leads to a situation where the corresponding energy function of equation (6) could not be well-defined on $H^1(\mathbb{R}^2)$. Inspired by [44], Cingolani & Weth [17] developed a variational framework for such kinds of problems, that is to consider the smaller Hilbert space

$$X_* = \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2}\ln(1+|x|)u^2\mathrm{d}x < \infty\right\},$$

which contributes to overcome the difficulty caused by the typical feature of equation (6). After this work, a series of subsequent studies have been done on the existence and multiplicity of the solutions to the nonlinear Choquard Logarithmic equations, see for example [13,14,16,19]. For problems with the exponential critical growth, we refer to [4,6,15,45]. Chen & Tang [15] first obtained the existence of solution with an axially symmetric setting, instead of radial symmetry. In particular, they investigated equation (6) under the conditions (V_0') and (F_0'), where

(V_0') $V \in \mathcal{C}(\mathbb{R}^2, [0,\infty))$, $V(x) = V(x_1, x_2) = V(|x_1|, |x_2|)$ for all $x \in \mathbb{R}^2$ and $\liminf_{|x|\to\infty} V(x) > 0$,
(F_0') $f(x,t) := f(x_1, x_2, t) = f(|x_1|, |x_2|, t)$ for all $(x,t) \in \mathbb{R}^2 \times \mathbb{R}$,

then they could introduce the following smaller space

$$H^1_{ps} = \{u \in H^1(\mathbb{R}^2) : u(x) := u(x_1, x_2) = u(|x_1|, |x_2|),\ \forall x \in \mathbb{R}^2\}.$$

With the help of [46, Theorem 1.28], they could show that $X_* \cap H^1_{ps}$ is a natural constraint of X. Another advantage of using axially symmetry is that they could control the part of the Logarithmic term

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln(|x-y|)u^2(x)u^2(y)\mathrm{d}x\mathrm{d}y$$

from below by a standard term which consists partially of the norm of space X_*. Applying certain new tricks, they obtained the existence of the nontrivial solution, the ground state solution of Nehari-type, and infinitely many solutions to the above system under some weak assumptions on V and f. Notice that very recently, Cao, Dai & Zhang [12] considered the following Schrödinger-Newton equation in $\mathbb{R}^2$,

$$-\Delta u + a(x)u + \frac{\gamma}{2\pi}\left(\ln(|\cdot|) * |u|^p\right)|u|^{p-2}u = b|u|^{q-2}u,$$

where $\inf_{\mathbb{R}^2} a > 0$, $\gamma > 0$, $b \geq 0$, $p \geq 2$ and $q \geq 2$. They obtained the existence of ground state solutions and mountain pass solutions to the above equations for $p \geq 2$ and $q \geq 2p - 2$ via variational methods.

Inspired by the axially symmetric setting explored in Chen & Tang [15], we make the hypotheses.

(V0) $V \in \mathcal{C}(\mathbb{R}^N, [0, \infty))$, $V(x) = V(x_1, x_2, ..., x_n) = V(|x_1|, |x_2|, ..., |x_n|)$ for all $x \in \mathbb{R}^N$ and $\liminf_{|x|\to\infty} V(x) > 0$.

(F0) $f(x,t) := f(x_1, x_2, ..., x_n, t) = f(|x_1|, |x_2|, ..., |x_n|, t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$.

Together with Proposition 6 illustrated in next section, we could avoid the problem i). However, it is far from understood whether the equation (5) still possesses nontrivial solutions under the exponential subcritical and critical cases since the appearance of the Stein-Weiss type convolution and whether the variational method developed in Chen & Tang [15] can be extended to the nonlinear Choquard Logarithmic equation (5). In particular, although we could rely on the approach explored in [15] to surmount difficulty ii), but there still exists a problem for verifying whether the fact that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u(x)}{|x|^\beta}\mathrm{d}x \to \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)}{|x|^\beta}\mathrm{d}x$$

holds when $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$. In order to obtain the above fact, there is the only paper that dealing with the nonlinear equation with Stein-Weiss convolution term [9], where they used the Radon-Nikodym theorem to obtain that when $u_n \rightharpoonup u$ in $W^{s,p}(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))\phi(x)}{|x|^\beta}\mathrm{d}x = \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))\phi(x)}{|x|^\beta}\mathrm{d}x$$

holds for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$. However, since the appearance of the Choquard Logarithmic term and since we can not prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\ln(|x-y|)|u_n(x)|^p|u_n(y)|^{p-2}u_n\phi\mathrm{d}x\mathrm{d}y \to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\ln(|x-y|)|u(x)|^p|u(y)|^{p-2}u\phi\mathrm{d}x\mathrm{d}y,$$

the arguments explored in [9] is not available in our problem, which implies that we need some new tricks and approaches.

1.3. Main results

To state our results, at the beginning, we introduce some assumptions. In the subcritical exponential growth case, namely (F1) holds, in addition to (V0) and (F0), we also assume that f satisfies the following conditions:

(F2) $\lim_{t\to 0} \frac{f(x,t)}{t^{\frac{(2N-2\beta-\mu)p}{2N}}} = 0$ uniformly in $x \in \mathbb{R}^N$;
(F3) there exists $\theta > 2p$ such that $f(x,t)t \geq \theta F(x,t) > 0$, for all $t > 0$.

The first main result in this paper establishes the existence of solutions of (5) in the subcritical case.

Theorem 4. *Assume that V and f satisfy* (V0) *and* (F0)-(F3). *Then* (5) *has an axially symmetric solution* $\bar{u} \in X \setminus \{0\}$.

In the critical exponential growth case, namely (F1′) holds, in order to find positive solutions of equation (5), we assume that $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and for a.e. $x \in \mathbb{R}^N$, $f(x,s) = 0$ for $s \leq 0$ and $f(s) > 0$ for $s > 0$. In addition to (F0), (F1′) and (F2), we also assume that V and f satisfy:

(V1) $V \in \mathcal{C}(\mathbb{R}^N, [1,\infty))$, $V(x) = V(x_1, x_2, ..., x_n) = V(|x_1|, |x_2|, ..., |x_n|)$ for all $x \in \mathbb{R}^N$ and $\liminf_{|x|\to\infty} V(x) > 0$;
(F4) $\frac{f(x,t)}{|t|^{2p-1}}$ is nondecreasing on $(0, +\infty)$;
(F5) there exist M_0 and $t_0 > 0$ such that for any $|t| \geq t_0$, $F(t) \leq M_0|f(t)|$;
(F6) $\liminf_{t\to\infty} f(t)/e^{\alpha_0 t^{\frac{N}{N-s}}} = \kappa$, and κ satisfies that

$$\left[\tilde{C} w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} + 1\right] \mathcal{O}\left(\frac{1}{\ln n}\right) + \frac{(N-s) + C_1 \ln[\Lambda \kappa^{-2} C_2 (1 + V_\infty \rho^N A \delta_n)]}{s(2N - 2\beta - \mu)\ln n}$$
$$- \frac{C_3 \alpha^*_{s,N}(N-s)}{2\alpha_0 N s \ln n} \left[\frac{2\alpha_0 N}{(2N-2\beta-\mu)\alpha^*_{s,N}}\right]^{\frac{N-s}{s}} + 1 - \left(\frac{\alpha_{N,s}}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} < 0,$$

where n is large enough and Λ, $\tilde{C}$, V_∞, ρ, $\alpha_{N,s}$, δ_n will be explained in Section 4.

Remark 2. By (F3)-(F4), we can obtain that $F(x,t) \geq 0$, $\frac{1}{2p} f(x,t)t - F(x,t) \geq 0$ and $\frac{F(x,t)}{t^{2p}}$ is nondecreasing on $(-\infty, 0) \cup (0, +\infty)$. We would like to point out that the conditions (F3)-(F4) can be weakened. However, we do not involve the weakening conditions in our work, since we are aiming to figure out the feature of the Choquard Logarithmic term $(\ln|\cdot| * |u|^p)|u|^{p-2}u$ and the feature of exponential critical growth on the Stein-Weiss nonlinearity, in which weakening conditions (F3)-(F4) do not bring any effective changes to these aspects.

Remark 3. We shall use the weaker assumption (F6) to control the energy level of the Cerami sequence by a fine threshold, which needs a more complicated process compared with (f) and (f'), see Lemma 16. This kind of condition like (F6) somehow reveals more essential features of the exponential critical growth given in (F1′).

Next, we give the second result in the exponential critical case.

Theorem 5. *Assume that V and f satisfy* (V1), (F0), (F1′), (F2) *and* (F4)-(F6). *Then* (5) *has an axially symmetric positive solution $\bar{u} \in \mathcal{N}$ such that $I(\bar{u}) = m := \inf_{\mathcal{N}} I$ when $\beta < \mu$, where*

$$\mathcal{N} := \{u \in E \setminus \{0\} : \langle I'(u), u\rangle = 0\},$$

$E := X \cap W^{s,p}_{as}$ *and*

$$W^{s,p}_{as} = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) := u(x_1, x_2, ..., x_N) = u(|x_1|, |x_2|, ..., |x_N|),\ \forall\ x \in \mathbb{R}^N\}.$$

Remark 4. In view of the [9, Remark 1.5], we could conclude that if potential term V satisfies the compact-coercive case, or radially symmetric case, or asymptotic case of a periodic function, then we could use the arguments explored in [9]. It is worth mentioning here that by (V0)(or (V1)) our framework is far from the radially symmetric case. Compared with the condition (V0), condition (V1) ensures that $\inf_{x\in\mathbb{R}^N} V(x) \geq 1$, which contributes to establish the relationship between two norms illustrated later.

The present paper is organized as follows. In Section 2, we discuss the variational setting and some preliminary results. Section 3 is devoted to establishing the mountain pass geometry and some analysis with the mountain pass level. In Section 4, we prove the existence of the axially symmetric solution of the equation (5) in the exponential subcritical case. In Section 5, we show the existence of the axially symmetric positive solution of the equation (5) in the exponential critical case after some refined analysis.

Notation. Throughout this paper, we make use of the following notations:

- C, c, C_i, c_i $(i = 1, 2, ...,)$ denote positive constants which may vary from line to line.
- For any exponent $p > 1$, p' denotes the conjugate of p and is given as $p' = p/(p-1)$.
- $B_r(x)$ denotes the ball of radius r centered at $x \in \mathbb{R}^N$.
- The arrows $\rightharpoonup$ and $\to$ denote the weak convergence and strong convergence, respectively.
- $L^s(\mathbb{R}^N)(1 \leq s < +\infty)$ denotes the Lebesgue space with the norm $\|u\|_s = (\int_{\mathbb{R}^N} |u|^s \mathrm{d}x)^{1/s}$.

2. Preliminary results

In this section we recall some preliminary results. We first recall that the fact $0 < s < 1$ and $sp = N$, and the Sobolev space $W^{s,p}(\mathbb{R}^N)$ is defined by

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]^p_{W^{s,p}(\mathbb{R}^N)} < +\infty\},$$

where $[u]_{W^{s,p}(\mathbb{R}^N)}$ is the Gagliardo seminorm

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y\right)^{\frac{1}{p}}.$$

It is well-known that the space $(W^{s,p}(\mathbb{R}^N), \|\cdot\|_{W^{s,p}(\mathbb{R}^N)})$, where $\|\cdot\|^p_{W^{s,p}(\mathbb{R}^N)} = [\cdot]^p_{s,p} + \|\cdot\|^p_p$, is an uniformly convex Banach space, particularly reflexive, and separable. We also remind the reader that $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, see [2, Theorem 7.38].

Note that if $u \in W^{s,p}(\mathbb{R}^N)$ is a weak solution for (5), that is for every $v \in W^{s,p}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\mathrm{d}x$$

$$+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \ln(|x-y|)|u(x)|^p|u(y)|^{p-2}u(y)v(y)\mathrm{d}x\mathrm{d}y$$

$$-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))v(x)}{|x|^\beta}\mathrm{d}x=0.$$

In this sense, we consider the associated energy functional associated to the problem (5),

$$\begin{aligned}I(u)=&\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y+\frac{1}{p}\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x\\&+\frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\ln(|x-y|)|u(x)|^p|u(y)|^p\mathrm{d}x\mathrm{d}y\\&-\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x.\end{aligned}\tag{7}$$

Then, the critical points of I will be weak solutions for (5). However, one can see that I is not well defined over the whole space $W^{s,p}(\mathbb{R}^N)$ due to the appearance of the Choquard logarithmic term. Hence, following the ideas introduced by Stubbe [44], we consider the slightly smaller space

$$X=\left\{u\in W^{s,p}(\mathbb{R}^N):\int_{\mathbb{R}^N}[V(x)+\ln(1+|x|)]\,|u(x)|^p\mathrm{d}x<+\infty\right\}.\tag{8}$$

Let

$$\|u\|^p:=[\,\cdot\,]_{s,p}^p+\|\cdot\|_{V,p}^p,$$

where $\|u\|_{V,p}^p=\int_{\mathbb{R}^N}V(x)|u|^p\mathrm{d}x$. We define, for any measurable function $u:\mathbb{R}^N\to\mathbb{R}$

$$\|u\|_*=\int_{\mathbb{R}^N}\ln(1+|x|)|u(x)|^p\mathrm{d}x\in[0,\infty].\tag{9}$$

Then $\|u\|_E:=(\|u\|^p+\|u\|_*^p)^{\frac{1}{p}}$ is a norm on X. The space $(X,\|\cdot\|_E)$ is uniformly convex and reflexive. Here, we give the useful embedding properties in X as [9, Proposition 2.1] and [15, Lemma 2.1].

Proposition 6. *Assume that* (V0) *holds, then there exists $\gamma_0>0$ such that*

$$\gamma_0\|u\|_{W^{s,p}(\mathbb{R}^N)}\le\|u\|,\forall\ u\in X,$$

and the space X is continuous embedded in $W^{s,p}(\mathbb{R}^N)$. Moreover, the space X is compactly embedded in $L^\omega(\mathbb{R}^N)$, for all $\omega\ge p$.

Next, we consider the property of the continuity of the functional I. The following lemma contributes to controlling the exponential term.

Lemma 7. *Let $(\phi_n)\subset X$ and $\phi\in X$. Then,*

(a) *if* $\phi_n \to \phi$ *in* $W^{s,p}$ *or* $\phi_n \rightharpoonup \phi$ *in* X*, then there exists a subsequence* $(\phi_{n_k}) \subset (\phi_n)$ *and a function* $h \in W^{s,p}(\mathbb{R}^N)$ *such that* $\phi_{n_k} \to \phi(x)$ *a.e. in* $\mathbb{R}^N$ *and* $|\phi_{n_k}(x)| \leq h(x)$ *for all* $k \in \mathbb{N}$ *and a.e. in* $\mathbb{R}^N$*.*

(b) *if* $\phi_n \to \phi$ *in* X*, then there exists a subsequence* $(\phi_{n_k}) \subset (\phi_n)$ *and a function* $h \in X$ *such that* $\phi_{n_k}(x) \to \phi(x)$ *a.e. in* $\mathbb{R}^N$ *and* $|\phi_{n_k}| \leq h(x)$*, for all* $k \in \mathbb{N}$ *and a.e. in* $\mathbb{R}^N$*.*

We define the operators $\Upsilon : W^{s,p}(\mathbb{R}^N) \times W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ by

$$\Upsilon(u,v) := \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y, \quad \forall u,v \in W^{s,p}(\mathbb{R}^N)$$

and $\tilde{\Upsilon} : W^{s,p}(\mathbb{R}^N) \times W^{s,p}(\mathbb{R}^N) \to \mathbb{R}$ is defined by

$$\tilde{\Upsilon}(u,v) = \Upsilon(u,v) + \int\limits_{\mathbb{R}^N} V(x)|u|^{p-2}uv\mathrm{d}x, \quad \forall u,v \in W^{s,p}(\mathbb{R}^N).$$

One can easily verify that $\tilde{\Upsilon}(u,v) \leq \|u\|^{p-1}\|v\|$ and $\tilde{\Upsilon}(u,u) = \|u\|$ for all $u,v \in W^{s,p}(\mathbb{R}^N)$.

Inspired by [17], we define the following symmetric bilinear forms

$$(u,v) \mapsto A_1(u,v) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln(1+|x-y|)u(x)v(y)\mathrm{d}x\mathrm{d}y,$$

$$(u,v) \mapsto A_2(u,v) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln\left(1+\frac{1}{|x-y|}\right)u(x)v(y)\mathrm{d}x\mathrm{d}y,$$

$$u \mapsto A_0(u,v) = A_1(u,v) - A_2(u,v) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln(|x-y|)u(x)v(y)\mathrm{d}x\mathrm{d}y,$$

where the definition is restricted, in each case, to measurable functions $u,v : \mathbb{R}^N \to \mathbb{R}$ such that the corresponding double integral is well defined in Lebesgue sense. Noting that $0 \leq \ln(1+r) \leq r$ for $r \geq 0$, it follows from the Hardy-Littlewood-Sobolev inequality that

$$|A_2(u,v)| \leq \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{1}{|x-y|}|u(x)||v(y)|\mathrm{d}x\mathrm{d}y \leq \mathcal{C}_1\|u\|_{2N/(2N-1)}\|v\|_{2N/(2N-1)} \tag{10}$$

with a constant $\mathcal{C}_1 > 0$. Using above three symmetric bilinear forms, we define three auxiliary functions $V_1 : W^{s,p}(\mathbb{R}^N) \to [0,\infty)$, $V_2 : L^{\frac{2Np}{2N-1}}(\mathbb{R}^N) \to [0,\infty)$ and $V_0 : W^{s,p}(\mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$, given by

$$u \mapsto V_1(u) = A_1(|u|^p,|u|^p) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln(1+|x-y|)|u(x)|^p|u(y)|^p\mathrm{d}x\mathrm{d}y,$$

$$u \mapsto V_2(u) = A_2(|u|^p,|u|^p) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln\left(1+\frac{1}{|x-y|}\right)|u(x)|^p|u(y)|^p\mathrm{d}x\mathrm{d}y,$$

$$u \mapsto V_0(u) = A_0(|u|^p,|u|^p) = V_1(u) - V_2(u) = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln(|x-y|)|u(x)|^p|u(y)|^p\mathrm{d}x\mathrm{d}y.$$

Remark 5. (i) As a consequence of Hardy-Littlewood-Sobolev Inequality (HLS) [29], with $\alpha = \beta = 0$ and $\lambda = 1$, we have $\frac{1}{q} + \frac{1}{t} + \frac{1}{N} = 2$. So, making a natural choice for q and t, that is $q = t = 2N/(2N-1)$, we obtain that

$$|V_2(u)| \leq K_0 \|u\|^{2p}_{\frac{2Np}{2N-1}}, \quad \forall\, u \in L^{\frac{2Np}{2N-1}}(\mathbb{R}^N), \tag{11}$$

so V_2 takes finite values over $L^{\frac{2Np}{2N-1}}(\mathbb{R}^N)$;

(ii) $V_1(u) \leq 2\|u\|_*^p \|u\|_{p'}^p$, since $\ln(1+|x-y|) \leq \ln(1+|x|) + \ln(1+|y|)$;

(iii) $\int_{\mathbb{R}^{2N}} \ln(1+|x-y|)|u(x)|^p|v(y)|^p \mathrm{d}x\mathrm{d}y \leq \|u\|_*^p\|v\|_p^p + \|v\|_*^p\|u\|_{p'}^p$.

According to the arguments explored in [9] and [10], we can obtain that V_0, V_1, V_2 are of class $\mathcal{C}^1(X,\mathbb{R})$, with

$$V_1'(u)(v) = 2p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \ln(1+|x-y|)|u(x)|^p|u(y)|^{p-2}u(y)v(y)\mathrm{d}x\mathrm{d}y$$

and

$$V_2'(u)(v) = 2p \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \ln\left(1+\frac{1}{|x-y|}\right)|u(x)|^p|u(y)|^{p-2}u(y)v(y)\mathrm{d}x\mathrm{d}y.$$

From the assumptions (F1)(or (F1$'$)) and (F2), we obtain that for any $\varepsilon > 0$, $q \geq p$, there exist positive constants $C(q,\varepsilon) > 0$ and $\alpha > 0$ (or $\alpha > \alpha_0 > 0$) such that

$$|F(x,s)| \leq \varepsilon |u|^{\frac{(2N-2\beta-\mu)p}{2N}} + C(q,\varepsilon)|s|^q\left[\exp(\alpha|s|^{\frac{N}{N-s}}) - S_{k_p-2}(\alpha,s)\right] \text{ for all } (x,s) \in \mathbb{R}^N \times \mathbb{R}. \tag{12}$$

Thus, in light of the Sobolev embedding, for any $u \in W^{s,p}(\mathbb{R}^N)$, $F(x,u) \in L^q(\mathbb{R}^N)$ for any $q \geq 2N/(2N-2\beta-\mu)$. In view of Proposition 1 with $t = s$ and $\vartheta = \beta$, we can obtain

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x \leq C(N,\mu,\beta)\|F(\cdot,u)\|^2_{2N/(2N-2\beta-\mu)}. \tag{13}$$

By Lemma 3, (12), (13) and the standard arguments, we can obtain that I is well defined and $I \in \mathcal{C}^1(X,\mathbb{R})$, and

$$I(u) = \frac{1}{p}\|u\|^p + \frac{1}{2p}V_0(u) - \frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x, \ \forall\, u \in X, \tag{14}$$

$$\begin{aligned}\langle I'(u),v\rangle = &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^N} V(x)|u|^{p-2}uv\mathrm{d}x \\ &+ A_0(|u|^p,|u|^{p-2}uv) - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))v(x)}{|x|^\beta}\mathrm{d}x, \ \forall\, u,v \in X,\end{aligned} \tag{15}$$

and

$$\langle I'(u),u\rangle = \|u\|^p + V_0(u) - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)}{|x|^\beta}\mathrm{d}x, \ \forall\, u \in X. \tag{16}$$

Hence, the solutions of the equation (5) are the critical points of the functional (7).

3. Mountain pass geometry and technical results

In this section we will investigate the geometry of I and provide some technical results. First, we give the following version of the mountain pass Theorem, by which we can verify the mountain pass geometry of I.

Lemma 8. *Let Y be a real Banach space and let $I \in \mathcal{C}^1(Y, \mathbb{R})$. Let S be a closed subset of Y which disconnects (archwise) Y in distinct connect components Y_1 and Y_2. Suppose further that $I(0) = 0$ and*
(i) *$0 \in Y_1$ and there is $\alpha > 0$ such that $I|_S \geq \alpha$,*
(ii) *there is $e \in Y_2$ such that $I(e) \leq 0$.*
Then I possesses a $(Ce)_c$ sequence with $c \geq \alpha > 0$ given by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in \mathcal{C}([0, 1], Y) : \gamma(0) = 0, \gamma(1) = e\}.$$

Now we choose $w_0 \in E \setminus \{0\}$, then it is easy to show that $\lim_{t\to\infty} I(tw_0) = -\infty$ due to (F3). To make the notation concise, we set, for $\alpha > 0$ and $t \in \mathbb{R}$,

$$\mathcal{H}(\alpha, t) = \exp(\alpha|t|^{\frac{N}{N-s}}) - S_{k_p-2}(\alpha, t) = \sum_{k=k_p-1}^{+\infty} \frac{\alpha^k}{k!}|t|^{\frac{Nk}{N-s}},$$

where $S_{k_p-2}(\alpha, t) = \sum_{k=0}^{k_p-2} \frac{\alpha^k}{k!}|t|^{\frac{Nk}{N-s}}$ and $k_p = \min\{k \in \mathbb{N}; k \geq p\}$.

Lemma 9. *Assume that* (V0), (F0), (F1) *(or* (F1′)*),* (F2) *and* (F3) *hold, then there exist a constant $\bar{c} \in (0, \sup_{t\geq0} \Phi(tw_0)]$ and a sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to \bar{c}, \quad \|\Phi'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0. \tag{17}$$

Proof. By Proposition 6, the embedding $X \hookrightarrow L^w(\mathbb{R}^N)$ is continuous for $w \in [p, \infty)$, which implies that there exists $\gamma_w > 0$ such that

$$\|u\|_w \leq \gamma_w\|u\|, \quad \forall\, u \in X.$$

By (F1)(or (F1′)) and (F2), for each $\varepsilon > 0$, there exists some constants $\alpha > 0$, $q > p$ and $C_\varepsilon > 0$ such that

$$|F(x, u)| \leq \varepsilon|u|^{\frac{(2N-2\beta-\mu)p}{2N}} + C_\varepsilon|u|^q\mathcal{H}(\alpha, u).$$

In view of Lemma 3, for all $\|u\|_{W^{s,p}}$ satisfies

$$\frac{4N\alpha\|u\|^{\frac{N}{N-s}}_{W^{s,p}(\mathbb{R}^N)}}{2N - 2\beta - \mu} < \alpha_*,$$

we have

$$\left\{\int\limits_{\mathbb{R}^N} \mathcal{H}\left(\frac{4N\alpha\|u\|^{\frac{N}{N-s}}_{W^{s,p}(\mathbb{R}^N)}}{2N - 2\beta - \mu}, \left|\frac{u}{\|u\|_{W^{s,p}(\mathbb{R}^N)}}\right|\right) \mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{2N}} < +\infty.$$

Hence, jointly with the Sobolev embedding, we can obtain that

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x\\
&\le C(N,\mu,\beta)\left\|\varepsilon|u|^{\frac{(2N-2\beta-\mu)p}{2N}}+C_\varepsilon|u|^q\mathcal{H}(\alpha,u)\right\|^2_{L^{\frac{2N}{2N-2\beta-\mu}}(\mathbb{R}^N)}\\
&\le C(N,\mu,\beta)\left[2^{\frac{2N}{2N-2\beta-\mu}}\left\{\varepsilon^{\frac{2N}{2N-2\beta-\mu}}\int_{\mathbb{R}^N}|u|^p\mathrm{d}x\right.\right.\\
&\qquad\left.\left.+C_\varepsilon^{\frac{2N}{2N-2\beta-\mu}}\int_{\mathbb{R}^N}|u|^{\frac{2Nq}{2N-2\beta-\mu}}\mathcal{H}\left(\frac{2N\alpha}{2N-2\beta-\mu},u\right)\mathrm{d}x\right\}\right]^{\frac{2N-2\beta-\mu}{N}}\\
&\le C_1(N,\mu,\beta,\varepsilon)\left[\|u\|^{\frac{(2N-2\beta-\mu)p}{N}}_{L^p(\mathbb{R}^N)}+\|u\|^{2q}_{L^{\frac{4Nq}{2N-2\beta-\mu}}(\mathbb{R}^N)}\left\{\int_{\mathbb{R}^N}\mathcal{H}\left(\frac{4N\alpha}{2N-2\beta-\mu},u\right)\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{2N}}\right]\\
&\le C_2(N,\mu,\beta,\varepsilon)\left[\|u\|^{\frac{(2N-2\beta-\mu)p}{N}}+\|u\|^{2q}\left\{\int_{\mathbb{R}^N}\mathcal{H}\left(\frac{4N\alpha\|u\|_{W^{s,p}(\mathbb{R}^N)}}{2N-2\beta-\mu},\frac{u}{\|u\|_{W^{s,p}(\mathbb{R}^N)}}\right)\mathrm{d}x\right\}^{\frac{2N-2\beta-\mu}{2N}}\right]\\
&\le C_3(N,\mu,\beta,\varepsilon)\left(\|u\|^{\frac{(2N-2\beta-\mu)p}{N}}+\|u\|^{2q}\right).
\end{aligned}
$$

Using (11) and the Sobolev embedding, we have that for $\|u\|_{W^{s,p}(\mathbb{R}^N)}\le\left(\frac{(2N-2\beta-\mu)\alpha_*}{4N\alpha}\right)^{\frac{N-s}{N}}$, then

$$
\begin{aligned}
I(u)&=\frac{1}{p}\|u\|^p+\frac{1}{2p}\left[V_1(u)-V_2(u)\right]-\frac{1}{2}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x\\
&\ge\frac{1}{p}\|u\|^p-\frac{1}{2}C_3(N,\mu,\beta,\varepsilon)\left(\|u\|^{\frac{(2N-2\beta-\mu)p}{N}}+\|u\|^{2q}\right)-C_{K_0}\|u\|^{2p}.
\end{aligned}
$$

Hence, there exists $\kappa_0>0$ and $0<\rho<\left(\frac{(2N-2\beta-\mu)\alpha_*}{4N\alpha}\right)^{\frac{N-s}{N}}$ such that $I(u)\ge\kappa_0$ for all $u\in E$ with $\|u\|=\rho$. Since $\lim_{t\to\infty}I(tw_0)=-\infty$, we can choose $T>0$ such that $e=Tw_0\in Y_2:=\{u\in E:\|u\|>\rho\}$ and $I(e)<0$, then in view of Lemma 8 with $Y=E$ and $Y_1:=\{u\in E:\|u\|<\rho\}$, we deduce that there exist $\bar{c}\in[\kappa_0,\sup_{t\ge0}I(tw_0)]$ and a sequence $\{u_n\}\subset E$ satisfying (17). □

Lemma 10. *Let $(u_n)\subset X$ satisfying (17). Then, the sequence $\{u_n\}$ is bounded in X.*

Proof. From (17) and (F3), we have

$$
\begin{aligned}
d+o(1)\ge\;&I(u_n)-\frac{1}{2p}\langle I'(u_n),u_n\rangle\\
&\ge\frac{1}{2p}\|u_n\|^p+\frac{1}{2p}\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)-pF(x,u(x))}{|x|^\beta}\mathrm{d}x \qquad (18)\\
&\ge\frac{1}{2p}\|u_n\|^p
\end{aligned}
$$

for all $n \in \mathbb{N}$. Therefore, $\{u_n\}$ is bounded in X. $\square$

Next, to find the ground state solutions for (5), we shall establish the following inequality.

Lemma 11. *Assume that* (V0), (F0), (F1) (or (F1′)) *and* (F4) *hold, then*

$$I(u) \geq I(tu) + \frac{1-t^{2p}}{2p}\langle I'(u), u\rangle + \frac{(1-t^p)^2}{2p}\|u\|^p \tag{19}$$

for all $u \in E$ and $t \geq 0$.

Proof. First, we can verify that

$$\begin{aligned} I(u) - I(tu) = &\frac{1-t^p}{p}\int\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y + \frac{1-t^p}{p}\int\limits_{\mathbb{R}^N} V(x)|u|^p\mathrm{d}x \\ &+ \frac{1-t^{2p}}{2p}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln(|x-y|)|u(x)|^p|u(y)|^p\mathrm{d}x\mathrm{d}y \\ &- \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))F(x,u(x)) - F(y,tu(y))F(x,tu(x))}{|x|^\beta|x-y|^\mu||y|^\beta}\mathrm{d}x\mathrm{d}y \\ = &\frac{1-t^{2p}}{2p}\langle I'(u),u\rangle + \frac{(1-t^p)^2}{2p}\|u\|^p \\ &+ \frac{1}{2}\Bigg\{\frac{1-t^{2p}}{p}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))u(x)}{|x|^\beta|x-y|^\mu|y|^\beta}\mathrm{d}x\mathrm{d}y \\ &+ \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(y,tu(y))F(x,tu(x)) - F(y,u(y))F(x,u(x))}{|x|^\beta|x-y|^\mu|y|^\beta}\mathrm{d}x\mathrm{d}y\Bigg\}. \end{aligned}$$

To verify the Lemma 11, next, we just need to claim that for any $t > 0$ and any $u \in W^{s,p}(\mathbb{R}^N)$, we have

$$\begin{aligned} \Phi(t,u) = &\frac{1-t^{2p}}{2p}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))f(x,u(x))u(x)}{|x|^\beta|x-y|^\mu|y|^\beta}\mathrm{d}x\mathrm{d}y \\ &+ \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(y,tu(y))F(x,tu(x)) - F(y,u(y))F(x,u(x))}{|x|^\beta|x-y|^\mu|y|^\beta}\mathrm{d}x\mathrm{d}y \geq 0. \end{aligned}$$

Indeed, by (F4) and Remark 2, we can obtain that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,u) = &\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{-t^{2p-1}F(y,u(y))f(x,u(x))u(x)}{|x|^\beta|x-y|^\mu|y|^\beta}\mathrm{d}x\mathrm{d}y \\ &+ \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{F(y,tu(y))f(x,tu(x))u(x)}{|x|^\beta|x-y|^\mu|y|^\beta}\mathrm{d}x\mathrm{d}y \\ = &\, t^{4p-1}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{[u(x)]^{2p}}{|x|^\beta|x-y|^\mu|y|^\beta}\frac{F(y,tu(y))}{t^{2p}}\left(\frac{f(x,tu(x))}{t^{2p-1}[u(x)]^{2p-1}} - \frac{f(x,u(x))}{[u(x)]^{2p-1}}\right)\mathrm{d}x\mathrm{d}y \end{aligned}$$

$$+ t^{4p-1}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{[u(x)]^{2p}|u(y)|^{2p}}{|x|^{\beta}|x-y|^{\mu}|y|^{\beta}}\frac{f(x,u(x))}{[u(x)]^{2p-1}}\left[\frac{F(y,tu(y))}{t^{2p}|u(y)|^{2p}}-\frac{F(y,u(y))}{|u(y)|^{2p}}\right]\mathrm{d}x\mathrm{d}y$$

$$+ \left(t^{4p-1}-t^{2p-1}\right)\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))f(x,u(x))u(x)}{|x|^{\beta}|x-y|^{\mu}|y|^{\beta}}\mathrm{d}x\mathrm{d}y$$

$$\begin{cases}\geq 0, & t\geq 1;\\ \leq 0, & 0<t<1,\end{cases}$$

which implies that $\Phi(t,u)\geq\Phi(1,u)=0$ for $t>0$. From Remark 2, we can easily know

$$\Phi(0,u)=\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))[\frac{1}{2p}f(x,u(x))u(x)-F(x,u(x))]}{|x|^{\beta}|x-y|^{\mu}|y|^{\beta}}\mathrm{d}x\mathrm{d}y\geq 0.$$

This shows that $\Phi(t,u)\geq 0$ holds for $t\geq 0$. Therefore, for any $u\in E$ and any $t\geq 0$, we have

$$I(u)-I(tu)\geq\frac{1-t^{2p}}{2p}\langle I'(u),u\rangle+\frac{(1-t^p)^2}{2p}\|u\|^p\ .$$

We have completed the proof. □

Lemma 12. *Assume that* (V0), (F0), (F1) (or (F1′)) *and* (F4) *hold, then for every* $u\in\mathcal{N}$,

$$I(u)\geq\max_{t\geq 0}I(tu).$$

Lemma 13. *Assume that* (V0), (F0), (F1) (*or* (F1′)) *and* (F4) *hold. Then for any* $u\in E\setminus\{0\}$, *there exists a unique* $t>0$ *such that* $tu\in\mathcal{N}$.

Proof. Let $u\in E\setminus\{0\}$ be fixed and define a function $\vartheta_0(t):=\langle I'(tu),tu\rangle$ on $[0,\infty)$. It is easy to verify that $\vartheta_0(0)=0$, $\vartheta_0(t)>0$ for $t>0$ small and $\vartheta_0(t)<0$ for t large. Therefore, there exists $t=t_u>0$ such that $\vartheta_0(t_u)=0$ and $t_uu\in\mathcal{N}$.

We can conclude that t_u is unique for any $u\in E\setminus\{0\}$. In fact, for any given $u\in E\setminus\{0\}$, let $t_1,t_2>0$ such that $\vartheta_0(t_1)=\vartheta_0(t_2)=0$. By (19) for all $t>0$, $u\in E\setminus\{0\}$, taking $t=t_2/t_1$, one has

$$\begin{aligned}I(t_1u)&\geq I(t_2u)+\frac{(t_1^p-t_2^p)^2}{2pt_1^p}\|u\|^p\\ &\geq I(t_1u)+\frac{(t_2^p-t_1^p)^2}{2pt_2^p}\|u\|^p+\frac{(t_1^p-t_2^p)^2}{2pt_1^p}\|u\|^p,\end{aligned}$$

which implies that $t_1=t_2$. □

In view of Lemmas 12 and 13, we obtain that

$$\inf_{u\in\mathcal{N}}I(u):=m=\inf_{u\in E\setminus\{0\}}\max_{t\geq 0}I(tu). \tag{20}$$

For any $u\in\mathcal{N}$, it is easy to verify that $I(tu)<0$ for large $t>0$ by (F1′). Hence, we can prove the following lemma by the standard argument explored by [14, Lemma 3.2].

Lemma 14. *Assume that* (V0), (F0), (F1′), (F2) *and* (F4) *hold. Then there exist a constant* $c_* \in (0, m)$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$I(u_n) \to c_*, \quad \|I'(u_n)\|_{E^*}(1 + \|u_n\|_E) \to 0, \tag{21}$$

where m *is defined in* Theorem 5.

4. Subcritical case

This section is devoted to proving the Theorem 4. We first give the following property, which contributes to proving the boundedness of the sequence in E, in the sense of the norm $\|\cdot\|_*$.

Lemma 15. *Assume that* (V0) *holds. Then*

$$A_1(|u|^p, |v|^p) \geq \frac{1}{2^N}\|u\|_p^p\|v\|_*^p, \quad \forall\, u, v \in E. \tag{22}$$

Proof. Let $I^{(i_1,i_2,\dots,i_n)} = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^N : x_1(i_1), x_2(i_2), ..., x_n(i_n)\}$, and

$$x_j(i_j) = \begin{cases} x_j \geq 0, & \text{when } i_j = 1, \\ x_j < 0, & \text{when } i_j = -1, \end{cases}$$

where $j = 1, ..., n$. Then it follows from the definition of E that

$$\begin{aligned} A_1(|u|^p, |v|^p) &= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \ln(1 + |x-y|)|u(x)|^p|v(y)|^p \mathrm{d}x\mathrm{d}y \\ &\geq \sum_{i_j\in\{-1,1\}, j=1,\dots,n} \int\limits_{I^{(i_1,i_2,\dots,i_n)}} |u(y)|^p \mathrm{d}y \int\limits_{I^{(-i_1,-i_2,\dots,-i_n)}} \ln(1+|x-y|)|v(x)|^p \mathrm{d}x \\ &\geq \sum_{i_j\in\{-1,1\}, j=1,\dots,n} \int\limits_{I^{(i_1,i_2,\dots,i_n)}} |u(y)|^p \mathrm{d}y \int\limits_{I^{(-i_1,-i_2,\dots,-i_n)}} \ln(1+|x|)|v(x)|^p \mathrm{d}x \\ &\geq \frac{1}{2^N}\int\limits_{\mathbb{R}^N} |u(y)|^p \mathrm{d}y \int\limits_{\mathbb{R}^N} \ln(1+|x|)|v(x)|^p \mathrm{d}x \\ &\geq \frac{1}{2^N}\|u\|_p^p\|v\|_*^p, \ \forall\, u, v \in E. \end{aligned}$$

We have completed the proof. □

Proof of Theorem 4. Applying Lemmas 9 and 10, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (17). Thus there exists a constant $C > 0$ such that $\|u_n\|_p \leq C$. If $\delta_0 := \limsup_{n\to\infty}\|u_n\|_p = 0$, then from the Gagliardo-Nirenberg inequality explored in [36, Proposition, page 261]:

$$\|u_n\|_q^q \leq C_q\|u_n\|_p^p\|(-\Delta)^{\frac{s}{2}}u_n\|_p^{q-p}, \tag{23}$$

we derive that $u_n \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in [p, +\infty)$.

Taking α enough small in (12), and then it follows from the arbitrariness of the ε that

$$\begin{aligned}
&\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x\\
&\le C(N,\mu,\beta)\left\|\varepsilon|u_n|^{\frac{(2N-2\beta-\mu)p}{2N}}+C_\varepsilon|u_n|^q\mathcal{H}(\alpha,u_n)\right\|^2_{L^{\frac{2N}{2N-2\beta-\mu}}(\mathbb{R}^N)}\\
&\le C(N,\mu,\beta)\left[2^{\frac{2N}{2N-2\beta-\mu}}\left\{\varepsilon^{\frac{2N}{2N-2\beta-\mu}}\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x\right.\right.\\
&\qquad\left.\left.+C_\varepsilon^{\frac{2N}{2N-2\beta-\mu}}\int_{\mathbb{R}^N}|u_n|^{\frac{2Nq}{2N-2\beta-\mu}}\left\{\mathcal{H}\left(\frac{2N\alpha}{2N-2\beta-\mu},u_n\right)\right\}\mathrm{d}x\right\}\right]^{\frac{2N-2\beta-\mu}{N}}\\
&\le 4\varepsilon^2C_4(N,\mu,\beta)\left(\int_{\mathbb{R}^N}|u_n|^p\mathrm{d}x\right)^{\frac{2N-2\beta-\mu}{N}}\\
&\quad+4C_4(N,\mu,\beta)C_\varepsilon^2\left(\int_{\mathbb{R}^N}|u_n|^{\frac{4Nq}{2N-2\beta-\mu}}\mathrm{d}x\right)^{\frac{2N-2\beta-\mu}{2N}}\left[\int_{\mathbb{R}^N}\left\{\mathcal{H}\left(\frac{4N\alpha}{2N-2\beta-\mu},u_n\right)\right\}\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{2N}}\\
&\le 4\varepsilon^2C_4(N,\mu,\beta)\|u_n\|_p^{\frac{(2N-2\beta-\mu)p}{N}}\\
&\quad+4C_4(N,\mu,\beta)C_\varepsilon^2\|u_n\|^{2q}_{\frac{4Nq}{2N-2\beta-\mu}}\left[\int_{\mathbb{R}^N}\left\{\mathcal{H}\left(\frac{4N\alpha}{2N-2\beta-\mu},u_n\right)\right\}\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{2N}}\\
&= 4\varepsilon^2C_4(N,\mu,\beta)\|u_n\|_p^{\frac{(2N-2\beta-\mu)p}{N}}\\
&\quad+4C_4(N,\mu,\beta)C_\varepsilon^2\|u_n\|^{2q}_{\frac{4Nq}{2N-2\beta-\mu}}\left[\int_{\mathbb{R}^N}\left\{\mathcal{H}\left(\frac{4N\alpha\|u_n\|^{\frac{N}{N-s}}_{W^{s,p}(\mathbb{R}^N)}}{2N-2\beta-\mu},\frac{u_n}{\|u_n\|_{W^{s,p}(\mathbb{R}^N)}}\right)\right\}\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{2N}}\\
&\le\frac{\bar{c}}{2}+o(1).
\end{aligned}$$

Now from the inequality (11), one has

$$\begin{aligned}
&\bar{c}+o(1)\\
&=I(u_n)-\frac{1}{p}\langle I'(u_n),u_n\rangle\\
&=\frac{1}{2p}[V_2(u_n)-V_1(u_n)]+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{\frac{1}{p}f(x,u_n(x))u_n(x)-\frac{1}{2}F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\\
&\le\frac{\bar{c}}{2p}+o(1).
\end{aligned}$$

This contradiction show that $\delta_0 > 0$. By (11), (16), (17) and (18), one has

$$V_1(u_n) \leq V_2(u_n) + \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u(x))u(x)}{|x|^\beta} \mathrm{d}x + o(1) \leq C,$$

which together with Lemma 15, implies that $\{\|u_n\|_*\}$ is bounded, and thus $\{u_n\}$ is bounded in E. We may thus assume, passing to a subsequence again if necessary, that $u_n \rightharpoonup \bar{u}$ in E, $u_n \to \bar{u}$ in $L^q(\mathbb{R}^N)$, $q \in [p, +\infty)$ and $u_n \to \bar{u}$ a.e. on $\mathbb{R}^N$.

By Proposition 1, we have

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,\bar{u}(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,\bar{u}(x))(u_n(x)-\bar{u}(x))}{|x|^\beta} \mathrm{d}x \\
&\quad - \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))(u_n(x)-\bar{u}(x))}{|x|^\beta} \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,\bar{u}(y)) - F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))(u_n(x)-\bar{u}(x))}{|x|^\beta} \mathrm{d}x \\
&\quad + \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,\bar{u}(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{(f(x,\bar{u}(x)) - f(x,u_n(x)))(u_n(x)-\bar{u}(x))}{|x|^\beta} \mathrm{d}x \\
&\leq C(N,\mu,\beta) \|F(y,\bar{u}(y)) - F(y,u_n(y))\|_{\frac{2N}{2N-2\beta-\mu}} \|f(x,u_n(x))(u_n(x)-\bar{u}(x))\|_{\frac{2N}{2N-2\beta-\mu}} \\
&\quad + C(N,\mu,\beta) \|F(y,\bar{u}(y))\|_{\frac{2N}{2N-2\beta-\mu}} \|(f(x,\bar{u}(x)) - f(x,u_n(x)))(u_n(x)-\bar{u}(x))\|_{\frac{2N}{2N-2\beta-\mu}}.
\end{aligned}$$

In particular, by (F1), we can take enough small α in (12), and then we have

$$\begin{aligned}
&\|F(y,\bar{u}(y)) - F(y,u_n(y))\|_{\frac{2N}{2N-2\beta-\mu}} \|f(x,u_n(x))(u_n(x)-\bar{u}(x))\|_{\frac{2N}{2N-2\beta-\mu}} \\
\leq &\left(\int\limits_{\mathbb{R}^N} \left\{ 4^{\frac{2\beta+\mu}{2N-2\beta-\mu}} \varepsilon^{\frac{2N}{2N-2\beta-\mu}} (|\bar{u}|^p + |u_n|^p) \right.\right. \\
&+ 4^{\frac{2\beta+\mu}{2N-2\beta-\mu}} C_\varepsilon^{\frac{2N}{2N-2\beta-\mu}} \left[|\bar{u}|^{\frac{2Nq}{2N-2\beta-\mu}} \mathcal{H}\left(\frac{2N\alpha}{2N-2\beta-\mu}, \bar{u} \right) \right. \\
&\qquad \left.\left.\left. + |u_n|^{\frac{2Nq}{2N-2\beta-\mu}} \mathcal{H}\left(\frac{2N\alpha}{2N-2\beta-\mu}, u_n \right) \right] \right\} \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{2N}} \\
&\times \left(\int\limits_{\mathbb{R}^N} \left| \varepsilon |u_n|^{\frac{(2N-2\beta-\mu)p}{2N}-1} + C_\varepsilon |u_n|^{q-1} \mathcal{H}(\alpha, u_n) \right|^{\frac{4N}{2N-2\beta-\mu}} \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{4N}} \\
&\times \left(\int\limits_{\mathbb{R}^N} |u_n - \bar{u}|^{\frac{4N}{2N-2\beta-\mu}} \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{4N}} \\
\leq &\left\{ C_\varepsilon^1 \left(\|\bar{u}\|_p^{\frac{(2N-2\beta-\mu)p}{2N}} + \|u_n\|_p^{\frac{(2N-2\beta-\mu)p}{2N}} \right) + C_\varepsilon^2 \left(\int\limits_{\mathbb{R}^N} |\bar{u}|^{\frac{2Nq}{2N-2\beta-\mu}} \mathcal{H}\left(\frac{2N\alpha}{2N-2\beta-\mu}, \bar{u} \right) \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{2N}} \right.
\end{aligned}$$

$$
\begin{aligned}
&+C_\varepsilon^2\left(\int_{\mathbb{R}^N}|u_n|^{\frac{2Nq}{2N-2\beta-\mu}}\mathcal{H}\left(\frac{2N\alpha}{2N-2\beta-\mu},u_n\right)\mathrm{d}x\right)^{\frac{2N-2\beta-\mu}{2N}}\Bigg\}\\
&\times\|u_n-\bar u\|_{\frac{4N}{2N-2\beta-\mu}}\Bigg\{C_\varepsilon^3\|u_n\|_{2p-\frac{4N}{2N-2\beta-\mu}}^{\frac{(2N-2\beta-\mu)}{2N}-1}\\
&\qquad+C_\varepsilon^4\left[\int_{\mathbb{R}^N}|u_n|^{\frac{4N(q-1)}{2N-2\beta-\mu}}\mathcal{H}\left(\frac{4N\alpha}{2N-2\beta-\mu},u_n\right)\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{4N}}\Bigg\}.\\
\le&\Bigg\{C_\varepsilon^1\left(\|\bar u\|_p^{\frac{(2N-2\beta-\mu)p}{2N}}+\|u_n\|_p^{\frac{(2N-2\beta-\mu)p}{2N}}\right)+C_\varepsilon^2\|\bar u\|_{\frac{4Nq}{2N-2\beta-\mu}}^q\left[\int_{\mathbb{R}^N}\mathcal{H}\left(\frac{4N\alpha}{2N-2\beta-\mu},\bar u\right)\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{4N}}\\
&\qquad+C_\varepsilon^2\|u_n\|_{\frac{4Nq}{2N-2\beta-\mu}}^q\left[\int_{\mathbb{R}^N}\mathcal{H}\left(\frac{4N\alpha}{2N-2\beta-\mu},u_n\right)\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{4N}}\Bigg\}\\
&\times\|u_n-\bar u\|_{\frac{4N}{4N-2\beta-\mu}}\Bigg\{C_\varepsilon^3\|u_n\|_{2p-\frac{4N}{2N-2\beta-\mu}}^{\frac{(2N-2\beta-\mu)}{2N}-1}\\
&\qquad+C_\varepsilon^4\|u_n\|_{\frac{8N(q-1)}{2N-2\beta-\mu}}^{q-1}\left[\int_{\mathbb{R}^N}\mathcal{H}\left(\frac{8N\alpha}{2N-2\beta-\mu},u_n\right)\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{8N}}\Bigg\}.\\
&\le C(\varepsilon,N,\alpha,\beta,\mu)\|u_n-\bar u\|_{\frac{4N}{4N-2\beta-\mu}}=o(1).
\end{aligned}
$$

Similarly, we can deduce that

$$\|F(y,\bar u(y))\|_{\frac{2N}{2N-2\beta-\mu}}\|(f(x,\bar u(x))-f(x,u_n(x)))(u_n(x)-\bar u(x))\|_{\frac{2N}{2N-2\beta-\mu}}=o(1).$$

To prove our results, we need to recall the well-known Simon inequality

$$(|a|^{p-2}a-|b|^{p-2}b)(a-b)\ge k_p|a-b|^p,\quad k_p>0,\quad\forall\, a,b\in\mathbb{R}$$

for $p\ge 2$. By the Simon inequality, we thus have

$$\Upsilon(u_n,u_n-u)-\Upsilon(u,u_n-u)\ge k_p\Upsilon(u_n-u,u_n-u)=k_p[u_n-u]_{s,p}^p$$

and

$$\int_{\mathbb{R}^N}V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u)\mathrm{d}x\ge k_p\int_{\mathbb{R}^N}V(x)|u_n-u|^p\mathrm{d}x.$$

It follows from (15) that

$$
\begin{aligned}
o(1)&=\langle I'(u_n)-I'(\bar u),u_n-\bar u\rangle\\
&=\Upsilon(u_n,u_n-\bar u)-\Upsilon(\bar u,u_n-\bar u)+\int_{\mathbb{R}^N}V(x)(|u_n|^{p-2}u_n-|\bar u|^{p-2}\bar u)(u_n-\bar u)\mathrm{d}x\\
&\quad+A_0(|u_n|^p,|u_n|^{p-2}u_n(u_n-\bar u))-A_0(|\bar u|^p,|\bar u|^{p-2}\bar u(u_n-\bar u))
\end{aligned}
$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))(u_n(x)-\bar{u}(x))}{|x|^\beta} \mathrm{d}x \\
&+ \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{F(y,\bar{u}(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,\bar{u}(x))(u_n(x)-\bar{u}(x))}{|x|^\beta} \mathrm{d}x \\
\geq\ & k_p\|u_n-\bar{u}\|^p + k_p A_1(|u_n|^p, |u_n-\bar{u}|^p) \\
&+ A_1(|u_n|^p, |\bar{u}|^{p-2}\bar{u}(u_n-\bar{u})) - A_1(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n-\bar{u})) \\
&- A_2(|u_n|^p, |u_n|^{p-2}u_n(u_n-\bar{u})) + A_2(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n-\bar{u})) + o(1).
\end{aligned}
$$

It is easy to verify that

$$
\begin{aligned}
& |A_1(|u_n(x)|^p, |\bar{u}(y)|^{p-2}\bar{u}(y)(u_n(y)-\bar{u}(y)))| \\
=\ & \left| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \ln(1+|x-y|)|u_n(x)|^p |\bar{u}(y)|^{p-2}\bar{u}(y)(u_n(y)-\bar{u}(y))\mathrm{d}x\mathrm{d}y \right| \\
\leq\ & \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [\ln(1+|x|)+\ln(1+|y|)]|u_n(x)|^p |\bar{u}(y)|^{p-2}\bar{u}(y)(u_n(y)-\bar{u}(y))\mathrm{d}x\mathrm{d}y \\
\leq\ & \|u_n\|_* \|\bar{u}\|_p^{p-1}\|u_n-\bar{u}\|_p + \|u_n\|_p^p \int_{\mathbb{R}^N} \ln(1+|y|)|\bar{u}(y)|^{p-1}|u_n(y)-\bar{u}(y)|\mathrm{d}y \\
\leq\ & \|u_n\|_* \|\bar{u}\|_p^{p-1}\|u_n-\bar{u}\|_p + \|u_n\|_p^p \ln(1+R)\|\bar{u}(y)\|_p^{p-1}\|u_n-\bar{u}\|_p \\
& \qquad + \|u_n\|_p^p \|u_n-\bar{u}\|_*^{\frac{1}{p}} \left[\int_{\mathbb{R}^N\setminus B_R(0)} \ln(1+|y|)|\bar{u}(y)|^p \mathrm{d}y \right]^{p-1} \\
\leq\ & \|u_n\|_* \|\bar{u}\|_p^{\frac{p-1}{p}} \|u_n-\bar{u}\|_p + o_n(1) + o_R(1), \quad \text{as } n\to\infty, R\to\infty.
\end{aligned}
$$

Similarly,

$$
A_1(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n-\bar{u}))] = o(1).
$$

By Hölder inequality and (10), we have

$$
A_2(|u_n|^p, |u_n|^{p-2}u_n(u_n-\bar{u})) \leq K_1\|u_n\|_{\frac{2Np}{2N-1}}^{p^2} \||u_n|^{p-2}u_n(u_n-\bar{u})\|_{\frac{2N}{2N-1}}^p = o(1).
$$

Similarly,

$$
A_2(|\bar{u}|^p, |\bar{u}|^{p-2}\bar{u}(u_n-\bar{u}) = o(1).
$$

Thus, by Lemma 15,

$$
\begin{aligned}
o(1) &= \langle I'(u_n) - I'(\bar{u}), u_n-\bar{u}\rangle \\
&\geq\ k_p\|u_n-\bar{u}\|^p + k_p A_1(|u_n|^p, |u_n-\bar{u}|^p) + o(1) \\
&\geq\ k_p\|u_n-\bar{u}\|^p + \frac{k_p}{2^N}\|u_n\|_p^p\|u_n-\bar{u}\|_*^p + o(1),
\end{aligned}
$$

which together with $\delta_0 = \limsup_{n\to\infty} \|u_n\|_p > 0$, implies that $u_n \to \bar{u}$ in E. Hence, $0 < \bar{c} = \lim_{n\to\infty} I(u_n) = I(\bar{u})$ and $I'(\bar{u}) = 0$.

Next, we point out that the $\bar{u}$ we obtained on E is actually the critical point of I on X. We define a set $\{as_1, ..., as_n\}$ where as_i is the reflection at the coordinate axis x_i for $i = 1, ..., n$. Let $G = P(\{as_1, ..., as_n\})$, which means that G is the power set of $\{as_1, ..., as_n\}$, and then define a map Θ on the set $A = \{F_1, ..., F_n\}$, where $F_i (i = 1, 2, ..., n)$ are maps.

$$\Theta(A) = \Pi_{i=1}^{i=n} F_i.$$

Thus, $\bar{G} = \bigcup_{o\in G} \Theta(o)$ is a transform group. The action of $\bar{G}$ on $W^{s,p}(\mathbb{R}^N)$ is defined by

$$g(u)(x) := u(g^{-1}x).$$

Then it is easy to verify that for any $g_1, g_2, g \in \bar{G}$ and $u \in W^{s,p}(\mathbb{R}^N)$,

$$(id)u = u, \quad (g_1g_2)u = g_1(g_2u), \quad u \mapsto \; gu \text{ is linear}, \quad \|gu\|_{W^{s,p}(\mathbb{R}^N)} = \|u\|_{W^{s,p}(\mathbb{R}^N)};$$
$$\mathrm{Fix}(\bar{G}) := \{u \in W^{s,p}(\mathbb{R}^N) : gu = u, \forall\ g \in \bar{G}\} = W^{s,p}_{as}(\mathbb{R}^N).$$

In virtue of (V0) and (F0), we have $I(gu) = I(u)$ for all $g \in \bar{G}$ and $u \in W^{s,p}(\mathbb{R}^N)$. Therefore, in view of [46, Theorem 1.28], one has that if u is a critical point of I restricted to E, then u is a critical point of I on X, which shows that E is a natural constraint of X. Thus, we have completed the proof of Theorem 4. □

5. Exponential critical

In this Section, we shall consider the problem (5) in the exponential critical case. In the previous work [25], Kozono et al. proved that for all $\alpha > 0$ and $u \in W^{s,\frac{N}{s}}(\mathbb{R}^N)$, there holds

$$\int\limits_{\mathbb{R}^N} \left(e^{\alpha|t|^{\frac{N}{N-s}}} - \sum_{j=0}^{j_p-2} \frac{\alpha^j |t|^{\frac{jN}{N-s}}}{j!} \right) \mathrm{d}x \le \infty,$$

where $j_p = \min\{j \in \mathbb{N} : j \ge p\}$. Moreover, there exist positive constants $\alpha_{N,s}$ and $C_{N,s}$ depending on N and s such that

$$\int\limits_{\mathbb{R}^N} \left(e^{\alpha|t|^{\frac{N}{N-s}}} - \sum_{j=0}^{j_p-2} \frac{\alpha^j |t|^{\frac{jN}{N-s}}}{j!} \right) \mathrm{d}x \le C_{N,s},\ \forall\ \alpha \in (0, \alpha_{N,s}), \tag{24}$$

for all $u \in W^{s,N/s}(\mathbb{R}^N)$ with $\|u\|_{W^{s,N/s}(\mathbb{R}^N)} \le 1$. The inequality (24) is better for us to consider the behaviors of a sequence compared with Lemma 3, and it is obvious that $\alpha_{N,s} \le \alpha_* \le \alpha^*_{s,N}$.

Let $\rho \in (0, 1/2)$. Similar to [37, Section 5], we define the Moser type functions $w_n(x)$ supported in $B_\rho(0)$ as follows:

$$w_n(x) = \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{N}} \begin{cases} |\ln n|^{\frac{N-s}{N}} & \text{if } |x| \le \frac{\rho}{n}, \\ \dfrac{\left|\ln \frac{\rho}{|x|}\right|}{|\ln n|^{\frac{s}{N}}} & \text{if } \frac{\rho}{n} \le |x| \le \rho, \\ 0 & \text{if } |x| \ge \rho, \end{cases}$$

which belong to $W^{s,p}_0(\mathbb{R}^N)$. For $s \in (0, 1)$, as explored in [37], we cannot expect that $[w_n]_{s,p}$ is constant. Following the estimation in [37] and after some basic calculations, we know that

$$[w_n]^p_{W^{s,p}(\mathbb{R}^N)} = 1 + \mathcal{O}\left(\frac{1}{\log n}\right) + \mathcal{O}\left(\frac{1}{n^{N-1}}\right) + \mathcal{O}\left(\frac{(\log n)^{p-1}}{n^N}\right) \le 1 + \mathcal{O}\left(\frac{1}{\log n}\right) \tag{25}$$

and

$$\begin{aligned}\int\limits_{\mathbb{R}^N} V(x)|w_n(x)|^p \mathrm{d}x \le\; & w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} \frac{(\ln n)^{\frac{N-s}{s}}}{n^N} - w_N \frac{(\ln n)^{\frac{N-s}{s}}}{n^N} \\ & - \sum_{m=2}^{k+1} C_m(N,s) w_N \frac{(\ln n)^{\frac{N}{s}-m}}{n^N} + \tilde{C}_{N,s} \frac{\frac{N}{s}-(k+1)}{N} \int\limits_0^{\ln n} t^{\frac{N}{s}-(k+1)} e^{-Nt} \mathrm{d}t \\ \le\; & w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} \frac{(\ln n)^{\frac{N-s}{s}}}{n^N} - w_N \frac{(\ln n)^{\frac{N-s}{s}}}{n^N} \\ & - \sum_{m=2}^{k+1} C_m(N,s) w_N \frac{(\ln n)^{\frac{N}{s}-m}}{n^N} \\ \le\; & \tilde{C} w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} \mathcal{O}\left(\frac{1}{\ln n}\right),\end{aligned} \tag{26}$$

where $V_\infty := \sup_{|x|\le\rho} V(x)$, and $k = \left[\frac{N}{s}\right]$, which implies that $\frac{N}{s} - (k+1) < 0$. Thus we have

$$\begin{aligned}\|w_n\|^p \le & 1 + \left[\tilde{C} w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} + 1\right] \mathcal{O}\left(\frac{1}{\ln n}\right) \\ := & 1 + \tilde{C} w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} \delta_n,\end{aligned} \tag{27}$$

where $\delta_n = \left(1 + \frac{(\alpha^*_{s,N})^{\frac{N-s}{s}}}{\tilde{C} w_N V_\infty \rho^N N^{\frac{N-s}{s}}}\right) \mathcal{O}\left(\frac{1}{\ln n}\right)$.

Following the arguments explored by [9], we know that

$$\int\limits_{B_{\rho/n}(0)} \int\limits_{B_{\rho/n}(0)} \frac{1}{|x|^\beta |y|^\beta |x-y|^\mu} \mathrm{d}x\mathrm{d}y \ge C(\mu,\beta,N) \left(\frac{\rho}{n}\right)^{2N-2\beta-\mu},$$

where $C(\mu,\beta,N)$ is a positive constant. Inspired by the work [40,52], we have the following lemma.

Lemma 16. *Assume that* (F1′), (V0), (F0), (F2) *and* (F5)-(F6) *hold. Then there exists* $\bar{n} \in \mathbb{N}$ *such that*

$$\max_{t\ge 0} I(tw_{\bar{n}}) < \frac{s}{N} \cdot \left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2N\alpha_0}\right]^{\frac{N-s}{s}}. \tag{28}$$

Proof. Let us argue by contradiction and suppose (28) does not hold, so that for all n let $t_n > 0$ be such that

$$I(t_nw_n) = \max_{t\geq 0} I(tw_n) \geq \frac{s}{N} \cdot \left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2N\alpha_0}\right]^{\frac{N-s}{s}}, \tag{29}$$

where t_n satisfies $\frac{d}{dt}I(tw_n)\big|_{t=t_n} = 0$ and together with the estimation (27), then we have

$$t_n^p\left(1+\tilde{C}w_NV_\infty\rho^N\left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\delta_n\right) \geq \left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2N\alpha_0}\right]^{\frac{N-s}{s}}. \tag{30}$$

From now on, in the sequel, all inequalities hold for large $n \in \mathbb{N}$, and it is obvious $t_nw_n \geq t_\varepsilon$ under this condition. From (F6) and (29) we have

$$\begin{aligned}
&t_n^p\left(1+\tilde{C}w_NV_\infty\rho^N\left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\delta_n\right)\\
&\geq t_n^p\|w_n\|^p\\
&\geq \int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,t_nw_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,t_nw_n(x))t_nw_n(x)}{|x|^\beta}\mathrm{d}x\\
&\geq \int\limits_{B_{\rho/n}(0)}\left(\int\limits_{B_{\rho/n}(0)}\frac{F(y,t_nw_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,t_nw_n(x))t_nw_n(x)}{|x|^\beta}\mathrm{d}x\\
&\geq (\kappa-\varepsilon)^2t_n^{\frac{N-2s}{N-s}}(\ln n)^{\frac{N-2s}{N}}\frac{C(\mu,\beta,N)(N-s)\rho^{2N-2\beta-\mu}}{N^{\frac{2s}{N}}\alpha_0(\alpha^*_{s,N})^{\frac{N-2s}{s}}}\exp\left[\left(\frac{2N\alpha_0t_n^{\frac{N}{N-s}}}{a^*_{s,N}}\right)\ln n\right]n^{-(2N-2\beta-\mu)}\\
&= (\kappa-\varepsilon)^2t_n^{\frac{N-2s}{N-s}}(\ln n)^{\frac{N-2s}{N}}\frac{C(\mu,\beta,N)(N-s)\rho^{2N-2\beta-\mu}}{N^{\frac{2s}{N}}\alpha_0(\alpha^*_{s,N})^{\frac{N-2s}{s}}}\exp\left[\frac{2\alpha_0t_n^{\frac{N}{N-s}}N\ln n}{\alpha^*_{s,N}}-(2N-2\beta-\mu)\ln n\right],
\end{aligned}$$

which implies that there exists a constant $C_1 > 0$ such that

$$\left[\frac{2\alpha_0t_n^{\frac{N}{N-s}}N}{\alpha^*_{s,N}}-(2N-2\beta-\mu)\right]\ln n \leq C_1,$$

that is

$$t_n^p \leq \left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2N\alpha_0}\right]^{\frac{N-s}{s}}\left(1+\frac{C_2}{\log n}\right). \tag{31}$$

Combining (30) with (31), one can obtain that for any small $\varepsilon > 0$ we have

$$\left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2N\alpha_0}\right]^{\frac{N-s}{s}}(1-\varepsilon) \leq t_n^p \leq \left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2N\alpha_0}\right]^{\frac{N-s}{s}}(1+\varepsilon),$$

then we have

$$I(t_nw_n) = \frac{1}{p}t_n^p\|w_n\|^p - \frac{1}{2}\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N}\frac{F(y,t_nw_n(y))F(x,t_nw_n(x))}{|x|^\beta|y|^\beta|x-y|^\mu}\mathrm{d}x\mathrm{d}y$$

$$
\begin{aligned}
&\le \frac{1}{p}t_n^p\left(1+\tilde{C}w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\delta_n\right)\\
&\quad -\frac{1}{2}\int\limits_{B_{\rho/n}(0)}\int\limits_{B_{\rho/n}(0)}\frac{F(y,t_nw_n(y))F(x,t_nw_n(x))}{|x|^\beta|y|^\beta|x-y|^\mu}\mathrm{d}x\mathrm{d}y\\
&\le \frac{1}{p}t_n^p\left(1+\tilde{C}w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\delta_n\right)\\
&\quad -\frac{(N-s)^2}{2N^2\alpha_0^2 C(\mu,\beta,N)}\left(\frac{\rho}{n}\right)^{2N-2\beta-\mu}\frac{(k-\varepsilon)^2 e^{2\alpha_0 t_n^{\frac{N}{N-s}}N(\alpha^*_{s,N})^{-1}\ln n}}{t_n^{\frac{2s}{N-s}}\left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{2s}{N}}(\ln n)^{\frac{2s}{N}}}\\
&\le \frac{1}{p}t_n^p\left(1+\tilde{C}w_N V_\infty \rho^N \left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\delta_n\right)\\
&\quad -\frac{(N-s)^2}{2^{1-\frac{2s}{N}}N^2\alpha_0^{2-\frac{2s}{N}}C(\mu,\beta,N)(2N-2\beta-\mu)^{\frac{2s}{N}}}\left(\frac{\rho}{n}\right)^{2N-2\beta-\mu}\frac{(k-\varepsilon)^2 e^{2\alpha_0 t_n^{\frac{N}{N-s}}N(\alpha^*_{s,N})^{-1}\ln n}}{(\ln n)^{\frac{2s}{N}}}\\
&:=\varphi(t_n),
\end{aligned}
$$

and now we take the following notations.

Let

$$
A := \tilde{C}w_N\left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}
$$

and

$$
B := \frac{(N-s)^2\rho^{2N-2\beta-\mu}}{2^{1-\frac{2s}{N}}N^2\alpha_0^{2-\frac{2s}{N}}C(\mu,\beta,N)(2N-2\beta-\mu)^{\frac{2s}{N}}}.
$$

Thus,

$$
\varphi(t_n)=\frac{1}{p}t_n^p(1+AV_\infty\rho^N\delta_n)-B\frac{(k-\varepsilon)^2 e^{2\alpha_0 t_n^{\frac{N}{N-s}}N(\alpha^*_{s,N})^{-1}\ln n}}{(\ln n)^{\frac{2s}{N}}n^{2N-2\beta-\mu}},
$$

and there exists $\hat{t}_n$ such that $\varphi'(\hat{t}_n)=0$. Thus we have

$$
\hat{t}_n^{\frac{N-s}{s}}(1+AV_\infty\rho^N\delta_n)=B\frac{(k-\varepsilon)^2 e^{2\alpha_0 \hat{t}_n^{\frac{N}{N-s}}N(\alpha^*_{s,N})^{-1}\ln n}}{(\ln n)^{\frac{2s}{N}}n^{2N-2\beta-\mu}}\frac{2\alpha_0N^2\ln n\,\hat{t}_n^{\frac{s}{N-s}}}{\alpha^*_{s,N}(N-s)},
$$

which implies that

$$
e^{2\alpha_0 \hat{t}_n^{\frac{N}{N-s}}N(\alpha^*_{s,N})^{-1}\ln n}=\frac{\hat{t}_n^{\frac{N-s}{s}-\frac{s}{N-s}}(1+AV_\infty\rho^N\delta_n)(\ln n)^{\frac{2s}{N}}n^{2N-2\beta-\mu}\alpha^*_{s,N}(N-s)}{B(k-\varepsilon)^2 2\alpha_0N^2\ln n\,\hat{t}_n^{\frac{s}{N-s}}}.
$$

Thus,

$$\hat{t}_n^{\frac{N}{s}} \leq \left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}} \left[1+\frac{(N-s)C_1\ln[\Lambda(\kappa-\varepsilon)^{-2}C_2(1+V_\infty\rho^N A\delta_n)]}{s(2N-2\beta-\mu)\ln n}\right],$$

where $\Lambda = \frac{\alpha^*_{s,N}(N-s)}{2B\alpha_0 N^2}$. Furthermore, one has

$$\begin{aligned}
\varphi(\hat{t}_n) &= \frac{1}{p}\hat{t}_n^p(1+AV_\infty\rho^N\delta_n) - \frac{\hat{t}_n^{\frac{N-s}{s}-\frac{s}{N-s}}(1+AV_\infty\rho^N\delta_n)\alpha^*_{s,N}(N-s)}{2\alpha_0 N^2\ln n}\\
&\leq \frac{1}{p}\hat{t}_n^p(1+AV_\infty\rho^N\delta_n) - (1+AV_\infty\rho^N\delta_n)\frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 N^2\ln n}\\
&= \frac{1}{p}(1+AV_\infty\rho^N\delta_n)\left[\hat{t}_n^p - \frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 Ns\ln n}\right]\\
&\leq \frac{1}{p}(1+AV_\infty\rho^N\delta_n)\left[\left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}}\right.\\
&\quad \left.\left(1+\frac{(N-s)C_1\ln[\Lambda(\kappa-\varepsilon)^{-2}C_2(1+V_\infty\rho^N A\delta_n)]}{s(2N-2\beta-\mu)\ln n}\right) - \frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 Ns\ln n}\right]\\
&=\frac{1}{p}\left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}}(1+AV_\infty\rho^N\delta_n)\\
&\quad \left[\left(1+\frac{(N-s)C_1\ln[\Lambda(\kappa-\varepsilon)^{-2}C_2(1+V_\infty\rho^N A\delta_n)]}{s(2N-2\beta-\mu)\ln n}\right)\right.\\
&\qquad \left.-\frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 Ns\ln n}\left(\frac{2\alpha_0 N}{(2N-2\beta-\mu)\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\right]\\
&=\frac{1}{p}\left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}} - \frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 N^2\ln n} + \frac{1}{p}\left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}} AV_\infty\rho^N\delta_n\\
&\quad + \frac{1}{p}\left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}} \frac{(N-s)C_1\ln[\Lambda(\kappa-\varepsilon)^{-2}C_2(1+V_\infty\rho^N A\delta_n)]}{s(2N-2\beta-\mu)\ln n} + \mathcal{O}\left(\frac{1}{\log^2 n}\right)\\
&=\frac{1}{p}\left[\frac{(2N-2\beta-\mu)\alpha^*_{s,N}}{2\alpha_0 N}\right]^{\frac{N-s}{s}}\\
&\quad \left[1+AV_\infty\rho^N\delta_n + \frac{(N-s)+C_1\ln[\Lambda(\kappa-\varepsilon)^{-2}C_2(1+V_\infty\rho^N A\delta_n)]}{s(2N-2\beta-\mu)\ln n}\right.\\
&\qquad \left.-\frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 Ns\ln n}\left[\frac{2\alpha_0 N}{(2N-2\beta-\mu)\alpha^*_{s,N}}\right]^{\frac{N-s}{s}}\right] + \mathcal{O}\left(\frac{1}{\log^2 n}\right).
\end{aligned}$$

Recall that $\delta_n = \left(1+\frac{(\alpha^*_{s,N})^{\frac{N-s}{s}}}{\tilde{C}w_N V_\infty\rho^N N^{\frac{N-s}{s}}}\right)\mathcal{O}\left(\frac{1}{\ln n}\right)$. From (F6) and taking suitable $\rho \in (0,1/2)$ we know that there exists ε enough small such that

$$\left[Cw_N V_\infty\rho^N\left(\frac{N}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}+1\right]\mathcal{O}\left(\frac{1}{\ln n}\right) + \frac{(N-s)+C_1\ln[\Lambda(\kappa-\varepsilon)^{-2}C_2(1+V_\infty\rho^N A\delta_n)]}{s(2N-2\beta-\mu)\ln n}$$

$$-\frac{C_3\alpha^*_{s,N}(N-s)}{2\alpha_0 N s\ln n}\left[\frac{2\alpha_0 N}{(2N-2\beta-\mu)\alpha^*_{s,N}}\right]^{\frac{N-s}{s}} < \left(\frac{\alpha_{N,s}}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}} - 1,$$

then we have

$$I(t_n w_n) \le \varphi(t_n) < \frac{s}{N}\left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2\alpha_0 N}\right]^{\frac{N-s}{s}},$$

which is a contradiction with (29). Therefore we have finished this Lemma. □

From Lemmas 13, 16 and the definition of m, we have the following corollary immediately.

Corollary 17. *Assume that* (F1′), (V0), (F0), (F2) *and* (F5)-(F6) *hold. Then*

$$m < \frac{s}{N}\cdot\left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2N\alpha_0}\right]^{\frac{N-s}{s}}.$$

To finish the proof of Theorem 5, we need the following two Lemmas.

Lemma 18. *Assume that $\{u_n\}$ is bounded in E, $u_n \rightharpoonup u$ in E and*

$$\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x \le C. \tag{32}$$

Then we have

$$\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x \to \int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x.$$

Lemma 19. *Under the same condition as* Lemma 18, *we have*

$$\begin{aligned}&\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u(x)}{|x|^\beta}\mathrm{d}x\\ &\to \int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)}{|x|^\beta}\mathrm{d}x.\end{aligned} \tag{33}$$

To prove the Lemma 19, we need the following

Lemma 20. *Assume that $\beta < \mu$, $u_n \rightharpoonup u$ in E and*

$$\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x \le \mathcal{K}$$

for some constant $\mathcal{K} > 0$. Then for every $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty}\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))\phi(x)}{|x|^\beta}\mathrm{d}x = \int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))\phi(x)}{|x|^\beta}\mathrm{d}x.$$

Proof. By the Fatou's Lemma we have

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)}{|x|^\beta}\mathrm{d}x\le\mathcal{K}.$$

Take $\Omega=\operatorname{supp}\phi$, for any given $\varepsilon>0$, let $M_\varepsilon:=\mathcal{K}\|\phi\|_\infty\varepsilon^{-1}$, then it follows that for n large enough,

$$\begin{aligned}&\int_{(|u_n|\ge M_\varepsilon)\cup(|u|=M_\varepsilon)}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|f(x,u_n(x))\phi(x)|}{|x|^\beta}\mathrm{d}x\\ \le&\frac{2\varepsilon}{\mathcal{K}}\int_{|u_n|\ge\frac{M_\varepsilon}{2}}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x\le 2\varepsilon\end{aligned}$$

and

$$\begin{aligned}&\int_{|u|\ge M_\varepsilon}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|f(x,u(x))\phi(x)|}{|x|^\beta}\mathrm{d}x\\ \le&\frac{\varepsilon}{\mathcal{K}}\int_{|u|\ge M_\varepsilon}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)}{|x|^\beta}\mathrm{d}x\le\varepsilon.\end{aligned}$$

Since $|f(x,u_n)|\chi_{|u_n|\le M_\varepsilon}\to|f(x,u)|\chi_{|u|\le M_\varepsilon}$ a.e. in $\Omega\setminus D_\varepsilon$, where $D_\varepsilon=\{x\in\Omega:|u(x)|=M_\varepsilon\}$, and

$$|f(x,u_n)|\chi_{|u_n|\le M_\varepsilon}\le\max_{|t|\le M_\varepsilon}|f(x,t)|<\infty,\quad\forall\,x\in\Omega,$$

the Lebesgue dominated convergence theorem leads to

$$\lim_{n\to\infty}\int_{(\Omega\setminus D_\varepsilon)\cup\{|u_n|\le M_\varepsilon\}}|f(x,u_n(x))|^{\frac{2N}{2N-2\beta-\mu}}\mathrm{d}x=\int_{(\Omega\setminus D_\varepsilon)\cup\{|u|\le M_\varepsilon\}}|f(x,u(x))|^{\frac{2N}{2N-2\beta-\mu}}\mathrm{d}x.$$

Here, we choose $K_\varepsilon>t_0$ such that

$$\|\phi\|_\infty\left(\frac{M_0\mathcal{K}}{K_\varepsilon}\right)^{\frac{1}{2}}\left[2\mathcal{C}_0\int_\Omega|f(u)|^{\frac{2N}{2N-2\beta-\mu}}\mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{2N}}<\varepsilon$$

and

$$\int_{|u|\le M_\varepsilon}\left[\frac{F(y,u(y))\chi_{|u|\ge K_\varepsilon}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right]\frac{|f(x,u(x))\phi|}{|x|^\beta}\mathrm{d}x<\varepsilon.$$

With the help of the Cauchy-Schwartz inequality introduced in [40], we have

$$\int_{(|u_n|\ge M_\varepsilon)\cap(|u|\ne M_\varepsilon)}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))\chi_{|u_n|\ge K_\varepsilon}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|f(x,u_n(x))\phi|}{|x|^\beta}\mathrm{d}x$$

$$\leq \|\phi\|_\infty \left[\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))\chi_{|u_n|\geq K_\varepsilon}}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{F(x,u_n(x))\chi_{|u_n|\geq K_\varepsilon}}{|x|^\beta} \mathrm{d}x\right]^{\frac{1}{2}}$$
$$\times \left[\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{|f(y,u_n(y))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{|f(x,u_n(x))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|x|^\beta} \mathrm{d}x\right]^{\frac{1}{2}},$$

then from (F5) and Proposition 1, one has

$$\int\limits_{(|u_n|\geq M_\varepsilon)\cap(|u|\neq M_\varepsilon)} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))\chi_{|u_n|\geq K_\varepsilon}}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{|f(x,u_n(x))\phi|}{|x|^\beta} \mathrm{d}x$$
$$\leq \|\phi\|_\infty \left[\int\limits_{|u_n|\geq K_\varepsilon} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{F(x,u_n(x))}{|x|^\beta} \mathrm{d}x\right]^{\frac{1}{2}}$$
$$\times \left[\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{|f(y,u_n(y))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{|f(x,u_n(x))|\chi_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}}}{|x|^\beta} \mathrm{d}x\right]^{\frac{1}{2}}$$
$$\leq \|\phi\|_\infty \left[\int\limits_{|u_n|\geq K_\varepsilon} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{F(x,u_n(x))}{|x|^\beta} \mathrm{d}x\right]^{\frac{1}{2}}$$
$$\times \left[C(N,\mu,\beta) \int\limits_{(\Omega\setminus D_\varepsilon)\cap\{|u_n|\leq M_\varepsilon\}} |f(u_n)|^{\frac{2N}{2N-2\beta-\mu}} \mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{2N}}$$
$$\leq \|\phi\|_\infty \left[\frac{M_0}{K_\varepsilon} \int\limits_{|u_n|\geq K_\varepsilon} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y\right) \frac{f(x,u_n(x))u_n(x)}{|x|^\beta} \mathrm{d}x\right]^{\frac{1}{2}}$$
$$\times \left[2C(N,\mu,\beta) \int\limits_{\Omega} |f(u)|^{\frac{2N}{2N-2\beta-\mu}} \mathrm{d}x + o(1)\right]^{\frac{2N-2\beta-\mu}{2N}}$$
$$\leq \|\phi\|_\infty \left(\frac{M_0 \mathcal{K}}{K_\varepsilon}\right)^{\frac{1}{2}} \left[2C(N,\mu,\beta) \int\limits_{\Omega} |f(u)|^{\frac{2N}{2N-2\beta-\mu}} \mathrm{d}x\right]^{\frac{2N-2\beta-\mu}{2N}} + o(1) < \varepsilon + o(1).$$

For any $x \in \mathbb{R}^N$, define $\zeta_n(x)$ and $\bar{\zeta}(x)$ as follows,

$$\zeta_n(x) := \int\limits_{\mathbb{R}^N} \frac{|F(y,u_n(y))|\chi_{|u_n|\leq K_\varepsilon}}{|y|^\beta |x-y|^\mu} \mathrm{d}y$$

and

$$\bar{\zeta}(x) := \int\limits_{\mathbb{R}^N} \frac{|F(y,u(y))|\chi_{|u|\leq K_\varepsilon}}{|y|^\beta |x-y|^\mu} \mathrm{d}y.$$

Let us first point out some relationships here. For fixed $x \in \mathbb{R}^N$, we consider the term

$$\int_{|x-y|\le R} \frac{1}{|y|^{\beta p_1}|x-y|^{\mu p_1}}\mathrm{d}y.$$

When $x \in \mathbb{R}^N/B_{2R}(0)$, $y \in B_R(x)$, thus $|x-y|<|y|$. Select p_1 such that $(\mu+\beta)p_1 < N$, and thus we have,

$$\int_{|x-y|\le R} \frac{1}{|y|^{\beta p_1}|x-y|^{\mu p_1}}\mathrm{d}y \le \int_{|x-y|\le R} \frac{1}{|x-y|^{(\mu+\beta)p_1}}\mathrm{d}y = \mathcal{O}\left(R^{N-(\mu+\beta)p_1}\right).$$

When $x \in B_{2R}(0)$, one has

$$\int_{|x-y|\le R} \frac{1}{|y|^{\beta p_1}|x-y|^{\mu p_1}}\mathrm{d}y \le \int_{|y|\le R} \frac{1}{|y|^{(\mu+\beta)p_1}}\mathrm{d}y + \int_{|x-y|\le 3R} \frac{1}{|x-y|^{(\beta+\mu)p_1}}\mathrm{d}y = \mathcal{O}\left(R^{N-(\mu+\beta)p_1}\right).$$

That is

$$\int_{|x-y|\le R} \frac{1}{|y|^{\beta p_1}|x-y|^{\mu p_1}}\mathrm{d}y \le \mathcal{O}\left(R^{N-(\mu+\beta)p_1}\right).$$

Choosing q such that $\beta q < N < \mu q$, one has

$$\begin{aligned}
&\int_{|x-y|\ge R} \frac{1}{|y|^{\beta q}|x-y|^{\mu q}}\mathrm{d}y \\
=&\int_{(\mathbb{R}^N\setminus B_R(x))\cap B_R(0)} \frac{1}{|y|^{\beta q}|x-y|^{\mu q}}\mathrm{d}y + \int_{(\mathbb{R}^N\setminus B_R(x))\cap(\mathbb{R}^N\setminus B_R(0))} \frac{1}{|y|^{\beta q}|x-y|^{\mu q}}\mathrm{d}y \\
\le&\frac{1}{R^{\mu q}}\int_{|y|\le R} \frac{1}{|y|^{\beta q}}\mathrm{d}y + \frac{1}{R^{\beta q}}\int_{\mathbb{R}^N\setminus B_R(x)} \frac{1}{|x-y|^{\mu q}}\mathrm{d}y = \mathcal{O}\left(R^{N-(\beta+\mu)q}\right).
\end{aligned}$$

Then from (12), we have

$$\begin{aligned}
|\zeta_n(x)-\bar{\zeta}(x)| \le& \int_{\mathbb{R}} \frac{\left||F(y,u_n(y))|\chi_{|u_n|\le K_\varepsilon} - |F(y,u(y))|\chi_{|u|\le K_\varepsilon}\right|}{|y|^\beta|x-y|^\mu}\mathrm{d}y \\
\le& \left[\int_{|x-y|\le R} \left||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar{u})|\chi_{|\bar{u}|\le K_\varepsilon}\right|^{p_1'}\mathrm{d}y\right]^{\frac{1}{p_1'}} \\
&\times\left(\int_{|x-y|\le R} \frac{1}{|y|^{\beta p_1}|x-y|^{\mu p_1}}\mathrm{d}y\right)^{\frac{1}{p_1}} \\
&+\left[\int_{|x-y|>R} \left||F(u_n)|\chi_{|u_n|\le K_\varepsilon} - |F(\bar{u})|\chi_{|\bar{u}|\le K_\varepsilon}\right|^{q'}\mathrm{d}y\right]^{\frac{1}{q'}}
\end{aligned}$$

$$\times\left(\int\limits_{|x-y|>R}\frac{1}{|y|^{\beta q}|x-y|^{\mu q}}\mathrm{d}y\right)^{\frac{1}{q}}$$

$$\leq \mathcal{O}\left(R^{N/p_1-\beta-\mu}\right)\left[\int\limits_{|x-y|\leq R}\left||F(u_n)|\chi_{|u_n|\leq K_\varepsilon}-|F(\bar{u})|\chi_{|\bar{u}|\leq K_\varepsilon}\right|^{p_1'}\mathrm{d}y\right]^{\frac{1}{p_1'}}$$

$$+\mathcal{O}\left(R^{N/q-\beta-\mu}\right)\left(\int\limits_{|x-y|>R}\left||F(u_n)|\chi_{|u_n|\leq K_\varepsilon}-|F(\bar{u})|\chi_{|\bar{u}|\leq K_\varepsilon}\right|^{q'}\mathrm{d}y\right)^{\frac{1}{q'}}$$

$$\leq \mathcal{O}\left(R^{N/p_1-\beta-\mu}\right)\left[\int\limits_{|x-y|\leq R}\left||F(u_n)|\chi_{|u_n|\leq K_\varepsilon}-|F(\bar{u})|\chi_{|\bar{u}|\leq K_\varepsilon}\right|^{p_1'}\mathrm{d}y\right]^{\frac{1}{p_1'}}$$

$$+\mathcal{O}\left(R^{N/q-\beta-\mu}\right)\left[\|u_n\|^{\frac{(2N-2\beta-\mu)p}{2N}}_{\frac{(2N-2\beta-\mu)pq'}{2N}}+\|\bar{u}\|^{\frac{(2N-2\beta-\mu)p}{2N}}_{\frac{(2N-2\beta-\mu)pq'}{2N}}\right]$$

$$\leq \mathcal{O}\left(R^{N/p_1-\beta-\mu}\right)o_n(1)+\mathcal{O}\left(R^{N/q-\beta-\mu}\right),\quad \forall\ \ x\in\mathbb{R}^N,$$

which implies that for any $x\in\mathbb{R}^N$, we have $\zeta_n(x)\to\bar{\zeta}(x)$. For any $x\in\mathbb{R}^N$, we know that

$$|\zeta_n(x)|\leq\int\limits_{\mathbb{R}^N}\frac{|F(x,u_n(x))|\chi_{|u_n|\leq K_\varepsilon}}{|y|^\beta|x-y|^\mu}\mathrm{d}y$$

$$\leq\left[\int\limits_{|x-y|\leq R}\left|F(x,u_n(x))\chi_{|u_n|\leq K_\varepsilon}\right|^{p_2'}\mathrm{d}y\right]^{\frac{1}{p_2'}}\left[\int\limits_{|x-y|\leq R}\frac{1}{|y|^{\beta p_2}|x-y|^{\mu p_2}}\mathrm{d}y\right]^{\frac{1}{p_2}}$$

$$+\left[\int\limits_{|x-y|>R}\left|F(x,u_n(x))\chi_{|u_n|\leq K_\varepsilon}\right|^{p_3'}\mathrm{d}y\right]^{\frac{1}{p_3'}}\left[\int\limits_{|x-y|>R}\frac{1}{|y|^{\beta p_3}|x-y|^{\mu p_3}}\mathrm{d}y\right]^{\frac{1}{p_3}}$$

$$\leq\left(\omega(N)R^N\right)^{\frac{1}{p_2'}}\mathcal{O}\left(R^{N/p_2-\beta-\mu}\right)\max_{|t|\leq K_\varepsilon}|F(x,t)|+C\mathcal{O}\left(R^{N/p_3-\beta-\mu}\right)\|u_n\|^{\frac{(2N-2\beta-\mu)p}{2N}}_{\frac{(2N-2\beta-\mu)pp_3'}{2N}}$$

$$\leq C.$$

It follows that

$$\left|\frac{\zeta_n(x)f(x,u_n(x))\phi(x)\chi_{|u_n|\leq M_\varepsilon}}{|x|^\beta}\right|\leq C\left|\frac{\phi(x)\max_{|t|\leq M_\varepsilon}|f(x,t)|}{|x|^\beta}\right|\leq\frac{C'}{|x|^\beta},\quad\forall\ x\in\Omega.$$

By $\beta<N$, it is easy to verify that $\frac{1}{|x|^\beta}\in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore, together with $\zeta_n(x)\to\bar{\zeta}(x)$ and the Lebesgue dominated convergence theorem, we have

$$\int\limits_{(|u_n|\leq M_\varepsilon)\cap(|\bar{u}|\neq M_\varepsilon)}\left(\int\limits_{\mathbb{R}^N}\frac{F(x,u_n(x))\chi_{|u_n|\leq K_\varepsilon}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|f(x,u_n(x))\phi(x)|}{|x|^\beta}\mathrm{d}x$$

$$\to \int\limits_{|\bar{u}|<M_\varepsilon} \left(\int\limits_{\mathbb{R}^N} \frac{F(x,u(x))\chi_{|u|\le K_\varepsilon}}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{|f(x,u(x))\phi(x)|}{|x|^\beta} \mathrm{d}x.$$

From the arguments above all and by the arbitrariness of $\varepsilon > 0$, we can conclude this Lemma. □

The proof of Lemma 18. In view of Proposition 6, we know $u_n \to u$ in $L^q(\mathbb{R}^N)$, where $q \ge p$. By [46, Theorem A.1], there exists $g \in L^q(\mathbb{R}^N)$ such that

$$|u_n(x)| \le g(x), \quad |u(x)| \le g(x), \quad \text{a.e. } x \in \mathbb{R}^N.$$

For any given $\varepsilon \in (0, M_0/t_0)$, it follows from (F5) that

$$\begin{aligned}
&\int\limits_{|u_n|\ge M_0\varepsilon^{-1}} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{F(x,u_n(x))}{|x|^\beta} \mathrm{d}x \\
&\le M_0 \int\limits_{|u_n|\ge M_0\varepsilon^{-1}} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{|f(x,u_n(x))|}{|x|^\beta} \mathrm{d}x \\
&\le \varepsilon \int\limits_{|u_n|\ge M_0\varepsilon^{-1}} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))u_n(x)}{|x|^\beta} \mathrm{d}x \le C\varepsilon.
\end{aligned}$$

Similarly, one has

$$\int\limits_{|u|\ge M_0\varepsilon^{-1}} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{F(x,u(x))}{|x|^\beta} \mathrm{d}x \le C\varepsilon.$$

Now, we can choose $R_\varepsilon > 0$ such that

$$\int\limits_{\mathbb{R}^N\setminus B_{R_\varepsilon}} \left| \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{F(x,u(x))}{|x|^\beta} \right| \mathrm{d}x < \varepsilon$$

and

$$\int\limits_{\mathbb{R}^N\setminus B_{R_\varepsilon}} \left| \left(\int\limits_{\mathbb{R}^N} \frac{|u(y)|^q}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{|u(x)|^q}{|x|^\beta} \right| \mathrm{d}x < \varepsilon.$$

Let C be the constant in (32) and choose $K \ge \max\{CM_0/\varepsilon, t_0\}$ such that

$$\int\limits_{|u|\le M_0\varepsilon^{-1}} \left(\int\limits_{|u|\ge K} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{F(x,u(x))}{|x|^\beta} \mathrm{d}x < \varepsilon. \tag{34}$$

By (F5), one has

$$
\begin{aligned}
&\int\limits_{|u_n|\leq M_0\varepsilon^{-1}}\left(\int\limits_{|u_n|\geq K}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x\\
\leq&\frac{1}{K}\int\limits_{|u_n|\leq M_0\varepsilon^{-1}}\left(\int\limits_{|u_n|\geq K}\frac{F(y,u_n(y))u_n(y)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x\\
\leq&\frac{M_0}{K}\int\limits_{|u_n|\leq M_0\varepsilon^{-1}}\left(\int\limits_{|u_n|\geq K}\frac{f(y,u_n(y))u_n(y)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x\\
\leq&\frac{M_0}{K}\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x\leq\varepsilon.
\end{aligned}
\tag{35}
$$

By (F2), we know that there exist $C>0$ and $\tilde{q}>\frac{(2N-2\beta-\mu)p}{2N}$ such that for all $x\in\mathbb{R}^N$, $|t|\leq K$,

$$
|F(x,t)|\leq C|t|^{\tilde{q}+1}. \tag{36}
$$

Thus we have

$$
\begin{aligned}
&\int\limits_{\{\mathbb{R}^N\setminus B_{R_\varepsilon}\}\cap\{|u_n|\leq M_0\varepsilon^{-1}\}}\left(\int\limits_{|u_n|\leq K}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\\
\leq&C\int\limits_{\mathbb{R}^N\setminus B_{R_\varepsilon}}\left(\int\limits_{|u_n|\leq K}\frac{u_n^{\tilde{q}+1}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{u_n^{\tilde{q}+1}}{|x|^\beta}\mathrm{d}x\\
\leq&C\int\limits_{\mathbb{R}^N\setminus B_{R_\varepsilon}}\left(\int\limits_{|u_n|\leq K}\frac{g^{\tilde{q}+1}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{g^{\tilde{q}+1}}{|x|^\beta}\mathrm{d}x\leq C\varepsilon,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
&\left|\int\limits_{\{\mathbb{R}^N\setminus B_{R_\varepsilon}\}\cap\{|u_n|\leq M_0\varepsilon^{-1}\}}\left[\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\right.\right.\\
&\qquad\left.\left.-\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\right]\mathrm{d}x\right|\\
\leq&\left|\int\limits_{\{\mathbb{R}^N\setminus B_{R_\varepsilon}\}\cap\{|u_n|\leq M_0\varepsilon^{-1}\}}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\right|\\
&+\left|\int\limits_{\{\mathbb{R}^N\setminus B_{R_\varepsilon}\}\cap\{|u_n|\leq M_0\varepsilon^{-1}\}}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x\right|
\end{aligned}
$$

$$< \varepsilon + \int\limits_{\{\mathbb{R}^N \backslash B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}} \left(\int\limits_{|u_n|\le K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x$$
$$+ \int\limits_{\{\mathbb{R}^N \backslash B_{R_\varepsilon}\}\cap\{|u_n|\le M_0\varepsilon^{-1}\}} \left(\int\limits_{|u_n|\ge K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x < (2+C)\varepsilon.$$

On the other hand,

$$\left| \int\limits_{B_{R_\varepsilon}} \left[\left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u_n(x))}{|x|^\beta} - \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u(x))}{|x|^\beta} \right] \mathrm{d}x \right|$$
$$\le 2C\varepsilon + \left| \int\limits_{B_{R_\varepsilon}\cap\{|u_n|\le M_0\varepsilon^{-1}\}} \left(\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\beta}\mathrm{d}y \right) \frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x \right.$$
$$\left. - \int\limits_{B_{R_\varepsilon}\cap\{|u|\le M_0\varepsilon^{-1}\}} \left(\frac{F(y,u(y))}{|y|^\beta|x-y|^\beta}\mathrm{d}y \right) \frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x \right|.$$

It remains to prove that as $n \to \infty$,

$$\int\limits_{\{|u_n|\le M_0\varepsilon^{-1}\}} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x$$
$$\to \int\limits_{\{|u|\le M_0\varepsilon^{-1}\}} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x. \tag{37}$$

Combining (34) with (35), we can see that

$$\left| \int\limits_{|u_n|\le M_0\varepsilon^{-1}} \left\{ \int\limits_{|u_n|\ge K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \frac{F(x,u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta} \right.\right.$$
$$\left.\left. - \int\limits_{|u|\ge K} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta} \right\} \mathrm{d}x \right| \le 2\varepsilon.$$

In order to prove (37), it remains to verify that as $n \to +\infty$ there holds

$$\int\limits_{\{|u_n|\le M_0\varepsilon^{-1}\}} \left(\int\limits_{|u_n|\le K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u_n(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x$$
$$\to \int\limits_{\{|u|\le M_0\varepsilon^{-1}\}} \left(\int\limits_{|u|\le K} \frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y \right) \frac{F(x,u(x))\chi_{B_{R_\varepsilon}}}{|x|^\beta}\mathrm{d}x.$$

Indeed, it can be easily verified that as $n \to \infty$,

$$\left(\int\limits_{|u_n|\le K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\chi_{\{B_{R_\varepsilon}\cap|u_n|\le M_0\varepsilon^{-1}\}}$$
$$\to\left(\int\limits_{|u|\le K} \frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\chi_{\{B_{R_\varepsilon}\cap|u|\le M_0\varepsilon^{-1}\}} \quad \text{pointwise a.e.}$$

From (36), we have

$$\begin{aligned}&\int\limits_{B_{R_\varepsilon}\cap|u_n|\le M_0\varepsilon^{-1}}\left(\int\limits_{|u_n|\le K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\\
\le& C\int\limits_{B_{R_\varepsilon}\cap|u_n|\le M_0\varepsilon^{-1}}\left(\int\limits_{|u_n|\le K} \frac{|u_n(y)|^{\tilde q+1}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|u_n(x)|^{\tilde q+1}}{|x|^\beta}\mathrm{d}x\\
\le& C\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N} \frac{|u_n(y)|^{\tilde q+1}(y)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|u_n(x)|^{\tilde q+1}}{|x|^\beta}\mathrm{d}x\\
\le& C\cdot C(N,\mu,\beta)\|u_n\|^{2(\tilde q+1)}_{\frac{2N(\tilde q+1)}{2N-2\beta-\mu}} \to C\cdot C(N,\mu,\beta)\|u\|^{2(\tilde q+1)}_{\frac{2N(\tilde q+1)}{2N-2\beta-\mu}}, \quad \text{as} \quad n\to\infty.\end{aligned}$$

From [11, Theorem 4.9], there exists $\mathcal{F}\in L^1(\mathbb{R}^N)$ such that up to a subsequence, still denoted by $\{u_n\}$, for each $n\in\mathbb{N}$, we have

$$\left|\left(\int\limits_{|u_n|\le K} \frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\chi_{\{B_{R_\varepsilon}\cap|u_n|\le M_0\varepsilon^{-1}\}}\right|\le|\mathcal{F}(x)|.$$

So, using the Lebesgue dominated convergence theorem, we can conclude our proof of Lemma 18. □

The proof of Lemma 19. For any given $\varepsilon>0$, noting that $u\in E$, we can choose $\phi_\varepsilon\in C_0^\infty(\mathbb{R}^N)\cap E$ such that $\|\phi_\varepsilon-u\|_E<\varepsilon$. Hence, from and the fact that $\{\|u_n\|_E^2\}$ is bounded, we have

$$\begin{aligned}o(1)&=\langle I'(u_n),\phi_\varepsilon-u\rangle\\
&=\int\limits_{\mathbb{R}^{2N}}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))((\phi_\varepsilon-u)(x)-(\phi_\varepsilon-u)(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y\\
&\quad+\int\limits_{\mathbb{R}^N}V(x)|u_n|^{p-2}u_n(\phi_\varepsilon-u)\mathrm{d}x+A_1(|u_n|^p,|u_n|^{p-2}u_n(\phi_\varepsilon-u))\\
&\quad-A_2(|u_n|^p,|u_n|^{p-2}u_n(\phi_\varepsilon-u))-\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))((\phi_\varepsilon-u)(x))}{|x|^\beta}\mathrm{d}x\\
&\le\|u_n\|^{p-1}\|\phi_\varepsilon-u\|\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[\ln(1+|x|)+\ln(1+|y|)]|u_n(x)|^p|u_n(y)|^{p-2}|u_n(y)||\phi_\varepsilon(y)-u(y)|\mathrm{d}x\mathrm{d}y\\
&+\mathcal{C}_1\|u_n\|^p_{\frac{2Np}{2N-1}}\|u_n\|^{p-1}_{\frac{2Np}{2N-1}}\|\phi_\varepsilon-u\|_{\frac{2Np}{2N-1}}\\
&-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))((\phi_\varepsilon-u)(x))}{|x|^\beta}\mathrm{d}x\\
\le{}& \|u_n\|^{p-1}\|\phi_\varepsilon-u\|+\|u_n\|_*^p\|u_n\|_p^{p-1}\|\phi_\varepsilon-u\|_p+\|u_n\|_p^p\|u_n\|_*^{p-1}\|\phi_\varepsilon-u\|_*\\
&+\mathcal{C}_1\|u_n\|^p_{\frac{2Np}{2N-1}}\|u_n\|^{p-1}_{\frac{2Np}{2N-1}}\|\phi_\varepsilon-u\|_{\frac{2Np}{2N-1}}\\
&-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))((\phi_\varepsilon-u)(x))}{|x|^\beta}\mathrm{d}x.
\end{aligned}
$$

That is

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))(\phi_\varepsilon-u)(x))}{|x|^\beta}\mathrm{d}x\right|\\
&\le\|u_n\|^{p-1}\|\phi_\varepsilon-u\|+C\varepsilon+o(1)\le C\varepsilon+o(1).
\end{aligned}
$$

It is easy to verify that

$$
\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))((\phi_\varepsilon-u)(x))}{|x|^\beta}\mathrm{d}x\right|\le C\varepsilon.
$$

By Lemma 20, we can obtain that for any $\phi\in\mathcal{C}_0^\infty(\mathbb{R}^N)$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u(x)}{|x|^\beta}\mathrm{d}x-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))u(x)}{|x|^\beta}\mathrm{d}x\right|\\
\le{}&\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))(\phi_\varepsilon-u)(x)}{|x|^\beta}\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))(\phi_\varepsilon-u)(x)}{|x|^\beta}\mathrm{d}x\right|\\
&+\left|\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))\phi_\varepsilon(x)}{|x|^\beta}\mathrm{d}x\right.\\
&\left.-\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u(x))\phi_\varepsilon(x)}{|x|^\beta}\mathrm{d}x\right|\le 2C\varepsilon+o(1).
\end{aligned}
$$

Thus, we have concluded the proof of Lemma 19. □

Proof of Theorem 5. Note that by Lemmas 10 and 14, we can deduce that there exists a sequence $\{u_n\}\subset E$ satisfying $\|u_n\|\le\mathcal{C}_1$. It follows from (F3) and (18) that

$$\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{2pF(x,u_n(x))}{|x|^\beta}\mathrm{d}x\le\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x\le\mathcal{C}_2.$$

Let $u_n=u_n^+-u_n^-$, where $u_n^+=\max\{u_n(x),0\}$ and $u_n^-=-\min\{u_n(x),0\}$. Since $f(x,t)=0$ for all $t\le 0$, by taking $v_n=-u_n^-$ and using the fact that $\{u_n\}$ satisfies (17), we obtain

$$\begin{aligned}o_n(1)=&\langle I'(u_n),-u_n^-\rangle\\=&-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y\\&-\int_{\mathbb{R}^N}V(x)|u_n|^{p-2}u_nu_n^-\mathrm{d}x-A_0(|u_n|^p,|u_n|^{p-2}u_nu_n^-)\\\ge&\|u_n^-\|^p,\end{aligned}$$

where we have used that $u_n^+,u_n^-\ge 0$. Thus, $\|u_n^-\|\to 0$, as $n\to\infty$. Hence, we have that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_n^+(x)-u_n^+(y)|^{p-2}(u_n^+(x)-u_n^+(y))(u_n^-(x)-u_n^-(y))}{|x-y|^{2N}}\mathrm{d}x\mathrm{d}y\to 0,\quad\text{as } n\to\infty,$$

which implies that $\|u_n\|=\|u_n^+\|+o_n(1)$. Therefore, $\{u_n^+\}$ also satisfies (17). For this reason, we may suppose, without loss of generality, that $\{u_n\}$ is a nonnegative sequence.

If $\delta_0:=\limsup_{n\to\infty}\|u_n\|_p=0$, then from the Gagliardo-Nirenberg inequality explored in [36, Proposition]:

$$\|u_n\|_q^q\le C_q\|u_n\|_p^p\|(-\Delta)^{\frac{s}{2}}u_n\|_p^{q-p},\tag{38}$$

we derive that $u_n\to 0$ in $L^q(\mathbb{R}^N)$ for $q\in[p,+\infty)$. For any given $\varepsilon\in(0,M_0\mathcal{C}_2/t_0)$, we choose $M_\varepsilon>M_0\mathcal{C}_2/\varepsilon$, then it follows from (F5) that

$$\begin{aligned}&\int_{|u_n|\ge M_\varepsilon}\left(\int_{R^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\\&\le\frac{M_0}{M_\varepsilon}\int_{|u_n|\ge M_\varepsilon}\left(\int_{R^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x<\varepsilon.\end{aligned}$$

Using (F1$'$) and (F2), we have

$$\begin{aligned}&\int_{|u_n|\le M_\varepsilon}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{2pF(x,u_n(x))}{|x|^\beta}\mathrm{d}x\\&\le\int_{|u_n|\le M_\varepsilon}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x\\&\le\mathcal{C}_\varepsilon\int_{|u_n|\le M_\varepsilon}\left(\int_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|u_n(x)|^{\frac{(2N-2\beta-\mu)p}{2N}}}{|x|^\beta}\mathrm{d}x\end{aligned}$$

$$\leq \mathcal{C}_\varepsilon \left(\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{2}}$$

$$\times\left(\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{|u_n(y)|^{\frac{(2N-2\beta-\mu)p}{2N}}}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{|u_n(x)|^{\frac{(2N-2\beta-\mu)p}{2N}}}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{2}}$$

$$\leq \frac{\sqrt{\mathcal{C}_2\mathcal{C}_\varepsilon\mathcal{C}(N,\beta,\mu)}}{\mathcal{C}_2}\|u_n\|_p^{\frac{(2N-2\beta-\mu)p}{N}} = o(1).$$

Due to the arbitrariness of $\varepsilon > 0$, thus we obtain

$$\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u(x))}{|x|^\beta}\mathrm{d}x = o(1).$$

Hence,

$$\begin{aligned}\|u_n\|^p \leq&\|u_n\|^p + \frac{1}{2}A_1(|u_n|^p,|u_n|^p)\\ &\leq pm + \frac{1}{2}A_2(|u_n|^p,|u_n|^p) + o(1)\\ &\leq pm + K_0\|u_n\|^{2p}_{\frac{2Np}{2N-1}} + o(1) = pm + o(1) \leq \left[\frac{(2N-2\beta-\mu)\alpha_{N,s}}{2N\alpha_0}\right]^{\frac{N-s}{s}}(1-3\bar{\varepsilon}) + o(1).\end{aligned}$$

Now, we choose $\gamma \in (1,2)$ satisfying that

$$\frac{(1+\bar{\varepsilon})(1-3\bar{\varepsilon})\gamma}{1-\bar{\varepsilon}} < 1.$$

By (F1′), there exists $\mathcal{C}_3 > 0$ such that

$$|f(x,t)t| \leq \mathcal{C}_3|t|^q\mathcal{H}\left(\alpha_0(1+\bar{\varepsilon}),t\right), \quad \forall\, x \in \mathbb{R}^N,\ |t| \geq 1,$$

and

$$\int\limits_{|u_n|\geq 1}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x$$

$$\leq \left(\int\limits_{\mathbb{R}^N}\left(\int\limits_{\mathbb{R}^N}\frac{F(y,u_n(y))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{2}} \times$$

$$\left(\int\limits_{|u_n|\geq 1}\left(\int\limits_{|u_n|\geq 1}\frac{f(x,u_n(y))u_n(y)}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{f(x,u_n(x))u_n(x)}{|x|^\beta}\mathrm{d}x\right)^{\frac{1}{2}}$$

$$\leq \mathcal{C}_4^{\frac{1}{2}}C(N,\mu,\beta)^{\frac{1}{2}}\|f(x,u_n)u_n\|_{\frac{2N}{2N-2\beta-\mu}}$$

$$\leq \mathcal{C}_4^{\frac{1}{2}}C(N,\mu,\beta)^{\frac{1}{2}}\left(\int\limits_{\mathbb{R}^N}\mathcal{C}_3|u_n|^{\frac{2Nq}{2N-2\beta-\mu}}\mathcal{H}\left(\frac{2N\alpha_0(1+\bar{\varepsilon})}{2N-2\beta-\mu},u_n\right)\mathrm{d}x\right)^{\frac{2N-2\beta-\mu}{2N}}$$

$$= \mathcal{C}_4^{\frac{1}{2}} C(N,\mu,\beta)^{\frac{1}{2}} \left(\int\limits_{\mathbb{R}^N} \mathcal{C}_3 |u_n|^{\frac{2Nq}{2N-2\beta-\mu}} \mathcal{H}\left(\frac{2N\alpha_0(1+\bar{\varepsilon})\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^{N/(N-s)}}{2N-2\beta-\mu}, \frac{u_n}{\|u_n\|_{W^{s,p}(\mathbb{R}^N)}} \right) \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{2N}}$$

$$\leq \mathcal{C}_5 \left(\int\limits_{\mathbb{R}^N} |u_n|^{\frac{2Nq\gamma'}{2N-2\beta-\mu}} \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{2N\gamma'}} \left(\int\limits_{\mathbb{R}^N} \mathcal{H}\left(\frac{2N\alpha_0\gamma(1+\bar{\varepsilon})\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^{N/(N-s)}}{2N-2\beta-\mu}, \frac{u_n}{\|u_n\|_{W^{s,p}(\mathbb{R}^N)}} \right) \mathrm{d}x \right)^{\frac{2N-2\beta-\mu}{2N\gamma}} .$$

By (V1), we have

$$\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^p \leq \left[\frac{(2N-2\beta-\mu)\alpha_{N,s}(1-3\varepsilon)}{2N\alpha_0} \right]^{\frac{N-s}{s}} \Rightarrow \frac{2N\alpha_0\gamma(1+\bar{\varepsilon})\|u_n\|_{W^{s,p}(\mathbb{R}^N)}^{\frac{N}{N-s}}}{2N-2\beta-\mu} < 1-\bar{\varepsilon}.$$

Then we have

$$\int\limits_{|u_n|\geq 1} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))u_n(x)}{|x|^\beta} \mathrm{d}x \leq \mathcal{C}_6 \|u_n\|_{\frac{2Nq\gamma'}{2N-2\beta-\mu}}^q ,$$

and it is easy to verify

$$\int\limits_{|u_n|\leq 1} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))u_n(x)}{|x|^\beta} \mathrm{d}x = o(1).$$

Therefore, we get

$$\int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))u_n(x)}{|x|^\beta} \mathrm{d}x = o(1).$$

Thus we have

$$\begin{aligned} c_* + o(1) &= I(u_n) - \frac{1}{p}\langle I'(u_n), u_n \rangle \\ &= -\frac{1}{2p} A_1(|u_n|^p, |u_n|^p) + \frac{1}{2p} A_2(|u_n|^p, |u_n|^p) \\ &\quad + \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{\frac{1}{p} f(x,u_n(x))u_n(x) - \frac{1}{2} F(x,u_n(x))}{|x|^\beta} \mathrm{d}x \leq o(1), \end{aligned}$$

and this contradiction means that $\delta_0 > 0$. Furthermore, we can obtain that

$$V_1(u_n) \leq \|u_n\|^p + V_1(u_n) = V_2(u_n) + \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{F(x,u_n(x))}{|x|^\beta} \mathrm{d}x + o(1) \leq \mathcal{C}_7,$$

which together with (22) implies that $\{\|u_n\|_*\}$ is bounded. Hence, $\{u_n\}$ is bounded in E. We may thus assume passing to a subsequence again if necessary, that $u_n \rightharpoonup u > 0$ in E, $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $q \in [p, +\infty)$ and $u_n(x) \to u(x)$ a.e. on $\mathbb{R}^N$.

From (11), we can obtain

$$A_2(|u_n|^p, (|u_n|^{p-2}u_n - |u|^{p-2}u)u) = o(1) \quad \text{and} \quad A_2(|u_n|^p - |u|^p, |u|^p) = o(1).$$

Inspired by the arguments in [51], we could know that for any $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned}
&\lim_{n\to\infty} \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y \\
&= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y
\end{aligned}$$

and

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n\varphi \mathrm{d}x = \int\limits_{\mathbb{R}^N} V(x)|u|^{p-2}u\varphi \mathrm{d}x.$$

Recalling Lemma 7 and the property that $\tilde{\Upsilon}(u,v) \leq \|u\|^{p-1}\|v\|$, together with the Lemma 19 and the Fatou's lemma, we can obtain

$$\begin{aligned}
o(1) &= \langle I'(u_n), u\rangle \\
&= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u(x)-u(y))}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(x)|u_n|^{p-2}u_n u \mathrm{d}x \\
&\quad + A_1(|u_n|^p, |u_n|^{p-2}u_n u) - A_2(|u_n|^p, |u_n|^{p-2}u_n u) \\
&\quad - \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u_n(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u_n(x))u(x)}{|x|^\beta} \mathrm{d}x \\
&= \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x + A_1(|u_n|^p, |u_n|^{p-2}u_n u) - A_2(|u|^p, |u|^p) \\
&\quad - \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u(x))u(x)}{|x|^\beta} \mathrm{d}x + o(1) \\
&\geq \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{2N}} \mathrm{d}x\mathrm{d}y + \int\limits_{\mathbb{R}^N} V(x)|u|^p \mathrm{d}x + A_1(|u|^p, |u|^p) - A_2(|u|^p, |u|^p) \\
&\quad - \int\limits_{\mathbb{R}^N} \left(\int\limits_{\mathbb{R}^N} \frac{F(y,u(y))}{|y|^\beta |x-y|^\mu} \mathrm{d}y \right) \frac{f(x,u(x))u(x)}{|x|^\beta} \mathrm{d}x + o(1) \\
&= \langle I'(u), u\rangle + o(1),
\end{aligned}$$

which implies that $\langle I'(u), u\rangle \leq 0$. Since $u > 0$, it follows from Lemma 13 that there exists $t_u > 0$ such that $t_u u \in \mathcal{N}$. Noting that $\langle I'(u), u\rangle \leq 0$, then by the weak continuity of the norm and Fatou's lemma, we have

$$m \geq c_* = \lim_{n\to\infty} \left[I(u_n) - \frac{1}{2p}\langle I'(u_n), u_n\rangle \right]$$

$$
\begin{aligned}
&= \lim_{n\to\infty}\left\{\frac{1}{2p}\|u_n\|^p + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(x,u_n(x))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{\frac{1}{2p}f(x,u_n(x))u_n(x)-F(x,u_n(x))}{|x|^\beta}\mathrm{d}x\right\}\\
&\geq \frac{1}{2p}\|u\|^p + \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(x,u(x))}{|y|^\beta|x-y|^\mu}\mathrm{d}y\right)\frac{\frac{1}{2p}f(x,u(x))u(x)-F(x,u(x))}{|x|^\beta}\mathrm{d}x\\
&= I(u) - \frac{1}{2p}\langle I'(u),u\rangle\\
&\geq I(t_u u) - \frac{t_u^{2p}}{2p}\langle I'(u),u\rangle\\
&\geq m - \frac{t_u^{2p}}{2p}\langle I'(u),u\rangle \geq m,
\end{aligned}
$$

which implies that

$$
I(u) = m, \quad \langle I'(u),u\rangle = 0.
$$

Next, we verify that $V_1(|u_n|^p, |u|^{p-2}u(u_n-u)) = o_n(1)$. Indeed, from the boundedness of $\{\|u_n\|_*\}$ and $\{\|u_n\|\}$, we can easily know

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\ln(1+|y|)|u(y)|^{p-2}|u(y)||u_n(y)-u(y)|\mathrm{d}y\\
&\leq \ln(1+R)\|u\|_p^{p-1}\|u_n-u\|_p + \|u_n-u\|_*\left[\int_{\mathbb{R}^N\setminus B_R(0)}\ln(1+|y|)|u(y)|^p\mathrm{d}y\right]^{\frac{p-1}{p}}\\
&= o_n(1) + o_R(1), \quad \text{as} \quad n\to\infty, \quad R\to\infty,
\end{aligned}
$$

which implies

$$
\int_{\mathbb{R}^N}\ln(1+|y|)|u(y)|^{p-2}|u(y)||u_n(y)-u(y)|\mathrm{d}y = o(1).
$$

By $\|u_n-u\|_p \to 0$, we have

$$
\begin{aligned}
&\left|V_1(|u_n|^p, |u|^{p-2}|u||u_n-u|)\right|\\
&\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}[\ln(1+|x|)+\ln(1+|y|)]|u_n(x)|^p|u(y)|^{p-2}|u(y)||u_n(y)-u(y)|\mathrm{d}x\mathrm{d}y\\
&\leq \|u_n\|_*^p\|u\|_p^{p-1}\|u_n-u\|_p + \|u_n\|_p^p\int_{\mathbb{R}^N}\ln(1+|y|)|u(y)|^{p-2}|u(y)||u_n(y)-u(y)|\mathrm{d}y\\
&= o_n(1).
\end{aligned}
$$

Then by Fatou's lemma and the Simon inequality, we know that

$$
\begin{aligned}
&A_1(|u_n|^p,|u_n|^p) - A_1(|u|^p,|u|^p)\\
&= A_1(|u_n|^p,|u_n|^p) - A_1(|u_n|^p,|u|^{p-2}u_n u) - A_1(|u_n|^p,|u_n|^{p-2}u_n u) + A_1(|u_n|^p,|u|^p)
\end{aligned}
$$

$$
\begin{aligned}
&+ A_1(|u_n|^p, |u|^{p-2}u_n u) + A_1(|u_n|^p, |u_n|^{p-2}u_n u) - A_1(|u_n|^p, |u|^p) - A_1(|u|^p, |u|^p) \\
&\geq k_p A_1(|u_n|, |u_n - u|^p) + A_1(|u_n|^p, |u|^{p-2}u(u_n - u)) + A_1(|u_n|^p, |u_n|^{p-2}u_n u) - A_1(|u|^p, |u|^p) \\
&\geq \frac{k_p}{2^N}\|u_n\|_p^p\|u_n - u\|_*^p + A_1(|u_n|^p, |u_n|^{p-2}u_n u) - A_1(|u|^p, |u|^p) + o_n(1) \\
&\geq \frac{k_p}{2^N}\|u_n\|_p^p\|u_n - u\|_*^p + o_n(1).
\end{aligned}
$$

By (10) and a simple calculation, we know that

$$A_2(|u_n|^p, |u_n|^p) - A_2(|u|^p, |u|^p) = o_n(1).$$

From (22), and the Brezis-Lieb Lemma, we then derive that

$$
\begin{aligned}
m + o(1) \geq c_* + o(1) = I(u_n)& \\
&= \frac{1}{p}\|u_n\|^p + \frac{1}{2p}[V_1(u_n) - V_2(u_n)] - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y, u_n(y))}{|y|^\beta|x-y|^\mu}\right)\frac{F(x, u_n(x))}{|x|^\beta}\mathrm{d}x \\
&\geq \frac{1}{p}\|u\|^p + \frac{1}{2p}\left[V_1(u) - V_2(u)\right] - \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\frac{F(y, u(y))}{|y|^\beta|x-y|^\mu}\right)\frac{F(x, u(x))}{|x|^\beta}\mathrm{d}x \\
&\quad + \frac{1}{p}\|u_n - u\|^p + \frac{1}{2p}[A_1(|u_n|^p, |u_n|^p) - A_1(|u|^p, |u|^p)] \\
&\quad + \frac{1}{2p}[A_2(|u|^p, |u|^p) - A_2(|u_n|^p, |u_n|^p)] + o(1) \\
&\geq I(u) + \frac{1}{p}\|u_n - u\|^p + \frac{k_p}{2^{N+1}p}\|u_n\|_p^p\|u_n - u\|_*^p + o(1) \\
&= m + \frac{1}{p}\|u_n - u\|^p + \frac{k_p}{2^{N+1}p}\|u_n\|_p^p\|u_n - u\|_*^p + o(1),
\end{aligned}
$$

which implies that $u_n \to u$ in E. Hence, $I(u) = m$ and $I'(u) = 0$. □

Declaration of competing interest

The authors declare that they have no conflict of interest.

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References

[1] S. Adachi, K. Tanaka, Trudinger type inequalities in $\mathbb{R}^N$ and their best exponents, Proc. Am. Math. Soc. 128 (2000) 2051–2057.
[2] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, Elsevier Science, Oxford, 2014.
[3] Adimurthi, K. Sandeep, A singular Moser-Trudinger embedding and its applications, Nonlinear Differ. Equ. Appl. 13 (2007) 585–603.
[4] F. Albuquerque, J. Carvalho, G. Figueiredo, E. Medeiros, On a planar non-autonomous Schrödinger-Poisson system involving exponential critical growth, Calc. Var. Partial Differ. Equ. 60 (2021) 40.
[5] C.O. Alves, G.M. Figueiredo, On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$, J. Differ. Equ. 246 (2009) 1288–1311.
[6] C.O. Alves, G.M. Figueiredo, Existence of positive solution for a planar Schrödinger-Poisson system with exponential growth, J. Math. Phys. 60 (2019) 011503.
[7] C.O. Alves, L.J. Shen, Critical Schrödinger equations with Stein-Weiss convolution parts in $\mathbb{R}^2$, J. Differ. Equ. 344 (2023) 352–404.
[8] R. Arora, J. Giacomoni, T. Mukherjee, K. Sreenadh, Polyharmonic Kirchhoff problems involving exponential non-linearity of Choquard type with singular weights, Nonlinear Anal. 196 (2020) 111779.
[9] R. Biswas, S. Goyal, K. Sreenadh, Quasilinear Schrödinger equations with Stein-Weiss type convolution and critical exponential nonlinearity in $\mathbb{R}^N$, arXiv:2202.07611, 2022.
[10] E. de, S. Böer, H. Miyagaki, Existence and multiplicity of solutions for the fractional p-Laplacian Choquard logarithmic equation involving a nonlinearity with exponential critical and subcritical growth, J. Math. Phys. 62 (5) (2021) 051507.
[11] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Universitext, Springer, New York, 2011.
[12] D.M. Cao, W. Dai, Y. Zhang, Existence and symmetry of solutions to 2-D Schrödinger-Newton equations, Dyn. Partial Differ. Equ. 18 (2021) 113–156.
[13] S.T. Chen, J.P. Shi, X.H. Tang, Ground state solutions of Nehari-Pohožaev type for the planar Schrödinger-Poisson system with general nonlinearity, Discrete Contin. Dyn. Syst., Ser. A 39 (2019) 5867–5889.
[14] S.T. Chen, X.H. Tang, Existence of ground state solutions for the planar axially symmetric Schrödinger-Poisson system, Discrete Contin. Dyn. Syst., Ser. B 24 (2019) 4685–4702.
[15] S.T. Chen, X.H. Tang, Axially symmetric solutions for the planar Schrödinger-Poisson system with critical exponential growth, J. Differ. Equ. 269 (2020) 9144–9174.
[16] S.T. Chen, X.H. Tang, On the planar Schrödinger-Poisson system with the axially symmetric potential, J. Differ. Equ. 268 (2020) 945–976.
[17] S. Cingolani, T. Weth, On the planar Schrödinger-Poisson system, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 33 (2016) 169–197.
[18] L.L. Du, F.S. Gao, M.B. Yang, On elliptic equations with Stein-Weiss type convolution parts, Math. Z. 301 (2022) 2185–2225.
[19] M. Du, T. Weth, Ground states and high energy solutions of the planar Schrödinger-Poisson system, Nonlinearity 30 (2017) 3492–3515.
[20] J.M. do Ó, N-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. 2 (1997) 301–315.
[21] G. Franzina, G. Palatucci, Fractional p-eigenvalues, Riv. Mat. Univ. Parma 5 (2014) 315–328.
[22] R.L. de Freitas, Multiplicity of solutions for a class of quasilinear equations with exponential critical growth, Nonlinear Anal. 95 (2014) 607–624.
[23] G.H. Hardy, J.E. Littlewood, Some properties of fractional integrals. I, Math. Z. 27 (1928) 565–606.
[24] A. Iannizzotto, S. Liu, K. Perera, M. Squassina, Existence results for fractional p-Laplacian problems via Morse theory, Adv. Calc. Var. 9 (2016) 101–125.
[25] H. Kozono, T. Sato, H. Wadade, Upper bound of the best constant of a Trudinger-Moser inequality and its application to a Gagliardo-Nirenberg inequality, Indiana Univ. Math. J. 55 (2006) 1951–1974.
[26] N. Lam, G.Z. Lu, Existence and multiplicity of solutions to equations of N-Laplacian type with critical exponential growth in $\mathbb{R}^N$, J. Funct. Anal. 262 (2012) 1132–1165.
[27] N. Lam, G.Z. Lu, Existence of nontrivial solutions to polyharmonic equations with subcritical and critical exponential growth, Discrete Contin. Dyn. Syst. 32 (2012) 2187–2205.
[28] N. Lam, G.Z. Lu, Elliptic equations and systems with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition, J. Geom. Anal. 24 (2014) 118–143.
[29] E.H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. Math. 118 (1983) 349–374.
[30] E.H. Lieb, M. Loss, Analysis, Graduate Studies in Mathematics, AMS, Providence, Rhode Island, 2001.
[31] E. Lindgren, P. Lindqvist, Fractional eigenvalues, Calc. Var. Partial Differ. Equ. 49 (2014) 795–826.
[32] G. Molica Bisci, V.D. Rădulescu, R. Servadei, Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications, vol. 162, Cambridge University Press, Cambridge, 2016.
[33] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1970/1971) 1077–1092.
[34] L. Martinazzi, Fractional Adams-Moser-Trudinger type inequalities, Nonlinear Anal. 127 (2015) 263–278.
[35] T.V. Nguyen, Existence of solution to singular Schrödinger systems involving the fractional p-Laplacian with Trudinger-Moser nonlinearity in $\mathbb{R}^N$, Math. Methods Appl. Sci. 44 (2021) 6540–6570.
[36] T. Ozawa, On critical cases of Sobolev's inequalities, J. Funct. Anal. 127 (1995) 259–269.
[37] E. Parini, B. Ruf, On the Moser-Trudinger inequality in fractional Sobolev-Slobodeckij spaces, Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl. 29 (2018) 315–319.
[38] J. Peetre, Espaces d'interpolation et théorème de Soboleff, Ann. Inst. Fourier (Grenoble) 16 (1966) 279–317.

[39] R.C. Pei, Fractional p-Laplacian equations with subcritical and critical exponential growth without the Ambrosetti-Rabinowitz condition, Mediterr. J. Math. 15 (2) (2018) 66.

[40] D.D. Qin, X.H. Tang, On the planar Choquard equation with indefinite potential and critical exponential growth, J. Differ. Equ. 285 (2021) 40–98.

[41] S.L. Sobolev, On a theorem of functional analysis, Mat. Sb. (N.S.) 4 (1938) 471–479, English transl. in Transl. Am. Math. Soc. 34 (1963) 39–68.

[42] E. Stein, G. Weiss, Fractional integrals on n-dimensional Euclidean space, J. Math. Mech. 7 (1958) 503–514.

[43] W. Strauss, Existence of solitary waves in higher dimensions, Commun. Math. Phys. 55 (1977) 149–162.

[44] J. Stubbe, Bound states of two-dimensional Schrödinger-Newton equations, arXiv:0807.4059, 2008.

[45] W.B. Wang, Q.Q. Li, Y.K. Li, The sign-changing solutions and ground states for planar Schrödinger-Newton system with an exponential critical growth, J. Math. Phys. 61 (2020) 101513.

[46] M. Willem, Minimax Theorems, Progress in Nonlinear Differential Equations and Their Applications, vol. 24, Birkhäuser Boston Inc., Boston, MA, 1996.

[47] M.Q. Xiang, B.L. Zhang, V.D. Rădulescu, Existence of solutions for perturbed fractional p-Laplacian equations, J. Differ. Equ. 260 (2016) 1392–1413.

[48] M.Q. Xiang, B.L. Zhang, V.D. Rădulescu, Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent, Adv. Nonlinear Anal. 9 (2020) 690–709.

[49] M.B. Yang, V.D. Rădulescu, X.M. Zhou, Critical Stein-Weiss elliptic systems: symmetry, regularity and asymptotic properties of solutions, Calc. Var. Partial Differ. Equ. 61 (2022) 109.

[50] M.B. Yang, X.M. Zhou, On a coupled Schrödinger system with Stein-Weiss type convolution part, J. Geom. Anal. 31 (2021) 10263–10303.

[51] C. Zhang, Trudinger-Moser inequalities in fractional Sobolev-Slobodeckij spaces and multiplicity of weak solutions to the fractional-Laplacian equation, Adv. Nonlinear Stud. 19 (2019) 197–217.

[52] L.M. Zhang, X.H. Tang, P. Chen, On the planar Kirchhoff-type problem involving supercritical exponential growth, Adv. Nonlinear Anal. 11 (2022) 1412–1446.

[53] Y.P. Zhang, X.H. Tang, Large perturbations of a magnetic system with Stein-Weiss convolution nonlinearity, J. Geom. Anal. 32 (3) (2022) 102.

[54] Y.P. Zhang, X.H. Tang, V.D. Rădulescu, Anisotropic Choquard problems with Stein-Weiss potential: nonlinear patterns and stationary waves, C. R. Math. Acad. Sci. Paris 359 (2021) 959–968.