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STANDING WAVES WITH PRESCRIBED NORM FOR THE COUPLED HARTREE-FOCK SYSTEM

Xiao Luo[1], Vicenţiu D. Rădulescu[*2,3,4,5,6] and Maoding Zhen[1]

[1]School of Mathematics, Hefei University of Technology, Hefei 230009, China

[2]Faculty of Applied Mathematics
AGH University of Science and Technology, 30-059 Kraków, Poland

[3]Brno University of Technology, Faculty of Electrical Engineering and Communication
Technická 3058/10, Brno 61600, Czech Republic

[4]School of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

[5]Simion Stoilow Institute of Mathematics of the Romanian Academy
010702 Bucharest, Romania

[6]Department of Mathematics, University of Craiova, 200585 Craiova, Romania

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Abstract. In this paper, we develop an exhaustive analysis on standing waves with prescribed mass for the coupled Hartree-Fock system, which is introduced by Hartree in the 1920's and developed by Fock for describing large systems of identical fermions. By transforming this problem into different types of constrained variational problems based on the width of the Hartree kernel, we establish several existence, multiplicity and asymptotic properties under suitable conditions on the corresponding physical parameters.

1. Introduction and main results.

In this paper, we study the following coupled Hartree-Fock system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1(|x|^{-\alpha} * |u|^2)u + \rho(|x|^{-\beta} * |v|^2)u & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2(|x|^{-\alpha} * |v|^2)v + \rho(|x|^{-\beta} * |u|^2)v & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

under the additional conditions

$$\int_{\mathbb{R}^N} u^2 dx = b_1^2 \text{ and } \int_{\mathbb{R}^N} v^2 dx = b_2^2. \tag{2}$$

Here, $b_1, b_2 > 0$ are prescribed, μ_1, μ_2, $\rho > 0$ and the frequencies λ_1, λ_2 are unknown and will appear as Lagrange multipliers.

Problem (1) comes from the research of standing waves for the following nonlinear Hartree-Fock model of two particles in $\mathbb{R}^+ \times \mathbb{R}^N$

$$\begin{cases} -i\partial_t \varphi_1 = \Delta\varphi_1 + \mu_1(|x|^{-\alpha} * |\varphi_1|^2)\varphi_1 + \rho(|x|^{-\beta} * |\varphi_2|^2)\varphi_1, \\ -i\partial_t \varphi_2 = \Delta\varphi_2 + \mu_2(|x|^{-\alpha} * |\varphi_2|^2)\varphi_2 + \rho(|x|^{-\beta} * |\varphi_1|^2)\varphi_2, \end{cases} \tag{3}$$

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*Corresponding author: Vicenţiu D. Rădulescu.

where i is the imaginary unit, parameters $\mu_i(i = 1, 2)$ and ρ describe the self-interaction between charged particles as a repulsive force if positive, and an attractive force if negative. Physically, $|x|^{-\alpha}$ and $|x|^{-\beta}$ are the nonlocal response function which possess information on the mutual interaction between the particles. Their widths determine the degree of nonlocality, see [36]. (3) appears in several physical models, such as in the nonlinear optics [37] and in the study of a two-component Bose-Einstein Condensate [13]. The rigorous time-dependent Hartree-Fock theory has been developed first by Chadam and Glassey [9] for (3) with $\alpha = \beta$.

Recently, when $\alpha = \beta = 1, N = 3$, J. Wang and J. Shi [48] proved the existence and non-existence of positive ground state for (1) under optimal conditions along with various qualitative properties of ground states. The uniqueness of the positive solution or the positive ground state solution are also obtained in some special cases. When $\alpha = \beta = 4,\ \lambda_1 = V_1(x),\ \lambda_2 = V_2(x)$, F. Gao *et. al.* [15] studied the high energy positive solutions for (1). By using moving sphere arguments in integral form, they gave a complete classification of the positive solutions and proved the uniqueness of positive solutions up to translation and dilation when $\lambda_1 = \lambda_2 = 0$. Then using the uniqueness property, they established a nonlocal version of the global compactness lemma and proved the existence of a high energy positive solution for the system under suitable conditions.

When $N = 3$, $\alpha = \beta = 1$, or $N = 5$, $\alpha = \beta = 3$, by using a two-dimensional linking argument, Wang [47] proved the existence of a normalized solution for $0 < \rho < \rho_1$. D. Cao, H. Jia and X. Luo [8] studied standing waves with prescribed mass for the Schrödinger equations with van der Waals type potentials, under different assumptions, they proved several existence, multiplicity and asymptotic behavior of solutions and the stability of the corresponding standing waves for the related time-dependent problem was discussed. The case of standing waves for the pseudo-relativistic Hartree equation with Berestycki-Lions nonlinearity was studied by F. Gao, V. Rădulescu, M Yang and Y. Zheng [16]. Since it seems almost impossible for us to provided a complete list of references, we refer the readers only to [6, 10, 11, 12, 33, 34, 39, 41, 43, 44, 22, 49, 26, 27, 1, 2, 4, 3] and reference therein. Inspired by the analysis developed in [27, 26, 8], we focus in the present paper with qualitative analysis of standing waves with prescribed norm for the coupled Hartree-Fock system. Clearly, the energy functional associated with problem (1)–(2) is given by

$$
\begin{aligned}
J(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\
&- \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy
\end{aligned}
$$

on the constraint $\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, where for $b > 0$ we define

$$
\mathrm{T}_b := \left\{ u \in D^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^2 = b^2 \right\}.
$$

Firstly, we consider the case of $0 < \alpha, \beta < 2$, in which the energy functional is bounded from blow on the product of L^2-spheres, we define

$$
m(b_1, b_2) = \inf_{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}} J(u,v) \tag{4}
$$

and then we prove the following result.

Theorem 1.1. *When $0 < \alpha, \beta < 2$, ρ, $\mu_i > 0 (i = 1, 2)$, then $-\infty < m_\beta(b_1, b_2) < 0$ is achieved. In addition any minimizing sequence for (4) is, up to translation, strongly convergent in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to a solution of (1)-(2).*

Next, we turn to the case of $0 < \alpha < \beta = 2$. In this case, the energy functional is not always bounded on $\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, relating to the values b_1 and b_2. We prove the existence of normalized solutions to (1). From (1.6) of [8], for any $\xi \in (0, \min\{N, 4\})$,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\xi} dxdy \le \frac{2}{\|Q_\xi\|^2_{L^2(\mathbb{R}^N)}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{\xi}{2}} \left(\int_{\mathbb{R}^N} |u|^2 dx\right)^{\frac{4-\xi}{2}} \quad (5)$$

where equality holds for $u = Q_\xi$, and Q_ξ is a nontrivial solution of

$$-\frac{\xi}{2}\Delta Q_\xi + \frac{4-\xi}{2} Q_\xi = (|x|^{-\xi} * |Q_\xi|^2) Q_\xi, \quad x \in \mathbb{R}^N.$$

The main results on this aspect can be stated as follows.

Theorem 1.2. *When $0 < \alpha < \beta = 2$ and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, for any $\mu_i > 0 (i = 1, 2)$, then $0 > m(b_1, b_2) > -\infty$ is achieved. In addition, any minimizing sequence for (4) is, up to translation, strongly convergent in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to a solution of (1)-(2).*

Remark 1.3. When $0 < \beta < \alpha = 2$, the energy functional is bounded from below on the product of L^2-spheres under suitable conditions, therefore we expect the constrained minimization method developed by L. Jeanjean [18] can be used to obtained a normalized ground state. However, when $0 < \beta < \alpha = 2$, Theorem 1.1 in [52] indicates that there exists $b_1^* = \|Q_2\|_{L^2(\mathbb{R}^N)}$ such that when $b_1 > b_1^*$ then $m(b_1, 0) = -\infty$, when $0 < b_1 \le b_1^*$, then $m(b_1, 0) = 0$ and the single equation has no ground normalized solution if $b_1 \ne b_1^*$. To establish the compactness of the minimizing sequences, $m(b_1, 0) < 0$ and the single equation has ground normalized solution play crucial role. So, it seems difficult to prove the compactness of the minimizing sequences (or Palais-Smale sequences). We believe that when $0 < \beta < \alpha = 2$, the existence of normalized solutions to (1) is an expected interesting result.

Now, we deal with the case of $0 < \alpha < 2 < \beta < \min\{N, 4\}$ or $0 < \beta < 2 < \alpha < \min\{N, 4\}$. In this case, $m(b_1, b_2) = -\infty$ for any $b_1, b_2 > 0$ and the energy function $J(u, v)$ is unbounded both from above and from below on $\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$. In order to search two normalized solutions, we use the ideas introduced by N. Soave [44, 43] to study the corresponding fibering maps $\Psi_{u,v}(t)$ (see (11)), which has the same Mountain pass structure as the original functional. To show our main results, we give following conditions

$$\begin{aligned}&\left(\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}\right) \rho^{\frac{2-\alpha}{\beta-2}} b_1^{\frac{(4-\beta)(2-\alpha)}{2(\beta-2)}} b_2^{\frac{(4-\beta)(2-\alpha)}{2(\beta-2)}} \\ &< \frac{\beta-2}{\beta-\alpha}\left(\frac{2-\alpha}{2(\beta-\alpha)}\right)^{\frac{2-\alpha}{\beta-2}} \|Q_\beta\|^{\frac{2(2-\alpha)}{\beta-2}}_{L^2(\mathbb{R}^N)} \|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}.\end{aligned} \quad (6)$$

$$\rho\left(\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}\right)^{\frac{\beta-2}{2-\alpha}} b_1^{\frac{4-\beta}{2}} b_2^{\frac{4-\beta}{2}} < \frac{\alpha-2}{\alpha-\beta}\left(\frac{2-\beta}{2(\alpha-\beta)}\right)^{\frac{2-\beta}{\alpha-2}} \|Q_\beta\|^2_{L^2(\mathbb{R}^N)} \|Q_\alpha\|^{\frac{2(2-\beta)}{\alpha-2}}_{L^2(\mathbb{R}^N)}. \quad (7)$$

Theorem 1.4. *When $\rho, \mu_i > 0 (i = 1, 2)$, $0 < \alpha < 2 < \beta < \min\{N, 4\}$ and (6) holds, or $0 < \beta < 2 < \alpha < \min\{N, 4\}$ and (7) holds, then (1)-(2) has at least two positive normalized solutions, one is a ground state $(\widehat{u}_{\mu_1}, \widehat{v}_{\mu_2})$, the other is an excited* *state* $(\widetilde{u}_{\mu_1}, \widetilde{v}_{\mu_2})$. *Moreover,* $J(\widehat{u}_{\mu_1}, \widehat{v}_{\mu_2}) \to 0^+$, $\int_{\mathbb{R}^N}(|\nabla \widehat{u}_{\mu_1}|^2 + |\nabla \widehat{u}_{\mu_2}|^2)dx \to 0$ *as* $\mu_1,\ \mu_2 \to 0^+$, *and* $(\widetilde{u}_{\mu_1}, \widetilde{v}_{\mu_2}) \to (\widetilde{u}, \widetilde{v})$ *strongly in* $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ *as* $\mu_1,\ \mu_2 \to 0^+$, *where* $(\widetilde{u}, \widetilde{v})$ *is a positive radial ground state solution of following problem*

$$\begin{cases} -\Delta u + \lambda_1 u = \rho(|x|^{-\beta} * |v|^2)u & in\ \ \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \rho(|x|^{-\beta} * |u|^2)v & in\ \ \mathbb{R}^N, \end{cases}$$

satisfying the additional conditions

$$\int_{\mathbb{R}^N} u^2 dx = b_1^2 \ \ and \ \int_{\mathbb{R}^N} v^2 dx = b_2^2.$$

Remark 1.5. Theorem 1.4 indicates that when the lower power perturbation terms are mass subcritical with a product of the perturbation coefficients and masses being controlled from above, problem (1)-(2) possesses at least two normalized solutions, one ground state and one excited state (whose energy is strictly larger than that of ground state). Furthermore, the first solution will disappear and the second solution will converge to the normalized solution of system (1) with $\mu_1 = \mu_2 = 0$.

Furthermore, we give a mass collapse behavior of the ground states obtained in Theorem 1.4.

Theorem 1.6. *Assume that the assumptions in Theorem 1.4 hold,* $b_1, b_2 \to 0$ *with* $b_1 \sim b_2$ *and* $(u_{b_1,b_2}, v_{b_1,b_2})$ *is a ground state for* (1)-(2)*. Up to a subsequence, we have*

$$(L_1^{-1} u_{b_1,b_2}((\lambda_{1,b_1,\mu_1})^{-\frac{1}{2}}x),\ L_2^{-1} v_{b_1,b_2}((\lambda_{2,b_2,\mu_2})^{-\frac{1}{2}}x)) \to (\omega, \omega),$$

where

$$\lambda_{1,b_1,\mu_1} = \mu_1^{\frac{2}{2-\alpha}} \left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{2}{2-\alpha}},\ \ \lambda_{2,b_2,\mu_2} = \mu_2^{\frac{2}{2-\alpha}} \left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{2}{2-\alpha}},$$

$$L_1 = \mu_1^{\frac{N}{4-2\alpha}} \left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{N+2-\alpha}{4-2\alpha}},\ \ L_2 = \mu_2^{\frac{N}{4-2\alpha}} \left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{N+2-\alpha}{4-2\alpha}},$$

and ω *is a positive solution of* $-\Delta u + u = (|x|^{-\alpha} * |u|^2)u,\ u \in H^1(\mathbb{R}^N)$.

Now, we deal with the L^2-critical case and L^2-supercritical case. In this case, $m(b_1, b_2) = -\infty$ under suitable condition of b_1, b_2 and the energy function $J(u, v)$ is unbounded both from above and from below on $\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$. The main results on this aspect can be state as follows

Theorem 1.7. *When* $\rho, \mu_i (i = 1, 2) > 0$, *if one of following conditions holds:*

i) $\beta = 2 < \alpha < \min\{N, 4\}$ *and* $\rho b_1 b_2 < \|Q_2\|_{L^2(\mathbb{R}^N)}^2$,

ii) $\alpha = 2 < \beta < \min\{N, 4\}$ *and* $\mu_1 b_1 + \mu_2 b_2 < \|Q_2\|_{L^2(\mathbb{R}^N)}^2$,

iii) $2 < \alpha, \beta < \min\{N, 4\}$,

then (1)-(2) *has a positive ground state solution.*

Throughout the paper, H_r^1 denotes the subspace of functions in H^1 which are radially symmetric with respect to 0, and $\mathrm{T}_{b_{i,r}} = \mathrm{T}_{b_i} \cap H_r^1, i = 1, 2$. $a \sim b$ if $C_1 a \le b \le C_2 a$, where $C_i, i = 1, 2$ are constants. The rest of this paper is organized as follows. In section 2, we prove Theorem 1.1-Theorem 1.2. In section 3, we prove Theorem 1.4. In section 4, we prove Theorem 1.6. Finally, Theorem 1.7 will be proved in section 5.

2. **Proof of Theorem 1.1 and Theorem 1.2.** To prove Theorems 1.1-1.2, we use the ideas introduced in [17, 18]. We recall the definition and some properties of the coupled rearrangement results of M. Shibata [42] as presented in [20, 23]. Let u be a Borel measurable function on $\mathbb{R}^N$. It is said to vanish at infinity if the level set $|\{x \in \mathbb{R}^N : |u(x)| > t\}| < +\infty$ for every $t > 0$. Here $|A|$ stands for the N-dimensional Lebesgue measure of a Lebesgue measurable set $A \subset \mathbb{R}^N$. Considering two Borel mesurable functions u, v which vanish at infinity in $\mathbb{R}^N$, for $t > 0$, we define $A^\star(u, v : t) := \{x \in \mathbb{R}^N : |x| < r\}$, where $r > 0$ is chosen so that

$$B(0, r) = |\{x \in \mathbb{R}^N : |u(x)| > t\}| + |\{x \in \mathbb{R}^N : |v(x)| > t\}|,$$

and $\{u, v\}^\star$ by

$$\{u, v\}^\star(x) := \int_0^\infty \chi_{A^\star(u,v:t)}(x) dt,$$

where $\chi_A(x)$ is a characteristic function of the set $A \subset \mathbb{R}^N$.

Lemma 2.1 ([20] Lemma A.1, [23] Lemma 2.2)**.** *(1) The function $\{u, v\}^\star$ is radially symmetric, non-increasing and lower semi-continuous. Moreover, for each $t > 0$ there holds $\{x \in \mathbb{R}^N : \{u, v\}^\star > t\} = A^\star(u, v : t)$.*

(2) Let $\Phi : [0, \infty) \to [0, \infty)$ be non-decreasing lower semi-continuous, continuous at 0 and $\Phi(0) = 0$. Then $\{\Phi(u), \Phi(v)\}^\star = \Phi(\{u, v\}^\star)$.

(3) $\|\{u, v\}^\star\|^p_{L^p(\mathbb{R}^N)} = \|u\|^p_{L^p(\mathbb{R}^N)} + \|v\|^p_{L^p(\mathbb{R}^N)}$ for $1 \le p < \infty$.

(4) If $u, v \in H^1(\mathbb{R}^N)$, then $\{u, v\}^\star \in H^1(\mathbb{R}^N)$ and

$$\|\nabla\{u, v\}^\star\|^2_{L^2(\mathbb{R}^N)} \le \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v\|^2_{L^2(\mathbb{R}^N)}.$$

In addition, if $u, v \in (H^1(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)) \setminus \{0\}$ are radially symmetric, positive and non-increasing, then

$$\int_{\mathbb{R}^N} |\nabla\{u, v\}^\star|^2 dx < \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |\nabla v|^2 dx.$$

(5)

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\{u, v\}^\star)^2(x)(\{u, v\}^\star)^2(y)}{|x-y|^\alpha} dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy.
\end{aligned}$$

(6) Let $u_1, u_2, v_1, v_2 \ge 0$ be Borel measurable functions which vanish at infinity, then

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u_1^2(x)u_2^2(y)}{|x-y|^\alpha} dxdy + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v_1^2(x)v_2^2(y)}{|x-y|^\alpha} dxdy \\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\{u_1, v_1\}^\star)^2(x)(\{u_2, v_2\}^\star)^2(y)}{|x-y|^\alpha} dxdy.
\end{aligned}$$

Firstly, we show that $m(b_1, b_2) < 0$.

Lemma 2.2. *When $0 < \alpha, \beta < 2$, for any $\mu_i > 0 (i = 1, 2)$, $\rho > 0$, we have*

$$m(b_1, b_2) = \inf_{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}} J(u, v) < 0.$$

Proof. For any $(u,v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, we define $t \star u = e^{\frac{Nt}{2}} u(e^t x)$ and $t \star (u,v) = (t \star u, t \star v)$, it is easy to see that $(t \star u, t \star v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$ and

$$\begin{aligned} J(t \star (u,v)) = & \frac{e^{2t}}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1 e^{\alpha t}}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x) u^2(y)}{|x-y|^\alpha} dxdy \\ & - \frac{\mu_2 e^{\alpha t}}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x) v^2(y)}{|x-y|^\alpha} dxdy - \frac{\rho e^{\beta t}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x) v^2(y)}{|x-y|^\beta} dxdy. \end{aligned}$$

Since $0 < \alpha, \beta < 2$, we see that for some $t \ll -1$, $J(t \star (u,v)) < 0$, so $m(b_1, b_2) < 0$. □

By the similar argument as Lemma 2.4 in [26] can also see Lemma 3.1 in [17], we have following lemma.

Lemma 2.3. *(i) If (d_1^n, d_2^n) is such that $(d_1^n, d_2^n) \to (d_1, d_2)$ as $n \to +\infty$ with $0 \le d_i^n \le b_i$ for $i = 1, 2$, we have $m(d_1^n, d_2^n) \to m(d_1, d_2)$ as $n \to +\infty$.*
(ii) Let $d_i \ge 0$, $b_i \ge 0, i = 1, 2$ such that $b_1^2 + d_1^2 = c_1^2$, $b_2^2 + d_2^2 = c_2^2$, then $m(b_1, b_2) + m(d_1, d_2) \ge m(c_1, c_2)$.

When we obtain Lemma 2.1, by the very similar argument as Lemma 2.5 in [26], we have following lemma.

Lemma 2.4. *Let $(u_n, v_n) \subset H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be a sequence such that*

$$J(u_n, v_n) \to m(b_1, b_2) \text{ and } \int_{\mathbb{R}^N} u_n^2 dx = b_1^2, \ \int_{\mathbb{R}^N} v_n^2 dx = b_2^2,$$

then $\{(u_n, v_n)\}$ is relatively compact in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ up to translations, that is there exists a subsequence (u_{n_k}, v_{n_k}), a sequence of points $\{y_k\} \subset \mathbb{R}^N$ and a function $(\widetilde{u}, \widetilde{v}) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that $(u_{n_k}(\cdot + y_k), v_{n_k}(\cdot + y_k)) \to (\widetilde{u}, \widetilde{v})$ strongly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

To deal with the coupling terms, we recall the following classical Hardy-Littlewood-Sobolev inequality (see Lemma 2.1 of [46]).

Lemma 2.5. *Assume that $f \in L^p(\mathbb{R}^N)$ and $g \in L^p(\mathbb{R}^N)$. Then one has*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x) g(y)}{|x-y|^t} dxdy \le c(p,q,t) \|f\|_{L^p(\mathbb{R}^N)} \|g\|_{L^q(\mathbb{R}^N)},$$

where $1 < p, q < \infty$, $0 < t < N$ and $\frac{1}{p} + \frac{1}{q} + \frac{t}{N} = 2$.

By the same argument as Lemma 2.3 in [46], we have following lemma.

Lemma 2.6. *For $u, v \in H^1(\mathbb{R}^N)$ and $0 < \xi < N$, we have that*

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x) v^2(y)}{|x-y|^\xi} dxdy \le \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x) u^2(y)}{|x-y|^\xi} dxdy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x) v^2(y)}{|x-y|^\xi} dxdy \right)^{\frac{1}{2}}.$$

From (5) and Lemma 2.6, when $\rho > 0$, we can deduce that

$$\begin{aligned} J(u,v) = & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x) u^2(y)}{|x-y|^\alpha} dxdy \\ & - \frac{\mu_2}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x) v^2(y)}{|x-y|^\alpha} dxdy - \frac{\rho}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x) v^2(y)}{|x-y|^\beta} dxdy \\ \ge & \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx - \frac{\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}}{2 \|Q_\alpha\|_{L^2(\mathbb{R}^N)}^2} \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2) dx \right)^{\frac{\alpha}{2}} \end{aligned}$$

$$-\frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}}.$$

Thus, if $0<\alpha<\beta=2$ and $\rho b_1 b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$, then $J(u,v)$ is coercive and bounded from below.

Lemma 2.7. *When $0<\alpha<\beta=2$ and $\rho b_1b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$, for any $\mu_i>0(i=1,2),\rho>0$, we have*

$$m(b_1,b_2)=\inf_{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}}J(u,v)<0.$$

Proof. For any $(u,v)\in \mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$, it is easy to see that $(t\star u,t\star v)\in \mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$ and

$$\begin{aligned}J(t\star(u,v))=&\frac{e^{2t}}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy\\&-\frac{\mu_2e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho e^{\beta t}}{2}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy.\end{aligned}$$

When $0<\alpha<\beta=2$, we see that for some $t\ll -1$, $J(t\star(u,v))<0$, so $m(b_1,b_2)<0$. □

P*roof of Theorem 1.1.* Let $\{w_n,\sigma_n\}$ be any minimizing sequence for the functional J on $\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. By Lemma 2.4, we know that there $\{y_n\}\subset\mathbb{R}^N$ such that $(w_n,\sigma_n)\rightharpoonup(w,\sigma)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ and $(w_n,\sigma_n)\to(w,\sigma)$ in $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$ for $2<p<2^*$. Hence, by the weak lower semi-continuity of the norm, we have $J(w,\sigma)\le m(b_1,b_2)<0$, which implies that $(w,\sigma)\neq(0,0)$. To show the compactness of $\{w_n,\sigma_n\}$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, it suffices to prove that $(w,\sigma)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. If $\int_{\mathbb{R}^N}|w|^2dx=b_1^2$ and $\int_{\mathbb{R}^N}|\sigma|^2dx=b_2^2$, we are done. Assume by contradiction that there exists $\bar b_1<b_1$ and $\bar b_2<b_2$ such that $\int_{\mathbb{R}^N}|w|^2dx=\bar b_1^2<b_1^2$, or $\int_{\mathbb{R}^N}|\sigma|^2dx=\bar b_2^2<b_2^2$. Then, by the definition of $m(b_1,b_2)$, we have $m(\bar b_1,\bar b_2)\le J(w,\sigma)$. By the same argument as Lemma 2.2, we have $m(b_1,0)<0($ resp. $m(0,b_2)<0)$ for any $b_1(b_2)>0$. From [42], we know that $m(b_1,0)($ resp. $m(0,b_2))$ can be achieved by some function $u\in T_{b_1}(v\in T_{b_2})$ which are real valued, positive, radially symmetric and radially decreasing.

If $\int_{\mathbb{R}^N}|w|^2dx=\bar b_1^2<b_1^2$ and $\int_{\mathbb{R}^N}|\sigma|^2dx=\bar b_2^2<b_2^2$. By Lemma 2.2 and (ii) of Lemma 2.3, we have $J(w,\sigma)\le m(b_1,b_2)\le m(\bar b_1,\bar b_2)+m(\sqrt{b_1^2-\bar b_1^2},\sqrt{b_2^2-\bar b_2^2})<m(\bar b_1,\bar b_2)\le J(w,\sigma)$, a contradiction.

If $\int_{\mathbb{R}^N}|w|^2dx=b_1^2$ and $\int_{\mathbb{R}^N}|\sigma|^2dx=\bar b_2^2<b_2^2$. By Lemma 2.2 and (ii) of Lemma 2.3 and $m(0,\sqrt{b_2^2-\bar b_2^2})<0$, we have $J_\beta(w,\sigma)\le m(b_1,b_2)\le m(b_1,\bar b_2)+m(0,\sqrt{b_2^2-\bar b_2^2})<m(b_1,\bar b_2)\le J(w,\sigma)$, a contradiction.

If $\int_{\mathbb{R}^N}|w|^2dx=\bar b_1^2<b_1^2$ and $\int_{\mathbb{R}^N}|\sigma|^2dx=b_2^2$. By Lemma 2.2 and (ii) of Lemma 2.3 and $m(\sqrt{b_1^2-\bar b_1^2},0)<0$, we have $J(w,\sigma)\le m(b_1,b_2)\le m(\bar b_1,b_2)+m(\sqrt{b_1^2-\bar b_1^2},0)<m(\bar b_1,b_2)\le J(w,\sigma)$, a contradiction.

Therefore, $(w,\sigma)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. □

P*roof of Theorem 1.2.* The proof is similar as the proof of Lemma 2.4 and the proof of Theorem 1.1. Indeed, when $0<\alpha<\beta=2$, for the single equation, the problem is L^2 subcritical, for any $b_1, b_2 > 0$, we have

$$m(b_1, 0) < 0 \text{ and } m(0, b_2) < 0.$$

Moreover, $m(b_1, 0)$, $m(0, b_2)$ can be achieved by some function $u \in T_{b_1}$ and $v \in T_{b_2}$, which is real valued, positive, radially symmetric and radially decreasing. □

3. **Proof of Theorem 1.4.** Define the set

$$\mathcal{P}_{b_1,b_2} := \left\{(u,v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : P_{b_1,b_2}(u,v) = 0\right\},$$

where

$$P_{b_1,b_2}(u,v) = \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1 \alpha}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy$$
$$- \frac{\mu_2 \alpha}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy - \frac{\rho\beta}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta} dxdy.$$

The following auxiliary result shows the role of $\mathcal{P}_{b_1,b_2}$.

Lemma 3.1. *If (u,v) is a solution of problem (1)–(2) for some $\lambda_1, \lambda_2 \in \mathbb{R}$, then $(u,v) \in \mathcal{P}_{b_1,b_2}$.*

Proof. It is easy to obtain that

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx + \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)dx \tag{8}$$
$$= \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy + \mu_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy$$
$$+ 2\rho \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta} dxdy.$$

By the similar argument as Lemma 2.4 in [50], we can get

$$\frac{N-2}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx + \frac{N}{2} \int_{\mathbb{R}^N} (\lambda_1 u^2 + \lambda_2 v^2)dx \tag{9}$$
$$= \frac{(2N-\alpha)}{4} \mu_1 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy + \frac{(2N-\alpha)}{4} \mu_2 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy$$
$$+ \frac{(2N-\beta)}{2} \rho \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta} dxdy.$$

By (8) and (9), we have

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1 \alpha}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy \tag{10}$$
$$- \frac{\mu_2 \alpha}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy - \frac{\rho\beta}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta} dxdy = 0.$$

□

Let $t \star u = e^{\frac{Nt}{2}} u(e^t x)$, then $t \star u \in \mathrm{T}_b$, we define $t \star (u,v) = (t \star u, t \star v)$, then

$$\Psi_{u,v}(t) = J(t \star (u,v)) = \frac{e^{2t}}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1 e^{\alpha t}}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy \tag{11}$$
$$- \frac{\mu_2 e^{\alpha t}}{4} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy - \frac{\rho e^{\beta t}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta} dxdy,$$

$$\Psi'_{u,v}(t) = e^{2t}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\mu_1\alpha e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2\alpha e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho\beta e^{\beta t}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy$$

and

$$\Psi'_{u,v}(t) = P_{b_1,b_2}(t\star u, t\star v), \quad \mathcal{P}_{b_1,b_2} = \{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2} : \Psi'_{u,v}(0)=0\}.$$

We decompose $\mathcal{P}_{b_1,b_2}$ into three disjoint unions

$$\mathcal{P}_{b_1,b_2} = \mathcal{P}^+_{b_1,b_2}\cup\mathcal{P}^0_{b_1,b_2}\cup\mathcal{P}^-_{b_1,b_2},$$

where

$$\mathcal{P}^+_{b_1,b_2} := \{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2},\ (u,v)\in\mathcal{P}_{b_1,b_2} : \Psi''_{u,v}(0)>0\},$$
$$\mathcal{P}^0_{b_1,b_2} := \{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2},\ (u,v)\in\mathcal{P}_{b_1,b_2} : \Psi''_{u,v}(0)=0\},$$
$$\mathcal{P}^-_{b_1,b_2} := \{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2},\ (u,v)\in\mathcal{P}_{b_1,b_2} : \Psi''_{u,v}(0)<0\}.$$

From (5) and Lemma 2.6, when $\rho>0$ we have

$$\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \le \frac{\mu_1}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}b_1^{4-\alpha}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{\alpha}{2}}, \tag{12}$$

$$\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy \le \frac{\mu_2}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}b_2^{4-\alpha}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{\alpha}{2}}, \tag{13}$$

$$\begin{aligned}
&\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy \qquad (14)\\
&\le \frac{\rho}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\beta}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\beta}dxdy\right)^{\frac{1}{2}}\\
&\le \frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{\beta}{4}}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{\beta}{4}}\\
&\le \frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2+\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^2\right]^{\frac{\beta}{4}}\\
&\le \frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}}.
\end{aligned}$$

From (12) and (13), we have

$$\begin{aligned}
&\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy + \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy \qquad (15)\\
&\le \frac{\mu_1}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}b_1^{4-\alpha}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{\alpha}{2}} + \frac{\mu_2}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}b_2^{4-\alpha}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{\alpha}{2}}\\
&\le \frac{\mu_1 b_1^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left[\int_{\mathbb{R}^N}|\nabla u|^2dx + \int_{\mathbb{R}^N}|\nabla v|^2dx\right]^{\frac{\alpha}{2}}
\end{aligned}$$

$$+\frac{\mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left[\int_{\mathbb{R}^N}|\nabla u|^2dx+\int_{\mathbb{R}^N}|\nabla v|^2dx\right]^{\frac{\alpha}{2}}$$
$$\le \frac{\mu_1 b_1^{4-\alpha}+\mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}.$$

Thus, from (14) and (15), if $\rho>0$, we have

$$\begin{aligned}
J(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\
&\quad -\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy \\
&\ge \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1 b_1^{4-\alpha}+\mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}} \\
&\quad -\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}} \\
&\ge h\left(\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{1}{2}}\right),
\end{aligned} \tag{16}$$

where $h(t):(0,+\infty)\to\mathbb{R}$ defined by

$$h(t)=\frac{1}{2}t^2-\mathcal{D}_1 t^\alpha-\mathcal{D}_2 t^\beta, \tag{17}$$

and

$$\mathcal{D}_1=\frac{\mu_1 b_1^{4-\alpha}+\mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}},\quad \mathcal{D}_2=\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}.$$

Lemma 3.2. *When $0<\alpha<2<\beta<\min\{N,4\}$ and condition (6) or $0<\beta<2<\alpha<\min\{N,4\}$ and (7) hold, then $h(t)$ has exactly two critical points, one is a local minimum at negative level, the other one is a global maximum at positive level. Further, there exists $0<R_0<R_1$ such that $h(R_0)=h(R_1)=0$, $h(t)>0$ if and only if $t\in(R_0,R_1)$.*

Proof. When $0<\alpha<2<\beta<\min\{N,4\}$, from $h(t)=t^\alpha\left[\frac{1}{2}t^{2-\alpha}-\mathcal{D}_2t^{\beta-\alpha}-\mathcal{D}_1\right]$, we have $h(t)>0$ if and only if $\varphi(t)>\mathcal{D}_1$, where $\varphi(t)=\frac{1}{2}t^{2-\alpha}-\mathcal{D}_2t^{\beta-\alpha}$. Since $\varphi'(t)=\frac{2-\alpha}{2}t^{1-\alpha}-(\beta-\alpha)\mathcal{D}_2t^{\beta-\alpha-1}$, we see that $\varphi(t)$ has a unique global maximum point at $\widetilde{t}=\left(\frac{2(\beta-\alpha)}{2-\alpha}\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{1}{2-\beta}}b_1^{\frac{4-\beta}{2(2-\beta)}}b_2^{\frac{4-\beta}{2(2-\beta)}}$ and

$$\varphi(\widetilde{t})=\frac{1}{2}\frac{\beta-2}{\beta-\alpha}\left(\frac{2(\beta-\alpha)}{2-\alpha}\right)^{\frac{2-\alpha}{2-\beta}}\left(\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{2-\alpha}{2-\beta}}b_1^{\frac{(4-\beta)(2-\alpha)}{2(2-\beta)}}b_2^{\frac{(4-\beta)((2-\alpha))}{2(2-\beta)}}.$$

Therefore $h(t)$ is positive on an open interval (R_0,R_1) if and only if $\varphi(\widetilde{t})>\mathcal{D}_1$, which means that

$$\frac{1}{2}\frac{\beta-2}{\beta-\alpha}\left(\frac{2(\beta-\alpha)}{2-\alpha}\right)^{\frac{2-\alpha}{2-\beta}}\left(\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{2-\alpha}{2-\beta}}b_1^{\frac{(4-\beta)(2-\alpha)}{2(2-\beta)}}b_2^{\frac{(4-\beta)((2-\alpha))}{2(2-\beta)}}>\frac{\mu_1 b_1^{4-\alpha}+\mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}, \tag{18}$$

so,

$$\left(\mu_1 b_1^{4-\alpha}+\mu_2 b_2^{4-\alpha}\right)\rho^{\frac{2-\alpha}{\beta-2}}b_1^{\frac{(4-\beta)(2-\alpha)}{2(\beta-2)}}b_2^{\frac{(4-\beta)(2-\alpha)}{2(\beta-2)}}$$

$$< \frac{\beta-2}{\beta-\alpha}\left(\frac{2-\alpha}{2(\beta-\alpha)}\right)^{\frac{2-\alpha}{\beta-2}}\|Q_\beta\|_{L^2(\mathbb{R}^N)}^{\frac{2(2-\alpha)}{\beta-2}}\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}.$$

Since $h'(t) = t^{\alpha-1}\left[t^{2-\alpha} - \beta\mathcal{D}_2 t^{\beta-\alpha} - \alpha\mathcal{D}_1\right] = t^{\alpha-1}g(t)$, it is easy to see that $g(0) < 0$, $g(+\infty) = -\infty$ and $g(t)$ has a unique global maximum point at positive level in $\bar{t} = \left(\frac{\beta(\beta-\alpha)}{2-\alpha}\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{1}{2-\beta}} b_1^{\frac{4-\beta}{2(2-\beta)}} b_2^{\frac{4-\beta}{2(2-\beta)}}$ we can deduce that when

$$g(\bar{t}) = \frac{\beta-2}{\beta-\alpha}\left(\frac{\beta(\beta-\alpha)}{2-\alpha}\right)^{\frac{2-\alpha}{2-\beta}}\left(\frac{\rho}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{2-\alpha}{2-\beta}} b_1^{\frac{(4-\beta)(2-\alpha)}{2(2-\beta)}} b_2^{\frac{(4-\beta)((2-\alpha))}{2(2-\beta)}} \tag{19}$$
$$> \alpha\frac{\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}},$$

then $h(t)$ has exactly two critical points, one is a local minimum at negative level, the other one is a global maximum at positive level. It is easy to see that when (18) holds, then (19) also holds.

By the similar argument as above, we can deduce that when $0 < \beta < 2 < \alpha < \min\{N, 4\}$ and (7) hold, the conclusions also hold. □

Lemma 3.3. *When (6) or (7) holds, then $\mathcal{P}^0_{b_1,b_2} = \emptyset$ and $\mathcal{P}_{b_1,b_2}$ is a C^1 submanifold in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ with codimension 3.*

Proof. We only consider the case of (6), the discussion for case (7) is similar to the case of (6). Assume by contradiction that there exists a $(u, v) \in \mathcal{P}^0_{b_1,b_2}$ such that

$$\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \tag{20}$$
$$-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy = 0,$$

and

$$2\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1\alpha^2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \tag{21}$$
$$-\frac{\mu_2\alpha^2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho\beta^2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy = 0.$$

From (20) and (21), we obtain

$$\frac{\alpha}{2}(1-\frac{\alpha}{2})\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy + \frac{\alpha}{2}(1-\frac{\alpha}{2})\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy \tag{22}$$
$$= \beta(\frac{\beta}{2}-1)\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy.$$

If $0 < \alpha < 2 < \beta$, then from $P_{b_1,b_2}(u, v) = 0$, (22), Lemma 2.5 and Lemma 2.6, we have

$$\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx = \left(\frac{\beta-\alpha}{2-\alpha}\right)\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy \tag{23}$$
$$\leq \left(\frac{\beta-\alpha}{2-\alpha}\right)\frac{\rho\beta}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\beta}dxdy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\beta}dxdy\right)^{\frac{1}{2}}$$

$$\le \left(\frac{\beta-\alpha}{2-\alpha}\right)\frac{\rho\beta}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{\beta}{4}}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{\beta}{4}}$$
$$\le \left(\frac{\beta-\alpha}{2-\alpha}\right)\frac{\rho\beta}{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx \qquad (24)\\&=\left(\frac{\beta-\alpha}{\beta-2}\right)\left[\frac{\alpha\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+\frac{\alpha\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\right]\\&\le\left(\frac{\beta-\alpha}{\beta-2}\right)\left[\frac{\alpha\mu_1b_1^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^{\frac{\alpha}{2}}+\frac{\alpha\mu_2b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}|\nabla v|^2dx\right)^{\frac{\alpha}{2}}\right]\\&\le\frac{\alpha}{2}\left(\frac{\beta-\alpha}{\beta-2}\right)\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}.\end{aligned}$$

Thus, from (23) and (24), we have

$$\begin{aligned}\left(\frac{1}{\beta}\left(\frac{2-\alpha}{\beta-\alpha}\right)\frac{\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}{\rho}b_1^{\frac{\beta-4}{2}}b_2^{\frac{\beta-4}{2}}\right)^{\frac{2}{\beta-2}}&\le\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx \qquad (25)\\&\le\left(\frac{\alpha}{2}\left(\frac{\beta-\alpha}{\beta-2}\right)\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{2}{2-\alpha}},\end{aligned}$$

so

$$\begin{aligned}&\left(\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}\right)^{\frac{2}{2-\alpha}}\rho^{\frac{2}{\beta-2}}b_1^{\frac{4-\beta}{\beta-2}}b_2^{\frac{4-\beta}{\beta-2}} \qquad (26)\\&\ge\left(\frac{2-\alpha}{2(\beta-\alpha)}\frac{2}{\beta}\right)^{\frac{2}{\beta-2}}\left(\frac{2}{\alpha}\frac{(\beta-2)}{\beta-\alpha}\right)^{\frac{2}{2-\alpha}}\|Q_\beta\|^{\frac{4}{\beta-2}}_{L^2(\mathbb{R}^N)}\|Q_\alpha\|^{\frac{4}{2-\alpha}}_{L^2(\mathbb{R}^N)}\\&\ge\left(\frac{2-\alpha}{2(\beta-\alpha)}\right)^{\frac{2}{\beta-2}}\left(\frac{\beta-2}{\beta-\alpha}\right)^{\frac{2}{2-\alpha}}\|Q_\beta\|^{\frac{4}{\beta-2}}_{L^2(\mathbb{R}^N)}\|Q_\alpha\|^{\frac{4}{2-\alpha}}_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Thus,

$$\begin{aligned}&(\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha})\rho^{\frac{2-\alpha}{\beta-2}}b_1^{\frac{(4-\beta)(2-\alpha)}{2(\beta-2)}}b_2^{\frac{(4-\beta)(2-\alpha)}{2(\beta-2)}}\\&\ge\frac{\beta-2}{\beta-\alpha}\left(\frac{2-\alpha}{2(\beta-\alpha)}\right)^{\frac{2-\alpha}{\beta-2}}\|Q_\beta\|^{\frac{2(2-\alpha)}{\beta-2}}_{L^2(\mathbb{R}^N)}\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)},\end{aligned}$$

which contradicts with the assumption (6), where we use the fact that

$$\left(\frac{2}{\beta}\right)^{\frac{2}{\beta-2}}\left(\frac{2}{\alpha}\right)^{\frac{2}{2-\alpha}}\ge 1.$$

Indeed, for any $0<\alpha<\beta<N$, $\frac{\log x}{x-1}$ is a monotone decreasing function of $x>0$, so $\frac{2}{\beta-2}\log\frac{\beta}{2}-\frac{2}{\alpha-2}\log\frac{\alpha}{2}\le 0$, which implies that $\left(\frac{\beta}{2}\right)^{\frac{2}{\beta-2}}\left(\frac{\alpha}{2}\right)^{\frac{2}{2-\alpha}}\le 1$, thus $\left(\frac{2}{\beta}\right)^{\frac{2}{\beta-2}}\left(\frac{2}{\alpha}\right)^{\frac{2}{2-\alpha}}\ge 1$, which implies that $\mathcal{P}_{b_1,b_2}=\emptyset$. Next, we show that $\mathcal{P}_{b_1,b_2}$ is a smooth manifold of codimension 3 on $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Since $\mathcal{P}_{b_1,b_2}$ is defined by $P_{b_1,b_2}(u,v)=0$, $G(u)=0$, $F(u)=0$, where $G(u)=\int_{\mathbb{R}^N}u^2dx-b_1$, $F(v)=$

$\int_{\mathbb{R}^N} v^2 dx - b_2$. Since $P_{b_1,b_2}(u,v)$, $G(u)$ and $F(u)$, are class of C^1, we only need to check that $d(P_{b_1,b_2}(u,v), G(u), F(v)) : H \to \mathbb{R}^3$ is surjective. If this is not true, $dP_{b_1,b_2}(u,v)$ has to be linearly dependent from $dG(u)$ and $dF(v)$ i.e. there exist a $\nu_1,\ \nu_2 \in \mathbb{R}$ such that

$$\begin{cases} 2\int_{\mathbb{R}^N} \nabla u \nabla \varphi + \nu_1 u\varphi = \mu_1 \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^\alpha} u(x)\varphi(x)dydx \\ +\rho\beta \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^\beta} u(x)\varphi(x)dydx & \text{in } \mathbb{R}^N, \\ 2\int_{\mathbb{R}^N} \nabla v \nabla \psi + \nu_2 v\psi = \mu_2 \alpha \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(y)}{|x-y|^\alpha} v(x)\psi(x)dydx \\ +\rho\beta \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^\beta} v(x)\psi(x)dydx & \text{in } \mathbb{R}^N, \end{cases}$$

for every $(\varphi,\psi) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, so

$$\begin{cases} -2\Delta u + 2\nu_1 u = \mu_1\alpha(|x|^{-\alpha} * |u|^2)u + \rho\beta(|x|^{-\beta} * |v|^2)u & \text{in } \mathbb{R}^N, \\ -2\Delta v + 2\nu_2 v = \mu_2\alpha(|x|^{-\alpha} * |v|^2)v + \rho\beta(|x|^{-\beta} * |u|^2)v & \text{in } \mathbb{R}^N, \end{cases}$$

by the similar argument as (8), (9) and (10), we know that the Pohozaev identity for above system is

$$\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx = \frac{\mu_1\alpha^2}{8}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\ + \frac{\mu_2\alpha^2}{8}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy + \frac{\rho\beta^2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy.$$

that is $(u,v) \in \mathcal{P}^0_{b_1,b_2}$, a contradiction. Hence, $\mathcal{P}_{b_1,b_2}$ is a natural constraint. □

Lemma 3.4. *When $0 < \alpha < 2 < \beta$, for every $(u,v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, the function $\Psi_{u,v}(t)$ has exactly two critical points $s_{u,v} < t_{u,v} \in \mathbb{R}$ and two zeros $c_{u,v} < d_{u,v} \in \mathbb{R}$ with $s_{u,v} < c_{u,v} < t_{u,v} < d_{u,v}$. Moreover,*

(1) *$s_{u,v} \star (u,v) \in \mathcal{P}^+_{b_1,b_2}$ and $t_{u,v} \star (u,v) \in \mathcal{P}^-_{b_1,b_2}$, and if $t \star (u,v) \in \mathcal{P}_{b_1,b_2}$, then either $t = s_{u,v}$ or $t = t_{u,v}$.*

(2) *$(\int_{\mathbb{R}^N} (|\nabla(t\star u)|^2 + |\nabla(t\star v)|^2)dx)^{\frac{1}{2}} \le R_0$ for every $t \le c_{u,v}$, and*

$$J(u,v)(s_{u,v}\star(u,v)) \\ = \min\left\{ J(t\star(u,v) : t \in \mathbb{R} \text{ and } (\int_{\mathbb{R}^N} (|\nabla(t\star u)|^2 + |\nabla(t\star v)|^2)dx)^{\frac{1}{2}} < R_0 \right\} < 0.$$

(3) *We have*

$$J(t_{u,v}\star(u,v)) = \max\{J(t\star(u,v)) : t \in \mathbb{R}\} > 0$$

and $\Psi_{u,v}(t)$ is strictly decreasing and concave on $(t_{u,v}, +\infty)$. In particular, if $t_{u,v} < 0$, then $P_{b_1,b_2}(u,v) < 0$.

(4) *The maps $(u,v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : s_{u,v} \in \mathbb{R}$ and $(u,v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : t_{u,v} \in \mathbb{R}$ are of class C^1.*

Proof. The proof is similar as the proof of Lemma 4.4 in [26], so we omit the details here. □

For $k > 0$, set

$$A_k = \left\{ (u,v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx \right)^{\frac{1}{2}} < k \right\},$$

and

$$m^+(b_1,b_2)=\inf_{(u,v)\in A_{R_0}} J(u,v).$$

From Lemma 3.4, we have following corollary

Corollary 3.5. *The set* $\mathcal{P}^+_{b_1,b_2}$ *is contained in*

$$A_{R_0}=\left\{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}:\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{1}{2}}<R_0\right\},$$

and

$$\sup_{\mathcal{P}^+_{b_1,b_2}} J(u,v)\le 0\le \inf_{\mathcal{P}^-_{b_1,b_2}} J(u,v).$$

By the similar arguments as Lemma 4.5 in [26], we give following lemma without the details of the proof.

Lemma 3.6. *We have* $m^+(b_1,b_2)\in(-\infty,0)$ *that*

$$m^+(b_1,b_2)=\inf_{\mathcal{P}_{b_1,b_2}} J(u,v)=\inf_{\mathcal{P}^+_{b_1,b_2}} J(u,v) \text{ and } m^+(b_1,b_2)<\overline{A}_{R_0}\setminus A_{R_0-\sigma}J(u,v).$$

Lemma 3.7. *Under the assumption* (6) *or* (7)*, we have*

$$m^+(b_1,b_2)<\min\{m^+(b_1,0),m^+(0,b_2)\}.$$

Proof. From [25], we know that $m^+(0,b_2)$ can be achieved by $v^*\in \mathrm{T}_{b_2}$ and v^* is radially symmetric and decreasing. We choose aproper test function $u\in \mathrm{T}_{b_1}$ such that $(t\star u,v^*)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}$. From Lemma 3.4, we obtain

$$h(t)<h_1(t)=\frac{1}{2}t^2-\frac{\mu_2 b_2^{\frac{4-\alpha}{2}}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}t^\alpha. \tag{27}$$

By direct calculations, there exists $0<t^*<R_0$ such that $h_1(t^*)=0$. From [25], we have $m^+(0,b_2)=\inf_{v\in\mathcal{P}^+_{0,b_2}} J(0,v)=\inf_{v\in \mathrm{T}_{b_2}\cap B(t^*)} J(0,v)$. Therefore, from the analysis in Lemma 3.2, we have

$$\|\nabla v^*\|_{L^2}\le t^*<R_0<\widetilde{t}=\left(\frac{\beta(\beta-\alpha)}{2-\alpha}\frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{1}{2-\beta}} b_1^{\frac{4-\beta}{4(2-\beta)}}b_2^{\frac{4-\beta}{4(2-\beta)}}.$$

Since $h(R_0)=h(R_1)=0$ and the monotonicity of $h(t)$, we deduce that $(t\star u,v^*)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}\cap A_{R_0}$ for $t\ll -1$, therefore,

$$\begin{aligned}
m^+(b_1,b_2)&=\inf_{(u,v)\in \mathrm{T}_{b_1}\times \mathrm{T}_{b_2}\cap A_{R_0}} J(u,v)\le J(t\star u,v^*)\\
&=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla v^*|^2dx-\frac{1}{4}\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v^*)^2(x)(v^*)^2(y)}{|x-y|^\alpha}dxdy\\
&\quad-\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(t*u)^2(x)(v^*)^2(y)}{|x-y|^\beta}dxdy\\
&\quad+\frac{e^{2t}}{2}\int_{\mathbb{R}^3}|\nabla u|^2dx-\frac{\mu_1 e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy<J(0,v^*)=m^+(0,b_2).
\end{aligned}$$

Similarly, we have $m^+(b_1,b_2)<m^+(b_1,0)$. Hence, the proof is completed. □

Lemma 3.8. *Let* $\{(u_n,v_n)\}\subset S_{a,r}=\mathrm{T}_{b_1}\times \mathrm{T}_{b_2}\cap H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ *be a Palais-Smale sequence for* $J(u,v)|_{\mathrm{T}_{b_1}\times \mathrm{T}_{b_2}}$ *at level* $m^+(b_1,b_2)$*. Then* $\{(u_n,v_n)\}$ *is bounded in* $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$*.*

Proof. Since $P_{b_1,b_2}(u_n,v_n)\to 0$, we have

$$P_{b_1,b_2}(u_n,v_n)=\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx-\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dxdy=o_n(1).$$

Thus, from (15), we have

$$J(u_n,v_n)=(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx+(\frac{\alpha}{\beta}-1)\Big[\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy$$
$$+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\Big]+o_n(1)$$
$$\geq(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx-\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}.$$

Since $\{(u_n,v_n)\}$ is a Palais-Smale sequence for $J(u,v)|_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}$ at level $m^+(b_1,b_2)$, we have $J(u_n,v_n)\leq m^+(b_1,b_2)+1$ for n large. Hence

$$(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx$$
$$\leq\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}+m^+(b_1,b_2)+2$$

so $\{(u_n,v_n)\}$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. This completes the proof. □

Lemma 3.9. *Let $\{(u_n,v_n)\}\subset S_{a,r}=\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}\cap H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ be a Palais-Smale sequence for $J(u,v)|_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}$ at level $m^+(b_1,b_2)$ with additional properties $P_{b_1,b_2}(u_n,v_n)\to 0$ and $u_n^-,v_n^-\to 0$ a.e. in $\mathbb{R}^N$, then up to a subsequence $\{(u_n,v_n)\}\to(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, where (u,v) is a positive solution of (1) for some $\lambda_1,\lambda_2>0$.*

Proof. By principle of symmetric criticality (Theorem 1.28 in [45]), the solutions for (1)-(2) in function space $H^1_r(\mathbb{R}^N)\times H^1_r(\mathbb{R}^N)$ are also those in function space $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. So, we can chosen the radial minimizing sequence.

Step 1. We claim $\lambda_1>0,\ \lambda_2>0$. From Lemma 3.8, we known that $\{(u_n,v_n)\}$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, thus, by the Sobolev embedding theorem, we have $H^1_r\hookrightarrow\hookrightarrow L^p_r(\mathbb{R}^N)$ for $2<p<2^*=\frac{2N}{N-2}$, thus there exists a $(u,v)\in H^1_r(\mathbb{R}^N)\times H^1_r(\mathbb{R}^N)$ such that $(u_n,v_n)\to(u,v)$ in $L^p_r(\mathbb{R}^N)\times L^p_r(\mathbb{R}^N)$ for $2<p<2^*$ and $(u_n,v_n)\rightharpoonup(u,v)$ in $H^1_r(\mathbb{R}^N)\times H^1_r(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)\times L^p(\mathbb{R}^N)$, and $(u_n,v_n)\to(u,v)$ a.e in $\mathbb{R}^N$. Hence $u,v\geq 0$ are radial functions. Since $J'\mid_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}(u_n,v_n)\to 0$, by the Lagrange multipliers rule, we know that there exists a sequence $(\lambda_{1,n},\lambda_{2,n})\in\mathbb{R}^2$ such that

$$\int_{\mathbb{R}^N}(\nabla u_n\nabla\varphi+\nabla v_n\nabla\psi)dx+\int_{\mathbb{R}^N}(\lambda_{1,n}u_n\varphi+\lambda_{2,n}v_n\psi)dx \tag{28}$$
$$-\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(y)}{|x-y|^\alpha}u_n(x)\varphi(x)dydx-\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(y)}{|x-y|^\alpha}v_n(x)\psi(x)dydx$$
$$-\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(y)}{|x-y|^\beta}u_n(x)\varphi(x)dydx-\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(y)}{|x-y|^\beta}v_n(x)\psi(x)dydx$$
$$=o(1)\|(\psi,\varphi)\|_{H^1}\quad\text{in}\quad\mathbb{R}^N,$$

for every $(\varphi,\psi)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Using $(u_n,0)$ and $(0,v_n)$ as text function in (28), we have

$$\int_{\mathbb{R}^N}\lambda_{1,n}u_n^2dx=-\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy+\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dydx+o(1)\|\varphi\|_{H^1},$$

$$\int_{\mathbb{R}^N}\lambda_{2,n}v_n^2dx=-\int_{\mathbb{R}^N}|\nabla v_n|^2dx+\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy+\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(y)v_n^2(x)}{|x-y|^\beta}dydx+o(1)\|\psi\|_{H^1},$$

so

$$\int_{\mathbb{R}^N}(\lambda_{1,n}u_n^2+\lambda_{2,n}v_n^2)dx=-\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx \tag{29}$$

$$+\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy+\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy+2\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dydx, \tag{30}$$

by the boundedness of $\{(u_n,v_n)\}$ and (12), (14), (15), we have $(\lambda_{1,n},\lambda_{2,n})$ is bounded, hence up to a subsequence $(\lambda_{1,n},\lambda_{2,n})\to(\lambda_1,\lambda_2)\in\mathbb{R}^2$, passing to limits in (28), we can deduce that (u,v) is a nonnegative solutions of (1), so

$$\int_{\mathbb{R}^N}(\lambda_1u^2+\lambda_2v^2)dx+\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx=\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+2\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dydx.$$

From (9), we can get

$$\frac{N-2}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx+\frac{N}{2}\int_{\mathbb{R}^N}(\lambda_1u^2+\lambda_2v^2)dx=\frac{(2N-\alpha)}{4}\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+\frac{(2N-\alpha)}{4}\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+\frac{(2N-\beta)}{2}\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy,$$

so

$$\int_{\mathbb{R}^N}(\lambda_1u^2+\lambda_2v^2)dx=\frac{4-\alpha}{4}\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+\frac{4-\alpha}{4}\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+\frac{4-\beta}{2}\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy. \tag{31}$$

From $P_{b_1,b_2}(u_n,v_n)\to 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx-\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy \\ &-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dxdy=o_n(1).\end{aligned}\tag{32}$$

Together (29) with (32), we can get

$$\begin{aligned}&\lambda_{1,n}b_1^2+\lambda_{2,n}b_2^2=\frac{4-\alpha}{4}\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy \\ &+\frac{4-\alpha}{4}\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy+\frac{4-\beta}{2}\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dxdy,\end{aligned}\tag{33}$$

it is easy to see that at least one sequence of $(\lambda_{i,n})$ is positive and bounded away from 0. Let $n\to+\infty$ in (33), we have

$$\begin{aligned}&\lambda_1b_1^2+\lambda_2b_2^2=\frac{4-\alpha}{4}\mu_1\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\ &+\frac{4-\alpha}{4}\mu_2\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+\frac{4-\beta}{2}\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy.\end{aligned}\tag{34}$$

We know that at least one sequence of (λ_i) is positive and bounded away from 0. Without loss of generate, we assume $\lambda_1>0$, now we argue by contradiction and assume that $\lambda_2\le 0$ and

$$-\Delta v=-\lambda_2 v+\mu_2(|x|^{-\alpha}*|v|^2)v+\rho(|x|^{-\beta}*|u|^2)v\ge 0,$$

using a Liouville type theorem [[20], Lemma A.2] , we can deduce that $v=0$, so, u satisfies that

$$\begin{cases}-\Delta u+\lambda_1 u=\mu_1(|x|^{-\alpha}*|u|^2)u & \text{in } \mathbb{R}^N,\\ u>0, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}u^2dx=b_1^2 & \text{in } \mathbb{R}^N.\end{cases}$$

By Lemma 2.5, we have

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy\le c(N,\alpha)\|u_n\|^4_{L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)},$$

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy\le c(N,\alpha)\|v_n\|^4_{L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)},$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dxdy \\ &\le\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\beta}dxdy\right)^{\frac12}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\beta}dxdy\right)^{\frac12} \\ &\le c(N,\beta)\|u_n\|^2_{L^{\frac{4N}{2N-\beta}}(\mathbb{R}^N)}\|v_n\|^2_{L^{\frac{4N}{2N-\beta}}(\mathbb{R}^N)}.\end{aligned}$$

When $N\ge 3$, $H^1_r\hookrightarrow L^p$ is compact for $p\in(2,2^*)$, we have that $u_n\to u$ strongly in $L^{\frac{4N}{2N-\alpha}}$ and $L^{\frac{4N}{2N-\beta}}$. So, we obtain that

$$\begin{aligned}m^+(b_1,b_2)&=\lim_{n\to+\infty}J(u_n,v_n)\\ &=\lim_{n\to+\infty}\Bigg[\frac{\mu_1}{4}(\frac{\alpha}{2}-1)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy\end{aligned}$$

$$+\frac{\mu_2}{4}(\frac{\alpha}{2}-1)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy+\frac{\rho}{2}(\frac{\beta}{2}-1)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)v_n^2(y)}{|x-y|^\beta}dxdy\Big]$$
$$=\frac{\mu_1}{4}(\frac{\alpha}{2}-1)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy=m^+(b_1,0).$$

Thus, it contracts with Lemma 3.7. So $\lambda_1>0,\ \lambda_2>0$. Similarly, we have if $\lambda_2>0$, then $\lambda_1>0$. Hence $\lambda_1>0,\ \lambda_2>0$. Similarly, we have if $\lambda_2>0$, then $\lambda_1>0$, so $\lambda_1>0,\ \lambda_2>0$.

Step 2. Prove the L^2 convergence. From (31) and (34), we have

$$\lambda_1\left(b_1^2-\int_{\mathbb{R}^N}u^2dx\right)+\lambda_2\left(b_2^2-\int_{\mathbb{R}^N}v^2dx\right)=0,$$

so

$$\int_{\mathbb{R}^N}u^2=b_1^2>0,\ \int_{\mathbb{R}^N}v^2=b_2^2>0.$$

Step 3. Prove the H^1 convergence. From $P_{b_1,b_2}(u_n,v_n)\to 0$, we have

$$m^+(b_1,b_2)=\lim_{n\to+\infty}J(u_n,v_n)=\lim_{n\to+\infty}\Big[(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla u_n|^2+|\nabla v_n|^2)dx$$
$$-\frac{\beta-\alpha}{\beta}\Big(\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy\Big)\Big].$$

From Step 2, we know that $u\neq 0$ and $v\neq 0$. Let $\widetilde{u}_n=u_n-u,\ \widetilde{v}_n=v_n-v$, it is easy to see that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{u}_n^2(y)}{|x-y|^\alpha}dxdy \\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_n^2(x)u_n^2(y)}{|x-y|^\alpha}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+o_n(1),\end{aligned}\tag{35}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{v}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\alpha}dxdy \\ &=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_n^2(x)v_n^2(y)}{|x-y|^\alpha}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+o_n(1).\end{aligned}\tag{36}$$

So, by Brezis-Lieb lemma, (35) and (36), we have

$$0=P_{b_1,b_2}(u_n,v_n)+o(1)=P_{b_1,b_2}(\widetilde{u}_n,\widetilde{v}_n)+P_{b_1,b_2}(u,v)+o(1)$$
$$=\int_{\mathbb{R}^N}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx-\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{u}_n^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{v}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\beta}dxdy+o(1).$$

Hence, we know that

$$m^+(b_1,b_2)=\lim_{n\to+\infty}J(u_n,v_n)=\lim_{n\to+\infty}J(\widetilde{u}_n,\widetilde{v}_n)+J(u,v)$$
$$=\lim_{n\to+\infty}\Big[(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx$$
$$-\frac{\beta-\alpha}{\beta}\Big(\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{u}_n^2(y)}{|x-y|^\alpha}dxdy+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{v}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\alpha}dxdy\Big)\Big]+J(u,v)$$
$$\geq\lim_{n\to+\infty}\Big[(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx$$

$$- \frac{\beta-\alpha}{\beta}\Big(\frac{\mu_1}{4}c(N,\alpha)\|\widetilde{u}_n\|^4_{L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)} + \frac{\mu_2}{4}c(N,\alpha)\|\widetilde{v}_n\|^4_{L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)}\Big)\Big] + J(u,v)$$
$$\geq \lim_{n\to+\infty}(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx + J(u,v) \geq m^+(b_1,b_2).$$

Thus, $J(u,v)=m^+(b_1,b_2)$ and $(u_n,v_n)\to(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. □

Next, we prove the existence of second critical point of mountain pass type for $J(u,v)\mid_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}$. By the similar argument as Lemma 4.10 in [26], we give following lemmas without the details of the proof.

Lemma 3.10. *We have*

$$m^-(b_1,b_2) := \inf_{(u,v)\in\mathcal{P}^-_{b_1,b_2}} J(u,v) > 0.$$

To exclude the semi-trivial solution of the mountain psss solution, we give following lemma.

Lemma 3.11. *When $0<\alpha<2<\beta$ and the assumption (6) holds, we have*

$$m^-(b_1,b_2) < \min\{m^-(b_1,0), m^-(0,b_2)\}.$$

Proof. From [25], we know that $m^-(b_1,0)$ can be achieved by $u^*\in\mathrm{T}_{b_1}$. We choose aproper test function $v\in\mathrm{T}_{b_2}$ such that $(u^*,t\star v)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. From Lemma 3.4, there exists $t^*\in\mathbb{R}$ such that $t^*\star(u^*,t\star v)\in\mathcal{P}^-_{b_1,b_2}$, then we have

$$e^{(2-\alpha)t^*}\int_{\mathbb{R}^N}(|\nabla u^*|^2+|\nabla(t\star v)|^2)dx - \frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^*)^2(x)(u^*)^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(t\star v)^2(x)(t\star v)^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\rho\beta e^{(\beta-\alpha)t^*}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^*)^2(x)(t\star v)^2(y)}{|x-y|^\beta}dxdy = 0.$$

Thus, $e^{t^*}\to 1$ as $t\to-\infty$. If $t\ll -1$, we have

$$m^-(b_1,b_2) = \inf_{(u,v)\in\mathcal{P}^-_{b_1,b_2}} J(u,v) \leq J(t^*\star(u^*,t\star v)$$
$$= \frac{e^{2t^*}}{2}\int_{\mathbb{R}^N}(|\nabla u^*|^2+|\nabla(t\star v)|^2)dx - \frac{\mu_1\alpha e^{\alpha t^*}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^*)^2(x)(u^*)^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2\alpha e^{\alpha t^*}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(t\star v)^2(x)(t\star v)^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\rho\beta e^{\beta t^*}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u^*)^2(x)(t\star v)^2(y)}{|x-y|^\beta}dxdy$$
$$< J(u^*,0) = m^-(b_1,0).$$

Similarly, we have $m^-(b_1,b_2)<m^-(0,b_2)$. Hence, the proof is completed. □

Lemma 3.12. *There is a radial symmetric Palais-Smale sequence of $J|_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}$ at level $m^-(b_1,b_2)$ with the additional properties $P(u_n,v_n)\to 0$ and $u_n^-,v_n^-\to 0$ a.e. in $\mathbb{R}^N$. Then up to a subsequence $(u_n,v_n)\to(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, where (u,v) is a positive solution of (1) for some $\lambda_1,\lambda_2>0$.*

Proof. By the similar argument as Lemma 4.12 in [26], we can get a radial Palais-Smale sequence $(\widetilde{u}_n, \widetilde{v}_n)$ of $J|_{\mathrm{T}_{b_1,r}\times \mathrm{T}_{b_2,r}}$ and hence a radial symmetric Palais-Smale sequence of $J|_{\mathrm{T}_{b_1}\times \mathrm{T}_{b_2}}$ at level $m^-(b_1,b_2)$.

Next, we prove that up to a subsequence $(\widetilde{u}_n, \widetilde{v}_n) \to (u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, where (u,v) is a positive solution of (1) for some $\lambda_1,\lambda_2>0$. When we get Lemma 3.11, by the similar argument as Step 1 in Lemma 3.9, we can prove that $\lambda_i>0 (i=1,2)$. From (31) and (34), we have

$$\lambda_1\left(b_1^2-\int_{\mathbb{R}^N}u^2dx\right)+\lambda_2\left(b_2^2-\int_{\mathbb{R}^N}v^2dx\right)=0,$$

so

$$\int_{\mathbb{R}^N}u^2=b_1^2>0,\ \int_{\mathbb{R}^N}v^2=b_2^2>0.$$

From $P_{b_1,b_2}(\widetilde{u}_n,\widetilde{v}_n)=0$, we have

$$\begin{aligned}&m^-(b_1,b_2)\\&=\lim_{n\to+\infty}\left[(\frac{1}{2}-\frac{1}{\alpha})\int_{\mathbb{R}^N}(|\nabla\widetilde{u}_n|^2+|\nabla\widetilde{v}_n|^2)dx+\frac{\rho}{2}[\frac{\beta}{\alpha}-1]\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\beta}dxdy\right].\end{aligned}$$

Let $w_n=\widetilde{u}_n-u,\ \sigma_n=\widetilde{v}_n-v$, it is easy to see that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w_n^2(x)w_n^2(y)}{|x-y|^\alpha}dxdy\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{u}_n^2(y)}{|x-y|^\alpha}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+o_n(1),\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\sigma_n^2(x)\sigma_n^2(y)}{|x-y|^\alpha}dxdy\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{v}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\alpha}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+o_n(1).\end{aligned}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w_n^2(x)\sigma_n^2(y)}{|x-y|^\beta}dxdy\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_n^2(x)\widetilde{v}_n^2(y)}{|x-y|^\beta}dxdy-\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy+o(1).\end{aligned}$$

So, by Brezis-Lieb lemma, we have

$$\begin{aligned}0&=P_{b_1,b_2}(\widetilde{u}_n,\widetilde{v}_n)+o(1)=P_{b_1,b_2}(w_n,\sigma_n)+P_{b_1,b_2}(u,v)+o(1)\\&=\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla\sigma_n|^2)dx-\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w_n^2(x)w_n^2(y)}{|x-y|^\alpha}dxdy\\&\quad-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\sigma_n^2(x)\sigma_n^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w_n^2(x)\sigma_n^2(y)}{|x-y|^\beta}dxdy+o(1).\end{aligned}$$

Hence, we know that

$$\begin{aligned}m^-(b_1,b_2)&=\lim_{n\to+\infty}J(u_n,v_n)=\lim_{n\to+\infty}J(w_n,\sigma_n)+J(u,v)\\&=\lim_{n\to+\infty}\left[(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla\sigma_n|^2)dx\right.\\&\quad\left.-\frac{\beta-\alpha}{\beta}\left(\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{w_n^2(x)w_n^2(y)}{|x-y|^\alpha}dxdy+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\sigma_n^2(x)\sigma_n^2(y)}{|x-y|^\alpha}dxdy\right)\right]+J(u,v)\end{aligned}$$

$$\geq \lim_{n\to+\infty}\Bigg[(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla\sigma_n|^2)dx$$
$$-\frac{\beta-\alpha}{\beta}\left(\frac{\mu_1}{4}c(N,\alpha)\|w_n\|^4_{L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)}+\frac{\mu_2}{4}c(N,\alpha)\|\sigma_n\|^4_{L^{\frac{4N}{2N-\alpha}}(\mathbb{R}^N)}\right)\Bigg]+J(u,v)$$
$$\geq \lim_{n\to+\infty}(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla\sigma_n|^2)dx+J(u,v)\geq m^-(b_1,b_2).$$

Thus, $J(u,v)=m^-(b_1,b_2)$ and $(u_n,v_n)\to(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Thus, $J(u,v)=m^-(b_1,b_2)$ and $(\widetilde{u}_n,\widetilde{v}_n)\to(u,v)$ in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. □

P*roof of Theorem 1.4*. From Lemma 3.9, we only need to prove the existence of radial Palais-Smale sequence for $J\mid_{\mathrm{T}^r_{b_1}\times\mathrm{T}^r_{b_2}}$ at level $m(b_1,b_2)$ with additional properties $P_{b_1,b_2}(u_n,v_n)\to0$ and $u_n^-,v_n^-\to0$ a.e. in $\mathbb{R}^N$. Let us consider a minimizing sequence (u_n,v_n) for $J(u,v)\mid_{A_{R_0}}$, we assume that $(u_n,v_n)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$ is radial decreasing for every n. Furthermore, by Lemma 3.4, for every n we can take $s_{u_n,v_n}*(u_n,v_n)\in\mathcal{P}^+_{b_1,b_2}$ such that $(\int_{\mathbb{R}^N}(|\nabla(s_{u_n,v_n}\star u_n)|^2+|\nabla(s_{u_n,v_n}\star v_n)|^2)dx)^{\frac{1}{2}}\leq R_0$ and

$$J(s_{u,v}\star(u_n,v_n))$$
$$=\min\left\{J(t\star(u_n,v_n):t\in\mathbb{R}\text{ and }(\int_{\mathbb{R}^N}(|\nabla(t\star u_n)|^2+|\nabla(t\star v_n)|^2)dx)^{\frac{1}{2}}<R_0\right\}$$
$$<J(u_n,v_n).$$

Thus, we obtain a new minimizing sequence $\{w_n,\sigma_n\}=\{s_{u_n,u_n}\star v_n,s_{u_n,v_n}\star v_n\}$ with $(w_n,\sigma_n)\in\mathrm{T}^r_{b_1}\times\mathrm{T}^r_{b_2}\cap\mathcal{P}^+_{a,\mu}$ radially decreasing for every n. By Lemma 3.6, we have $\left(\int_{\mathbb{R}^N}(|\nabla w_n|^2+|\nabla\sigma_n|^2)dx\right)^{\frac{1}{2}}\leq R_0$ for every n and hence by Ekeland's variational principle in a standard way, we know the existence of a new minimizing sequence for $\{u_n,v_n\}\subset A_{R_0}$ for $m(b_1,b_2)$ with $\|(u_n,v_n)-(w_n,\sigma_n)\|\to0$ as $n\to+\infty$, which is also a Palais-Smale sequence for $J(u,v)$ on $\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. By the boundedness of $\{(w_n,\sigma_n)\}$, $\|(u_n,v_n)-(w_n,\sigma_n)\|\to0$, Brézis-Lieb lemma and Sobolev embedding theorem, we have

$$P_{b_1,b_2}(u_n,v_n)=P_{b_1,b_2}(w_n,\sigma_n)+o(1)\to0\ \text{ and }u_n^-,v_n^-\to0\ a.e.\text{in }\mathbb{R}^N.$$

From Lemma 3.6, we know that (u,v) is a ground normalized solution. From Lemma 3.12, we get a second critical point of mountain pass type for $J(u,v)\mid_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}$.

Next, we prove the second part of Theorem 1.4, that is the limit behavior of the normalized solution as $\mu_1\to0,\ \mu_2\to0$.

For $b_1,b_2>0$ fixed, from the proof Lemma 3.2, we can deduce that when $\mu_1\to0,\ \mu_2\to0$, then $R_0(b_1,b_2,\rho,\mu_1,\mu_2)\to0$. By corollary 3.5, when $(\widehat{u},\widehat{v})$ is the ground normalized solution obtained in Theorem 1.4, then $\left(\int_{\mathbb{R}^N}(|\nabla\widehat{u}|^2+|\nabla\widehat{u}|^2)dx\right)^{\frac{1}{2}}<R_0(b_1,b_2,\rho,\alpha_1,\alpha_2)\to0$, and

$$0>m^+(b_1,b_2)=J(\widehat{u},\widehat{v})\geq h\left(\left(\int_{\mathbb{R}^N}(|\nabla\widehat{u}|^2+|\nabla\widehat{v}|^2)dx\right)^{\frac{1}{2}}\right)\to0,$$

so $m^+(b_1,b_2)\to0$. The prove of the limit behavior of the second solution as $\mu_1,\mu_2\to0$ are similar as the proof Theorem 1.3 in [26], so we omit the details here. □

4. **Proof of Theorem 1.6.** We first give a refined upper of $m^+(b_1,b_2)$ and search for $(\lambda_{1,b_1,b_2},\lambda_{2,b_1,b_2},u_{b_1,b_2},v_{b_1,b_2})$ solving

$$\begin{cases}-\Delta u+\lambda_1 u=\mu_1(|x|^{-\alpha}*|u|^2)u & \text{in } \mathbb{R}^N,\\ -\Delta v+\lambda_2 v=\mu_2(|x|^{-\alpha}*|v|^2)v & \text{in } \mathbb{R}^N,\end{cases} \tag{37}$$

satisfying the additional conditions

$$\int_{\mathbb{R}^N} u^2dx=b_1^2 \text{ and } \int_{\mathbb{R}^N} v^2dx=b_2^2. \tag{38}$$

Denote

$$\widetilde{J}(u,v)=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy,$$

on the constraint $\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$.

$$\widetilde{\mathcal{P}}_{b_1,b_2}(u,v):=\Big\{(u,v)\in \mathrm{T}_{b_1}\times\mathrm{T}_{b_2}:\widetilde{P}_{b_1,b_2}(u,v)=0\Big\},$$

where

$$\widetilde{P}_{b_1,b_2}(u,v)=\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy$$
$$-\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy.$$

Then, the solution of (37)-(38) can be found as minimizers of

$$\widetilde{m}(b_1,b_2)=\inf_{\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}}J_0(u,v)>-\infty.$$

Let ω be a positive solution of the following equation

$$-\Delta u+u=(|x|^{-\alpha}*|u|^2)u \text{ in } \mathbb{R}^N.$$

Lemma 4.1. (37)-(38) *has a positive solution*

$$(\lambda_{1,b_1,\mu_1},\lambda_{2,b_2,\mu_2},L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x),L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x)),$$

where

$$\lambda_{1,b_1,\mu_1}=\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{2}{2-\alpha}},\ \lambda_{2,b_2,\mu_2}=\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{2}{2-\alpha}},$$

$$L_1=\mu_1^{\frac{N}{4-2\alpha}}\left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{N+2-\alpha}{4-2\alpha}},\ L_2=\mu_2^{\frac{N}{4-2\alpha}}\left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{N+2-\alpha}{4-2\alpha}}.$$

Moreover,

$$\widetilde{m}(b_1,b_2)=\widetilde{J}(L_1\omega((\lambda_{1,b_1,\alpha_1})^{\frac{1}{2}}x),L_2\omega((\lambda_{2,b_2,\alpha_2})^{\frac{1}{2}}x)$$
$$=-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}+\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|_{L^2(\mathbb{R}^N)}^2.$$

Proof. By elementary calculation, we have

$$(\lambda_{1,b_1,\alpha_1},\lambda_{2,b_2,\alpha_2},L_1\omega((\lambda_{1,b_1,\alpha_1})^{\frac{1}{2}}x),L_2\omega((\lambda_{2,b_2,\alpha_2})^{\frac{1}{2}}x)),$$

is a positive solution of (37)-(38). Furthermore, we have

$$\begin{aligned}\widetilde{m}(b_1,b_2)&=\widetilde{J}(L_1\omega((\lambda_{1,b_1,\alpha_1})^{\frac{1}{2}}x),L_2\omega((\lambda_{2,b_2,\alpha_2})^{\frac{1}{2}}x)\\&=-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\frac{L_1^2}{\lambda_{1,b_1,\mu_1}^{\frac{N-2}{2}}}\int_{\mathbb{R}^N}|\nabla\omega|^2dx+\frac{L_2^2}{\lambda_{2,b_2,\mu_2}^{\frac{N-2}{2}}}\int_{\mathbb{R}^3}|\nabla\omega|^2dx\right]\\&=-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\frac{L_1^2}{\lambda_{1,b_1,\mu_1}^{\frac{N-2}{2}}}+\frac{L_2^2}{\lambda_{2,b_2,\mu_2}^{\frac{N-2}{2}}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}\\&=-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}+\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

□

Lemma 4.2. *We have*

$$\begin{aligned}&m^+(b_1,b_2)\\&<\widetilde{m}(b_1,b_2)=-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}+\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Proof. Since

$$\begin{aligned}\widetilde{J}(u,v)&=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy\\&\quad-\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\\&\geq\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}\\&\geq\widetilde{h}\left(\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{1}{2}}\right),\end{aligned}$$

where $\widetilde{h}(t):(0,+\infty)\to\mathbb{R}$ defined by $\widetilde{h}(t)=\frac{1}{2}t^2-\mathcal{D}_1t^\alpha$, and $\mathcal{D}_1=\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}$. So, when $t\in(0,\bar{t})$ we have $\widetilde{h}(t)<0$ and $t\in(\bar{t},+\infty)$ we have $\widetilde{h}(t)>0$, where $\bar{t}=\left(\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{1}{2-\alpha}}$. So, $\widetilde{J}(L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x),L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x)=\widetilde{m}(b_1,b_2)<0$, implies that

$$\left[\int_{\mathbb{R}^N}|\nabla L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x)|^2dx+\int_{\mathbb{R}^N}|\nabla L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x)|^2dx\right]^{\frac{1}{2}}<\widetilde{t}.$$

From the definition of $h(t)$(see (17)) and the definition of R_0 in Lemma 3.2, we have $\widetilde{t}<R_0$, thus

$$\|L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x)\|^2_{L^2(\mathbb{R}^N)}=b_1^2,\ \|L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x)\|^2_{L^2(\mathbb{R}^N)}=b_2^2,$$

$$\left[\int_{\mathbb{R}^N}|\nabla L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x)|^2dx+\int_{\mathbb{R}^N}|\nabla L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x)|^2dx\right]^{\frac{1}{2}}<\widetilde{t}<R_0.$$

Hence,

$$\begin{aligned}m^+(b_1,b_2) &= \inf_{A_{R_0}} J(u,v) \le J(L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x), L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x)\\ &< \widetilde{J}(L_1\omega((\lambda_{1,b_1,\mu_1})^{\frac{1}{2}}x), L_2\omega((\lambda_{2,b_2,\mu_2})^{\frac{1}{2}}x) = \widetilde{m}(b_1,b_2)\\ &= -\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_1^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}} + \mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_2^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

□

P*roof of Theorem 1.6*. Let $b_{1,k},\ b_{2,k} \to 0^+\ k\to+\infty$ and Let $(u_{b_{1,k},b_{2,k}}, v_{b_{1,k},b_{2,k}}) \in A_{R_0}$ be a positive minimizer of $m^+(b_{1,k}, b_{2,k}, R_0)$ for each $k\in\mathbb{N}$. From

$$P_{b_1,b_2}(u_{b_{1,k},b_{2,k}}, v_{b_{1,k},b_{2,k}}) = 0,$$

we have

$$\begin{aligned}J(u_{b_{1,k},b_{2,k}}, v_{b_{1,k},b_{2,k}}) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_{b_{1,k},b_{2,k}}|^2 + |\nabla v_{b_{1,k},b_{2,k}}|^2)dx\\ &- \frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)u^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy - \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy\\ &- \frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta}dxdy\\ &= \Bigg[(\frac{1}{2}-\frac{1}{\beta})\|\nabla u_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^3)}\\ &+ (\frac{\alpha}{\beta}-1)\Big[\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy + \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy\Big]\Bigg]\\ &= \Bigg[(\frac{1}{2}-\frac{1}{\alpha})\|\nabla u_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^3)}\\ &+ (\frac{\beta}{\alpha}-1)\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta}dxdy\Bigg]\\ &< -\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}} + \mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}} + \mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}\\ &\le \|\nabla u_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^3)}\end{aligned}$$

and

$$\|\nabla u_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v_{b_{1,k},b_{2,k}}\|^2_{L^2(\mathbb{R}^3)} \le \left(\frac{\beta-\alpha}{\beta-2}\frac{\mu_1 b_{1,k}^{4-\alpha}+\mu_2 b_{2,k}^{4-\alpha}}{\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{2}{2-\alpha}}. \tag{39}$$

When $b_{1,k}\sim b_{2,k}$, we have

$$\int_{\mathbb{R}^N}(|\nabla u_{b_{1,k},b_{2,k}}|^2 + |\nabla v_{b_{1,k},b_{2,k}}|^2)dx \sim b_{1,k}^{\frac{2(4-\alpha)}{2-\alpha}} + b_{2,k}^{\frac{2(4-\alpha)}{2-\alpha}}. \tag{40}$$

From (14) and (39), we get

$$\begin{aligned}&\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta}dxdy\\&\le\frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_{1,k}^{\frac{4-\beta}{2}}b_{2,k}^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u_{b_{1,k},b_{2,k}}|^2+|\nabla v_{b_{1,k},b_{2,k}}|^2)dx\right]^{\frac{\beta}{2}}\\&\le\frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_{1,k}^{\frac{4-\beta}{2}}b_{2,k}^{\frac{4-\beta}{2}}\left(\frac{\beta-\alpha}{\beta-2}\frac{\mu_1 b_{1,k}^{4-\alpha}+\mu_2 b_{2,k}^{4-\alpha}}{\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\right)^{\frac{\beta}{2-\alpha}}\to 0\end{aligned}$$

as $k\to+\infty$. From

$$m^+(b_{1,k},b_{2,k})<-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|^2_{L^2}}\right)^{\frac{4-\alpha}{2-\alpha}}+\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|^2_{L^2}}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)},$$

we get

$$\begin{aligned}&-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|^2_{L^2}}\right)^{\frac{4-\alpha}{2-\alpha}}+\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|^2_{L^2}}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)} \qquad (41)\\&>m^+(b_{1,k},b_{2,k})=J(u_{b_{1,k},b_{2,k}},v_{b_{1,k},b_{2,k}})\\&=\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_{b_{1,k},b_{2,k}}|^2+|\nabla v_{b_{1,k},b_{2,k}}|^2)dx-\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)u^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy\\&-\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy-\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta}dxdy\\&\ge\inf_{\mathrm{T}_{b_{1,k}}\times\mathrm{T}_{b_{2,k}}}\left\{\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy\right.\\&\left.-\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\right\}-\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta}dxdy \qquad (42)\\&=-\left(\frac{1}{\alpha}-\frac{1}{2}\right)\left[\mu_1^{\frac{2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|^2_{L^2}}\right)^{\frac{4-\alpha}{2-\alpha}}+\mu_2^{\frac{2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|^2_{L^2}}\right)^{\frac{4-\alpha}{2-\alpha}}\right]\|\nabla\omega\|^2_{L^2(\mathbb{R}^N)}\\&-\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta}dxdy.\end{aligned}$$

When $b_{1,k}\sim b_{2,k}$, we get

$$m^+(b_{1,k},b_{2,k})\sim b_{1,k}^{\frac{2(4-\alpha)}{2-\alpha}}+b_{2,k}^{\frac{2(4-\alpha)}{2-\alpha}}, \qquad (43)$$

and

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u_{b_{1,k},b_{2,k}}|^2+|\nabla v_{b_{1,k},b_{2,k}}|^2)dx-\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2_{b_{1,k},b_{2,k}}(x)u^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy \qquad (44)\\&-\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2_{b_{1,k},b_{2,k}}(x)v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha}dxdy\sim b_{1,k}^{\frac{2(4-\alpha)}{2-\alpha}}+b_{2,k}^{\frac{2(4-\alpha)}{2-\alpha}}\end{aligned}$$

as $k\to+\infty$. The Lagrange multipliers rule implies the existence of some $\lambda_{1,k},\lambda_{2,k}\in\mathbb{R}$ such that

$$\int_{\mathbb{R}^3}(\nabla u_{b_{1,k},b_{2,k}}\nabla\varphi)dx+\int_{\mathbb{R}^3}(\lambda_{1,k}u_{b_{1,k},b_{2,k}}\varphi)dx$$

$$= \mu_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha} u_{b_{1,k},b_{2,k}}(x)\varphi(x)dydx + \rho \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta} u_{b_{1,k},b_{2,k}}(x)\varphi(x)dydx,$$

$$\int_{\mathbb{R}^3} (\nabla v_{b_{1,k},b_{2,k}} \nabla \psi)dx + \int_{\mathbb{R}^3} (\lambda_{2,k} v_{b_{1,k},b_{2,k}}\psi)dx = \mu_2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha} v_{b_{1,k},b_{2,k}}(x)\psi(x)dydx + \rho \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta} v_{b_{1,k},b_{2,k}}(x)\psi(x)dydx,$$

for each $\varphi, \psi \in H^1(\mathbb{R}^3)$. Taking $\varphi = u_{b_{1,k},b_{2,k}}$ and $\psi = v_{b_{1,k},b_{2,k}}$, we have

$$\lambda_{1,k} b^2_{1,k} = -\int_{\mathbb{R}^3} |\nabla u_{b_{1,k},b_{2,k}}|^2 dx + \mu_1 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2_{b_{1,k},b_{2,k}}(x) u^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha} dydx + \rho \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2_{b_{1,k},b_{2,k}}(x) v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\beta} u_{b_{1,k},b_{2,k}}(x)dydx, \tag{45}$$

$$\lambda_{2,k} b^2_{2,k} = -\int_{\mathbb{R}^3} |\nabla v_{b_{1,k},b_{2,k}}|^2 dx + \mu_2 \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2_{b_{1,k},b_{2,k}}(x) v^2_{b_{1,k},b_{2,k}}(y)}{|x-y|^\alpha} dydx + \rho \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2_{b_{1,k},b_{2,k}}(y) v^2_{b_{1,k},b_{2,k}}(x)}{|x-y|^\beta} u_{b_{1,k},b_{2,k}}(x)dydx. \tag{46}$$

From $P_{b_1,b_2}(u_{b_{1,k},b_{2,k}}, v_{b_{1,k},b_{2,k}}) = 0$, (45), (46) and $b_{1,k} \sim b_{2,k}$, we have

$$\begin{aligned} &-\int_{\mathbb{R}^3} (|\nabla u_{b_{1,k},b_{2,k}}|^2 + |\nabla v_{b_{1,k},b_{2,k}}|^2)dx \\ &\le \lambda_{1,k} b^2_{1,k} + \lambda_{2,k} b^2_{2,k} = \frac{4-\alpha}{\alpha} \int_{\mathbb{R}^3} (|\nabla u_{b_{1,k},b_{2,k}}|^2 + |\nabla v_{b_{1,k},b_{2,k}}|^2)dx \\ &\quad - \frac{2(\beta-\alpha)}{\alpha}\rho \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2_{b_{1,k},b_{2,k}}(y) v^2_{b_{1,k},b_{2,k}}(x)}{|x-y|^\beta} u_{b_{1,k},b_{2,k}}(x)dydx, \end{aligned}$$

combing with (40) and (44), we get

$$\lambda_{1,k} b^2_{1,k} + \lambda_{2,k} b^2_{2,k} \sim b_{1,k}^{\frac{2(4-\alpha)}{2-\alpha}} + b_{2,k}^{\frac{2(4-\alpha)}{2-\alpha}}.$$

When $b_{1,k} \sim b_{2,k}$, we have

$$\lambda_{1,k} \sim b_{1,k}^{\frac{4}{2-\alpha}},\ \lambda_{2,k} \sim b_{2,k}^{\frac{4}{2-\alpha}}.$$

Denote

$$\lambda_{1,b_{1,k},\mu_1} = \mu_1^{\frac{2}{2-\alpha}} \left(\frac{b^2_{1,k}}{\|\omega\|^2_{L^2}}\right)^{\frac{2}{2-\alpha}},\ \lambda_{2,b_{2,k},\mu_2} = \mu_2^{\frac{2}{2-\alpha}} \left(\frac{b^2_{2,k}}{\|\omega\|^2_{L^2}}\right)^{\frac{2}{2-\alpha}},$$

$$L_{1,k} = \mu_1^{\frac{N}{4-2\alpha}} \left(\frac{b^2_{1,k}}{\|\omega\|^2_{L^2}}\right)^{\frac{N+2-\alpha}{4-2\alpha}},\ L_{2,k} = \mu_2^{\frac{N}{4-2\alpha}} \left(\frac{b^2_{2,k}}{\|\omega\|^2_{L^2}}\right)^{\frac{N+2-\alpha}{4-2\alpha}}.$$

Define

$$\widetilde{u}_{b_{1,k},b_{2,k}} = L^{-1}_{1,k} u_{b_{1,k},b_{2,k}}((\lambda_{1,b_{1,k},\alpha_1})^{-\frac{1}{2}}x),\quad \widetilde{v}_{b_{1,k},b_{2,k}} = L^{-1}_{2,k} v_{b_{1,k},b_{2,k}}((\lambda_{2,b_{2,k},\alpha_2})^{-\frac{1}{2}}x),$$

then

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla\widetilde{u}_{b_{1,k},b_{2,k}}|^2+|\nabla\widetilde{v}_{b_{1,k},b_{2,k}}|^2)dx \\
&=\frac{\lambda_{1,b_{1,k},\mu_1}^{\frac{N-2}{2}}}{L_{1,k}^2}\int_{\mathbb{R}^N}|\nabla u_{b_{1,k},b_{2,k}}|^2dx+\frac{\lambda_{2,b_{2,k},\mu_2}^{\frac{N-2}{2}}}{L_{2,k}^2}\int_{\mathbb{R}^N}|\nabla v_{b_{1,k},b_{2,k}}|^2dx \\
&\leq\left[\mu_1^{\frac{-2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{\alpha-4}{2-\alpha}}+\mu_2^{\frac{-2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{\alpha-4}{2-\alpha}}\right]\int_{\mathbb{R}^3}(|\nabla u_{b_{1,k},b_{2,k}}|^2+|\nabla v_{b_{1,k},b_{2,k}}|^2)dx \\
&\leq\left[\mu_1^{\frac{-2}{2-\alpha}}\left(\frac{b_{1,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{\alpha-4}{2-\alpha}}+\mu_2^{\frac{-2}{2-\alpha}}\left(\frac{b_{2,k}^2}{\|\omega\|_{L^2}^2}\right)^{\frac{\alpha-4}{2-\alpha}}\right]\left(\frac{\beta-\alpha}{\beta-2}\frac{\mu_1b_{1,k}^{4-\alpha}+\mu_2b_{2,k}^{4-\alpha}}{\|Q_\alpha\|_{L^2(\mathbb{R}^N)}^2}\right)^{\frac{2}{2-\alpha}},
\end{aligned}\tag{47}$$

$$\begin{aligned}
\int_{\mathbb{R}^3}(|\widetilde{u}_{b_{1,k},b_{2,k}}|^2+|\widetilde{v}_{b_{1,k},b_{2,k}}|^2)dx&=\frac{\lambda_{1,b_{1,k}}^{\frac{N}{2}}}{L_{1,k}^2}\int_{\mathbb{R}^3}|u_{b_{1,k},b_{2,k}}|^2dx+\frac{\lambda_{2,b_{2,k}}^{\frac{N}{2}}}{L_{2,k}^2}\int_{\mathbb{R}^3}|v_{b_{1,k},b_{2,k}}|^2dx \\
&=2\|\omega\|_{L^2(\mathbb{R}^3)}^2.
\end{aligned}\tag{48}$$

By the definition of $\lambda_{1,b_{1,k},\mu_1},\ \lambda_{2,b_{2,k},\mu_2},\ L_{1,k},\ L_{2,k}$, it is easy to see that

$$\begin{aligned}
&\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{u}_{b_{1,k},b_{2,k}}^2(x)\widetilde{u}_{b_{1,k},b_{2,k}}^2(y)}{|x-y|^\alpha}dxdy+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\widetilde{v}_{b_{1,k},b_{2,k}}^2(x)\widetilde{v}_{b_{1,k},b_{2,k}}^2(y)}{|x-y|^\alpha}dxdy \\
&=\frac{\mu_1}{4}\frac{\lambda_{1,b_{1,k}}^{N-\frac{\alpha}{2}}}{L_{1,k}^4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u_{b_{1,k},b_{2,k}}^2(x)u_{b_{1,k},b_{2,k}}^2(y)}{|x-y|^\alpha}dxdy \\
&+\frac{\mu_2}{4}\frac{\lambda_{2,b_{2,k}}^{N-\frac{\alpha}{2}}}{L_{2,k}^4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v_{b_{1,k},b_{2,k}}^2(x)v_{b_{1,k},b_{2,k}}^2(y)}{|x-y|^\alpha}dxdy\to+\infty\text{ as }k\to+\infty.
\end{aligned}\tag{49}$$

From (47)- (48), we know that $(\widetilde{u}_{b_{1,k},b_{2,k}},\widetilde{v}_{b_{1,k},b_{2,k}})$ is bounded in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Then, we have

$$\big(\widetilde{u}_{b_{1,k},b_{2,k}}(x),\widetilde{v}_{b_{1,k},b_{2,k}}(x)\big)\rightharpoonup(\bar{u},\bar{v})\neq(0,0),$$

for some $(\bar{u},\bar{v})\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$. Since $(\widetilde{u}_{b_{1,k},b_{2,k}}(x),\widetilde{v}_{b_{1,k},b_{2,k}}(x))$ satisfies

$$\begin{cases}
-\Delta\widetilde{u}_{b_{1,k},b_{2,k}}+\frac{\lambda_{1,k}}{\lambda_{1,b_{1,k},\mu_1}}\widetilde{u}_{b_{1,k},b_{2,k}}=\mu_1\frac{L_{1,k}^2}{\lambda_{1,b_{1,k},\mu_1}^{\frac{N+2-\alpha}{2}}}(|x|^{-\alpha}*|\widetilde{u}_{b_{1,k},b_{2,k}}|^2)\widetilde{u}_{b_{1,k},b_{2,k}} \\
\quad+\rho\frac{L_{2,k}^2}{\lambda_{1,b_{1,k},\mu_1}\lambda_{2,b_{2,k},\mu_2}^{\frac{N-\beta}{2}}}\int_{\mathbb{R}^N}\frac{|\widetilde{v}_{b_{1,k},b_{2,k}}(y)|^2}{|(\frac{\lambda_{1,b_{1,k},\mu_1}}{\lambda_{2,b_{2,k},\mu_2}})^{-\frac{1}{2}}x-y|^\beta}\widetilde{u}_{b_{1,k},b_{2,k}}(x) \\
-\Delta\widetilde{v}_{b_{1,k},b_{2,k}}+\frac{\lambda_{2,k}}{\lambda_{2,b_{2,k},\mu_2}}\widetilde{v}_{b_{1,k},b_{2,k}}=\mu_2\frac{L_{2,k}^2}{\lambda_{2,b_{2,k},\mu_2}^{\frac{N+2-\alpha}{2}}}(|x|^{-\alpha}*|\widetilde{v}_{b_{1,k},b_{2,k}}|^2)\widetilde{v}_{b_{1,k},b_{2,k}} \\
\quad+\rho\frac{L_{1,k}^2}{\lambda_{2,b_{2,k},\mu_2}\lambda_{1,b_{1,k},\mu_1}^{\frac{N-\beta}{2}}}\int_{\mathbb{R}^N}\frac{|\widetilde{u}_{b_{1,k},b_{2,k}}(y)|^2}{|x-(\frac{\lambda_{2,b_{2,k},\mu_2}}{\lambda_{1,b_{1,k},\mu_1}})^{-\frac{1}{2}}y|^\beta}\widetilde{v}_{b_{1,k},b_{2,k}}(x).
\end{cases}\tag{50}$$

By the definition of $\lambda_{1,b_{1,k},\mu_1},\ \lambda_{2,b_{2,k},\mu_2},\ L_{1,k},\ L_{2,k}$, when $b_{1,k}\to0,\ b_{2,k}\to0$ and $b_{1,k}\sim b_{2,k}$, we have

$$\mu_1\frac{L_{1,k}^2}{\lambda_{1,b_{1,k},\mu_1}^{\frac{N+2-\alpha}{2}}}=1,\ \mu_2\frac{L_{2,k}^2}{\lambda_{2,b_{2,k},\mu_2}^{\frac{N+2-\alpha}{2}}}=1,\ \rho\frac{L_{2,k}^2}{\lambda_{1,b_{1,k},\mu_1}\lambda_{2,b_{2,k},\mu_2}^{\frac{N-\beta}{2}}}\to0,\ \rho\frac{L_{1,k}^2}{\lambda_{2,b_{2,k},\mu_2}\lambda_{1,b_{1,k},\mu_1}^{\frac{N-\beta}{2}}}\to0,$$

and there exists $\lambda_1^*>0$ and $\lambda_2^*>0$ such that

$$\frac{\lambda_{1,k}}{\lambda_{1,b_{1,k},\mu_1}}\to\lambda_1^*,\ \frac{\lambda_{2,k}}{\lambda_{2,b_{2,k},\mu_2}}\to\lambda_2^*\text{ as }k\to+\infty.$$

Therefore, $(\bar{u},\bar{v})$ solves that

$$\begin{cases}-\Delta u+\lambda_1^*u=(|x|^{-\alpha}*|u|^2)u & \text{in }\ \mathbb{R}^N,\\ -\Delta v+\lambda_2^*v=(|x|^{-\alpha}*|v|^2)v & \text{in }\ \mathbb{R}^N.\end{cases}\tag{51}$$

Since $0\le\bar{u},\ 0\le\bar{v}$ and $(\bar{u},\bar{v})\neq(0,0)$, it follows from the strong maximum principle that $0<\bar{u}$, $0<\bar{v}$. We know that

$$\bar{u}=(\lambda_1^*)^{\frac{N+2-\alpha}{4}}\omega((\lambda_1^*)^{\frac12}x),\quad \bar{v}=(\lambda_2^*)^{\frac{N+2-\alpha}{4}}\omega((\lambda_2^*)^{\frac12}x).$$

Texting (50) and (51) with $\widetilde{u}_{b_{1,k},b_{2,k}}-\bar{u}$, $\widetilde{v}_{b_{1,k},b_{2,k}}-\bar{v}$ respectively, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(\widetilde{u}_{b_{1,k},b_{2,k}}-\bar{u})|^2dx+\int_{\mathbb{R}^N}(\frac{\lambda_{1,k}}{\lambda_{1,b_{1,k},\mu_1}}\widetilde{u}_{b_{1,k},b_{2,k}}-\lambda_1^*\bar{u})(\widetilde{u}_{b_{1,k},b_{2,k}}-\bar{u})\\&\quad=\int_{\mathbb{R}^N}|\nabla(\widetilde{u}_{b_{1,k},b_{2,k}}-\bar{u})|^2dx+\lambda_1^*\int_{\mathbb{R}^N}|\widetilde{u}_{b_{1,k},b_{2,k}}-\bar{u}|^2=o_k(1),\end{aligned}\tag{52}$$

$$\begin{aligned}&\int_{\mathbb{R}^N}|\nabla(\widetilde{v}_{b_{1,k},b_{2,k}}-\bar{v})|^2dx+\int_{\mathbb{R}^N}(\frac{\lambda_{2,k}}{\lambda_{2,b_{2,k},\mu_2}}\widetilde{v}_{b_{1,k},b_{2,k}}-\lambda_2^*\bar{v})(\widetilde{v}_{b_{1,k},b_{2,k}}-\bar{v})\\&\quad=\int_{\mathbb{R}^N}|\nabla(\widetilde{v}_{b_{1,k},b_{2,k}}-\bar{v})|^2dx+\lambda_2^*\int_{\mathbb{R}^N}|\widetilde{v}_{b_{1,k},b_{2,k}}-\bar{v}|^2=o_k(1).\end{aligned}\tag{53}$$

Therefore

$$\begin{aligned}&(\lambda_1^*)^{\frac{2-\alpha}{2}}\|\omega\|_{L^2}^2=\|(\lambda_1^*)^{\frac{N+2-\alpha}{4}}\omega((\lambda_1^*)^{\frac12}x)\|^2_{L^2(\mathbb{R}^N)}=\|\bar{u}\|^2_{L^2(\mathbb{R}^N)}\\&=\lim_{k\to+\infty}\int_{\mathbb{R}^N}|\widetilde{u}_{b_{1,k},b_{2,k}}|^2dx=\lim_{k\to+\infty}\frac{\lambda_{1,b_{1,k},\mu_1}^{\frac N2}}{L_{1,k}^2}\int_{\mathbb{R}^N}|u_{b_{1,k},b_{2,k}}|^2dx=\|\omega\|^2_{L^2},\end{aligned}$$

$$\begin{aligned}&(\lambda_2^*)^{\frac{2-\alpha}{2}}\|\omega\|_{L^2}^2=\|(\lambda_2^*)^{\frac{N+2-\alpha}{4}}\omega((\lambda_2^*)^{\frac12}x)\|^2_{L^2(\mathbb{R}^N)}=\|\bar{v}\|^2_{L^2(\mathbb{R}^3)}\\&=\lim_{k\to+\infty}\int_{\mathbb{R}^N}|\widetilde{v}_{b_{1,k},b_{2,k}}|^2dx=\lim_{k\to+\infty}\frac{\lambda_{2,b_{2,k},\mu_2}^{\frac N2}}{L_{2,k}^2}\int_{\mathbb{R}^N}|v_{b_{1,k},b_{2,k}}|^2dx=\|\omega\|^2_{L^2},\end{aligned}$$

therefore

$$\lambda_1^*=\lambda_2^*=1.$$

From (50), (51) ,(52) and (53), we have that

$$(\widetilde{u}_{b_{1,k},b_{2,k}},\widetilde{v}_{b_{1,k},b_{2,k}})\to(\omega,\omega)\ \text{ in }H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N).$$

Moreover, as the limit function (ω,ω) is independent of the sequence that we choose, so the convergence is true for the whole sequence. Therefore

$$(\widetilde{u}_{b_{1,k},b_{2,k}},\widetilde{v}_{b_{1,k},b_{2,k}})=(L_1^{-1}u_{b_1,b_2}((\lambda_{1,b_1,\alpha_1})^{-\frac12}x),\ L_2^{-1}v_{b_1,b_2}((\lambda_{2,b_2,\alpha_2})^{-\frac12}x)\to(\omega,\omega),$$

in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ as $b_{1,k},b_{2,k}\to0$ and $b_{1,k}\sim b_{2,k}$. □

5. **Proof of Theorem 1.7.**

Lemma 5.1. *When $\beta = 2 < \alpha < \min\{N, 4\}$ and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $\alpha = 2 < \beta < \min\{N, 4\}$ and $\max\{\mu_1 b_1, \mu_2 b_2\} < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $2 < \alpha, \beta < \min\{N, 4\}$, then for every $(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$, there exists $t_{(u,v)}$ such that $t_{(u,v)} \star (u, v) \in \mathcal{P}_{b_1,b_2}$. $t_{(u,v)}$ is the unique critical point of the function $\Psi_{u,v}$ and is a strict maximum point at positive level. Moreover:*

(1) *$\Psi''_{u,v}(0) < 0$ and $P_{b_1,b_2}(u, v) < 0$ iff $t_{(u,v)} < 0$.*
(2) *$\Psi_{u,v}$ is strictly increasing in $(-\infty, t_{(u,v)})$.*
(3) *The map $(u, v) \mapsto t_{(u,v)} \in \mathbb{R}$ is of class C^1.*

Proof. **Case 1.** $\beta = 2 < \alpha < \min\{N, 4\}$ and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, since

$$\begin{aligned}\Psi_{u,v}(t) &= \frac{e^{2t}}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx - \frac{\rho e^{2t}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy\\
&\quad - \frac{\mu_1 e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy - \frac{\mu_2 e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\\
&\geq e^{2t}\left[\frac{1}{2} - \frac{\rho}{2\|Q_2\|^2_{L^2(\mathbb{R}^N)}}b_1 b_2\right]\left[\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx\right]\\
&\quad - e^{\alpha t}\left[\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy + \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\right].\end{aligned}$$

When $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, it is easy to see that $\Psi_{u,v}(t)$ has a unique critical point $t_{(u,v)}$, which is a strict maximum point at positive level. Since $t \star (u, v) \in \mathcal{P}_{b_1,b_2}$ if and only if $\Psi'_{u,v}(t) = 0$. If $(u, v) \in \mathcal{P}_{b_1,b_2}$, then 0 is a maximum point, we have that $\Psi''_{u,v}(0) \leq 0$. We claim that $\Psi''_{u,v}(0) < 0$. Assume by contradiction, that is $\Psi'_{u,v}(0) = \Psi''_{u,v}(0) = 0$, then $\alpha(\alpha - 2)\left[\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy + \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\right] = 0$, which is not possible because $(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$. Thus, $\Psi''_{u,v}(0) < 0$.

Case 2. $\alpha = 2 < \beta < \min\{N, 4\}$ and $\mu_1 b_1 + \mu_2 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$,

$$\begin{aligned}\Psi_{u,v}(t) &= \frac{e^{2t}}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx - \frac{\rho e^{\beta t}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy\\
&\quad - \frac{\mu_1 e^{2t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^2}dxdy - \frac{\mu_2 e^{2t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^2}dxdy\\
&\geq e^{2t}\left[\frac{1}{2} - \frac{\mu_1 b_1 + \mu_2 b_2}{2\|Q_2\|^2_{L^2(\mathbb{R}^N)}}\right]\left[\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2)dx\right]\\
&\quad - \frac{\rho e^{\beta t}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy.\end{aligned}$$

When $\mu_1 b_1 + \mu_2 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, it is easy to see that $\Psi_{u,v}(t)$ has a unique critical point $t_{(u,v)}$, which is a strict maximum point at positive level. Since $t\star(u, v) \in \mathcal{P}_{b_1,b_2}$ if and only if $\Psi'_{u,v}(t) = 0$. If $(u, v) \in \mathcal{P}_{b_1,b_2}$, then 0 is a maximum point, we have that $\Psi''_{u,v}(0) \leq 0$. We claim that $\Psi''_{u,v}(0) < 0$. Assume by contradiction, that is $\Psi'_{u,v}(0) = \Psi''_{u,v}(0) = 0$, then $\beta(\beta - 2)\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy = 0$, which is not possible because $(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$. Thus, $\Psi''_{u,v}(0) < 0$.

Case 3. $2<\alpha<\beta<\min\{N,4\}$ or $2<\beta<\alpha<\min\{N,4\}$. Since

$$\begin{aligned}\Psi_{u,v}(t)=&\frac{e^{2t}}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx-\frac{\mu_1 e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy\\&-\frac{\mu_2 e^{\alpha t}}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy-\frac{\rho e^{\beta t}}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy.\end{aligned}$$

It is easy to see that $\Psi_{u,v}(t)$ has a unique critical point $t_{(u,v)}$, which is a strict maximum point at positive level. Since $t\star(u,v)\in\mathcal{P}_{b_1,b_2}$ if and only if $\Psi'_{u,v}(t)=0$. If $(u,v)\in\mathcal{P}_{b_1,b_2}$, then 0 is a maximum point, we have that $\Psi''_{u,v}(0)\le 0$. We claim that $\Psi''_{u,v}(0)<0$. Assume by contradiction, that is $\Psi'_{u,v}(0)=\Psi''_{u,v}(0)=0$, then $\alpha(\alpha-2)\left[\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\right]+\beta(\beta-2)\frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy=0$, which is not possible because $(u,v)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. Thus, $\Psi''_{u,v}(0)<0$.

As in the proof of Lemma 3.4 shows that the map $(u,v)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}\mapsto t_{(u,v)}\in\mathbb{R}$ is of class C^1. Finally, $\Psi'_{u,v}(t)<0$ if and only if $t>t_{(u,v)}$, then $P_{b_1,b_2}(u,v)=\Psi'_{u,v}(0)<0$ if and only if $t_{(u,v)}<0$. □

Lemma 5.2. *When $\beta=2<\alpha<\min\{N,4\}$ and $\rho b_1b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $\alpha=2<\beta<\min\{N,4\}$ and $\mu_1b_1+\mu_2b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $2<\alpha,\beta<\min\{N,4\}$, then the set $\mathcal{P}_{b_1,b_2}$ is a C^1-submanifold of codimension 1 in $\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$, and it is a C^1-submanifold of codimension 3 in $H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$.*

Proof. The proof is similar to that of Lemma 3.3. We can check that $\mathcal{P}_{b_1,b_2}$ is a smooth manifold of codimension 1 in $\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$. By Lemma 5.1, we have $\Psi''_{u,v}(0)<0$. Since $\mathcal{P}_{b_1,b_2}$ is defined by $P_{b_1,b_2}(u,v)=0$, $G_1(u)=0$ and $G_2(v)=0$, where $G_1(u)=\|u\|^2_{L^2(\mathbb{R}^N)}-b_1^2$ and $G_2(v)=\|v\|^2_{L^2(\mathbb{R}^N)}-b_2^2$. Since P_{b_1,b_2}, G_1 and G_2 are class of C^1, the proof is completed by showing that $d(P_{b_1,b_2},G_1,G_2):H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ to $\mathbb{R}^N$ is a surjective. If this is not true, then dP_{b_1,b_2} has to be linearly dependent from dG_1 and dG_2, i.e., there exist $\bar\nu_1,\bar\nu_2\in\mathbb{R}$ such that (u,v) is a solution to Since $P_{b_1,b_2}(u,v)$, $G(u)$ and $F(u)$, are class of C^1, we only need to check that $d(P_{b_1,b_2}(u,v),G(u),F(v)):H\to\mathbb{R}^N$ is surjective.
Case 1. $\beta=2<\alpha<\min\{N,4\}$ and $\rho b_1b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$. If this not true, $dP_{b_1,b_2}(u,v)$ has to be linearly dependent from $dG(u)$ and $dF(v)$ i.e. there exist a $\nu_1,\ \nu_2\in\mathbb{R}$ such that

$$\begin{cases}2\int_{\mathbb{R}^N}\nabla u\nabla\varphi+\nu_1u\varphi=\mu_1\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(y)}{|x-y|^\alpha}u(x)\varphi(x)dydx\\+2\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(y)}{|x-y|^2}u(x)\varphi(x)dydx & \text{in }\ \mathbb{R}^N,\\2\int_{\mathbb{R}^N}\nabla v\nabla\psi+\nu_2v\psi=\mu_2\alpha\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(y)}{|x-y|^\alpha}v(x)\psi(x)dydx\\+2\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(y)}{|x-y|^2}v(x)\psi(x)dydx & \text{in }\ \mathbb{R}^N,\end{cases}$$

for every $(\varphi,\psi)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, for every $(\varphi,\psi)\in H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$, so

$$\begin{cases}-2\Delta u+2\nu_1u=\mu_1\alpha(|x|^{-\alpha}*|u|^2)u+2\rho(|x|^{-2}*|v|^2)u & \text{in }\ \mathbb{R}^N,\\-2\Delta v+2\nu_2v=\mu_2\alpha(|x|^{-\alpha}*|v|^2)v+2\rho(|x|^{-2}*|u|^2)v & \text{in }\ \mathbb{R}^N,\end{cases}$$

by the similar arguments as (9), (8) and (10), we know that the Pohozaev identity for above system is

$$\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx=\frac{\mu_1\alpha^2}{8}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy$$
$$+\frac{\mu_2\alpha^2}{8}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy.$$

that is $\Psi''_{u,v}(0)=0$, which contradicts with $\Psi''_{u,v}(0)<0$. Hence, $\mathcal{P}_{b_1,b_2}(u,v)$ is a natural constraint.

Case 2. $\alpha=2<\beta<\min\{N,4\}$ and $\max\{\mu_1 b_1,\mu_2 b_2\}<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$. By the similar arguments as Case 1, we get $\Psi''_{u,v}(0)=0$, which contradicts with $\Psi''_{u,v}(0)<0$. Hence, $\mathcal{P}_{b_1,b_2}(u,v)$ is a natural constraint.

Case 3. $2<\alpha,\beta<\min\{N,4\}$. By the similar arguments as Case 1, we get $\Psi''_{u,v}(0)=0$, which contradicts with $\Psi''_{u,v}(0)<0$. Hence, $\mathcal{P}_{b_1,b_2}(u,v)$ is a natural constraint. □

Lemma 5.3. *When $\beta=2<\alpha<\min\{N,4\}$ and $\rho b_1b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $\alpha=2<\beta<\min\{N,4\}$ and $\mu_1b_1+\mu_2b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $2<\alpha,\beta<\min\{N,4\}$, then*

$$m(b_1,b_2):=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}J(u,v)>0.$$

Proof. **Case 1.** $\beta=2<\alpha<\min\{N,4\}$ and $\rho b_1b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$. If $(u,v)\in\mathcal{P}_{b_1,b_2}$, then by (14) and (15), we have

$$\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx=\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy$$
$$+\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy$$
$$\leq\left[\frac{\rho}{\|Q_2\|^2_{L^2(\mathbb{R}^N)}}b_1b_2\right]\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]$$
$$+\alpha\frac{\mu_1b_1^{4-\alpha}+\mu_2b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}.$$

Moreover, $\rho b_1b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$ and $\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\neq 0$ (since $(u,v)\in \mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$), we get

$$\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\geq C>0.$$

So

$$m(b_1,b_2)=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}J_\beta(u,v)$$
$$=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}(\frac{1}{2}-\frac{1}{\alpha})\left[\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^3)}-\rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy]\right]$$
$$\geq\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}(\frac{1}{2}-\frac{1}{\alpha})\left[1-\left[\frac{\rho}{\|Q_2\|^2_{L^2(\mathbb{R}^N)}}b_1b_2\right]\right]\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\geq C>0.$$

Case 2. $\alpha=2<\beta<\min\{N,4\}$ and $\mu_1b_1+\mu_2b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$. If $(u,v)\in\mathcal{P}_{b_1,b_2}$, then by (14) and (15), we have

$$\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx=\frac{\mu_1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^2}dxdy$$

$$
\begin{aligned}
&+\frac{\mu_2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^2}dxdy+\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy\\
\leq\ &\frac{\mu_1 b_1+\mu_2 b_2}{\|Q_2\|^2_{L^2(\mathbb{R}^N)}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]\\
&+\beta\frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}}.
\end{aligned}
$$

Moreover, $\mu_1 b_1+\mu_2 b_2<\|Q_2\|^2_{L^2(\mathbb{R}^N)}$ and $\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\neq 0$ (since $(u,v)\in \mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$), we get

$$
\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\geq C>0.
$$

So

$$
\begin{aligned}
m(b_1,b_2)&=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}J_\beta(u,v)\\
&=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}(\frac{1}{2}-\frac{1}{\beta})\Big[\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^3)}\\
&\quad-\left[\frac{\mu_1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^2}dxdy+\frac{\mu_2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^2}dxdy\right]\Big]\\
&\geq\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}(\frac{1}{2}-\frac{1}{\beta})\left[1-\frac{\mu_1 b_1+\mu_2 b_2}{\|Q_2\|^2_{L^2(\mathbb{R}^N)}}\right]\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|\nabla v\|^2_{L^2(\mathbb{R}^3)}\geq C>0.
\end{aligned}
$$

Case 3. $2<\alpha<\beta<\min\{N,4\}$ or $2<\beta<\alpha<\min\{N,4\}$. If $(u,v)\in\mathcal{P}_{b_1,b_2}$, then by (14) and (15), we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx=\frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy\\
&\quad+\frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy+\frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy\\
&\leq\frac{\rho\beta}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}}b_1^{\frac{4-\beta}{2}}b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}}\\
&\quad+\alpha\frac{\mu_1 b_1^{4-\alpha}+\mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}.
\end{aligned}
$$

Moreover, $\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\neq 0$ (since $(u,v)\in\mathrm{T}_{b_1}\times\mathrm{T}_{b_2}$), we get

$$
\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^N)}\geq C>0.
$$

If $2<\beta<\alpha<\min\{N,4\}$, we have

$$
\begin{aligned}
m(b_1,b_2)&=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}J_\beta(u,v)\\
&=\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}\Big[(\frac{1}{2}-\frac{1}{\beta})\|\nabla u\|^2_{L^2(\mathbb{R}^N)}+\|\nabla v\|^2_{L^2(\mathbb{R}^3)}\\
&\quad+(\frac{\alpha}{\beta}-1)\left[\frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy+\frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy\right]\Big]\\
&\geq\inf_{(u,v)\in\mathcal{P}_{b_1,b_2}}(\frac{1}{2}-\frac{1}{\beta})\|\nabla u\|^2_{L^2(\mathbb{R}^3)}+\|\nabla v\|^2_{L^2(\mathbb{R}^3)}\geq C>0.
\end{aligned}
$$

If $2 < \alpha < \beta < \min\{N, 4\}$, we have

$$\begin{aligned} m(b_1, b_2) &= \inf_{(u,v)\in\mathcal{P}_{b_1,b_2}} J_\beta(u, v) \\ &= \inf_{(u,v)\in\mathcal{P}_{b_1,b_2}} \Big[(\frac{1}{2} - \frac{1}{\alpha})\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v\|^2_{L^2(\mathbb{R}^3)} \\ &\quad + (\frac{\beta}{\alpha} - 1)\frac{\rho}{2} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^\beta} dxdy \Big] \\ &\geq \inf_{(u,v)\in\mathcal{P}_{b_1,b_2}} (\frac{1}{2} - \frac{1}{\alpha})\|\nabla u\|^2_{L^2(\mathbb{R}^3)} + \|\nabla v\|^2_{L^2(\mathbb{R}^3)} \geq C > 0. \end{aligned}$$

Thus,

$$m(b_1, b_2) := \inf_{(u,v)\in\mathcal{P}_{b_1,b_2}} J(u, v) > 0.$$

□

Lemma 5.4. *When $\beta = 2 < \alpha < \min\{N, 4\}$ and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $\alpha = 2 < \beta < \min\{N, 4\}$ and $\mu_1 b_1 + \mu_2 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $2 < \alpha, \beta < \min\{N, 4\}$, then there exists $k > 0$ sufficiently small such that*

$$0 < \sup_{\overline{A}_k} J(u, v) < m(b_1, b_2) \text{ and } (u, v) \in \overline{A}_k \Rightarrow J(u, v), P_{b_1,b_2}(u, v) > 0,$$

where $A_k := \{(u, v) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2} : \|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla v\|^2_{L^2(\mathbb{R}^N)} < k\}$.

Proof. **Case 1.** $\beta = 2 < \alpha < \min\{N, 4\}$ and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$. From (14), (15) and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, we get

$$\begin{aligned} J(u, v) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^\alpha} dxdy \\ &\quad - \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^\alpha} dxdy - \frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^2} dxdy \\ &\geq \left(\frac{1}{2} - \left[\frac{\rho}{2\|Q_2\|^2_{L^2(\mathbb{R}^N)}} b_1 b_2\right]\right)\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla v|^2)dx \\ &\quad - \frac{\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}} \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx\right)^{\frac{\alpha}{2}} > 0, \end{aligned}$$

$$\begin{aligned} P(u, v) &= \int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx - \frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)u^2(y)}{|x-y|^2} dxdy \\ &\quad - \frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{v^2(x)v^2(y)}{|x-y|^2} dxdy - \rho\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{u^2(x)v^2(y)}{|x-y|^2} dxdy \\ &\geq \left(1 - \left[\frac{\rho}{\|Q_2\|^2_{L^2(\mathbb{R}^N)}} b_1 b_2\right]\right)\int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla v|^2)dx \\ &\quad - \alpha\frac{\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}} \left(\int_{\mathbb{R}^N} (|\nabla u|^2 + |\nabla v|^2)dx\right)^{\frac{\alpha}{2}} > 0, \end{aligned}$$

Case 2. $\alpha = 2 < \beta < \min\{N,4\}$ and $\mu_1 b_1 + \mu_2 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$. From (14), (15), we get

$$\begin{aligned} J(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\ &\quad - \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy \\ &\geq \left(\frac{1}{2} - \frac{\mu_1 b_1 + \mu_2 b_2}{2\|Q_2\|^2_{L^2(\mathbb{R}^N)}}\right)\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx \\ &\quad - \frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}} b_1^{\frac{4-\beta}{2}} b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}} > 0, \end{aligned}$$

$$\begin{aligned} P(u,v) &= \int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\mu_1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\ &\quad - \frac{\mu_2}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^2}dxdy \\ &\geq \left(1 - \left[\frac{\rho}{\|Q_2\|^2_{L^2(\mathbb{R}^N)}} b_1 b_2\right]\right)\int_{\mathbb{R}^3}(|\nabla u|^2+|\nabla v|^2)dx \\ &\quad - \beta\frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}} b_1^{\frac{4-\beta}{2}} b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}} > 0, \end{aligned}$$

Case 3. $2 < \alpha, \beta < \min\{N,4\}$. From (14), (15), we get

$$\begin{aligned} J(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\mu_1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\ &- \frac{\mu_2}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\rho}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}} b_1^{\frac{4-\beta}{2}} b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}} \\ &- \frac{\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}, \end{aligned}$$

$$\begin{aligned} P(u,v) &= \int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\mu_1\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)u^2(y)}{|x-y|^\alpha}dxdy \\ &- \frac{\mu_2\alpha}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{v^2(x)v^2(y)}{|x-y|^\alpha}dxdy - \frac{\rho\beta}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{u^2(x)v^2(y)}{|x-y|^\beta}dxdy \\ &\geq \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx - \frac{\rho\beta}{2\|Q_\beta\|^2_{L^2(\mathbb{R}^N)}} b_1^{\frac{4-\beta}{2}} b_2^{\frac{4-\beta}{2}}\left[\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right]^{\frac{\beta}{2}} \\ &- \alpha\frac{\mu_1 b_1^{4-\alpha} + \mu_2 b_2^{4-\alpha}}{2\|Q_\alpha\|^2_{L^2(\mathbb{R}^N)}}\left(\int_{\mathbb{R}^N}(|\nabla u|^2+|\nabla v|^2)dx\right)^{\frac{\alpha}{2}}. \end{aligned}$$

If $u \in \overline{A}_k$ with k small enough. If necessary replacing k with a smaller quantity. From Lemma 5.3, we have $m(b_1, b_2) > 0$, so we also have

$$J(u,v) \leq \int_{\mathbb{R}^3} (|\nabla u|^2 + |\nabla v|^2)dx < m(b_1, b_2).$$

The proof is now complete. □

Now, we are ready to show that the infimum is attained by nontrivial positive radial functions.

Lemma 5.5. *Let $\mu_i, b_i > 0 (i = 1, 2)$, $\beta = 2 < \alpha < \min\{N, 4\}$ and $\rho b_1 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $\alpha = 2 < \beta < \min\{N, 4\}$ and $\mu_1 b_1 + \mu_2 b_2 < \|Q_2\|^2_{L^2(\mathbb{R}^N)}$, or $2 < \alpha, \beta < \min\{N, 4\}$, then $m(b_1, b_2)$ can be achieved by some function $(u_{b_1}, v_{b_2}) \in \mathrm{T}_{b_1} \times \mathrm{T}_{b_2}$ which is real valued, positive, radially symmetric and radially decreasing.*

Proof. By similar arguments as in Lemma 3.12, or similar arguments as in the proof of Theorem 1.6 in [43], we can also find a radial Palais-Smale sequence for $J\big|_{\mathrm{T}_{b_1} \times \mathrm{T}_{b_2}}$ at level $m(b_1, b_2)$ such that $P(u_n, v_n) \to 0$ and $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^N$. The rest of the proof is similar to that of Lemma 3.11 and Lemma 3.12. □

P*roof of Theorem 1.7.* The proof is completed by using Lemma 5.5. □

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