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Research paper

Convergence of least energy sign-changing solutions for logarithmic Schrödinger equations on locally finite graphs

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ABSTRACT

In this paper, we study the following logarithmic Schrödinger equation

$$-\Delta u + \lambda a(x)u = u \log u^2 \quad \text{in } V$$

on a connected locally finite graph $G = (V, E)$, where Δ denotes the graph Laplacian, $\lambda > 0$ is a constant, and $a(x) \geq 0$ represents the potential. Using variational techniques in combination with the Nehari manifold method based on directional derivative, we can prove that, there exists a constant $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, the above problem admits a least energy sign-changing solution u_λ. Moreover, as $\lambda \to +\infty$, we prove that the solution u_λ converges to a least energy sign-changing solution of the following Dirichlet problem

$$\begin{cases} -\Delta u = u \log u^2 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = \{x \in V : a(x) = 0\}$ is the potential well.

1. Introduction and main results

Theory of network (or graph) has a wide range of applications in various fields such as signal processing, image processing, data clustering and machine learning. (For example, see [1–3].) A graph $G = (V, E)$, where V denotes the vertex set and E denotes the edge set, is said to be locally finite if for any $x \in V$, there are only finite $y \in V$ such that $xy \in E$. A graph is connected if any two vertices x and y can be connected via finite edges. For any $xy \in E$, we assume that its weight $\omega_{xy} > 0$ and $\omega_{xy} = \omega_{yx}$. The degree of $x \in V$ is defined by $deg(x) = \sum_{y\sim x} \omega_{xy}$, where we write $y \sim x$ if

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$xy \in E$. The distance $d(x, y)$ of two vertices $x, y \in V$ is defined by the minimal number of edges which connect these two vertices. The measure $\mu : V \to \mathbb{R}^+$ is defined to be a finite positive function on G.

In recent years, there have been many studies on the existence and multiplicity of solutions to nonlinear elliptic equations on discrete graphs. For example, see [4–11] and their references. In [7], Grigor'yan, Lin and Yang studied nonlinear Schrödinger equations

$$-\Delta u + b(x)u = f(x, u) \quad \text{in } V \tag{1.1}$$

on a connected locally finite graph G. By applying the mountain pass theorem, they established the existence of strictly positive solutions of (1.1) when f satisfies the so-called Ambrosetti–Rabinowitz ((AR) for short) condition, and the potential $b : V \to \mathbb{R}^+$ has a positive lower bound and satisfies one of the following hypotheses:

(B_1) $b(x) \to +\infty$ as $d(x, x_0) \to +\infty$ for some fixed $x_0 \in V$;

(B_2) $1/b(x) \in L^1(V)$.

In [11], Zhang and Zhao established the existence and convergence (as $\lambda \to +\infty$) of ground state solutions for Eq. (1.1), when $b(x) = \lambda a(x) + 1$ and $f(x, u) = |u|^{p-1}u$, where $a(x) \geq 0$ satisfies (B_1) and the potential well $\Omega = \{x \in V : a(x) = 0\}$ is a non-empty connected and bounded domain in V. Similar results for p-Laplacian equations and biharmonic equations on locally finite graphs can be found in [12,13].

In this paper, we consider the following logarithmic Schrödinger equation

$$-\Delta u + \lambda a(x)u = u \log u^2 \quad \text{in } V \tag{1.2}$$

on a connected locally finite graph $G = (V, E)$, where the parameter $\lambda > 0$. We recall that the logarithmic Schrödinger equation in the Euclidean space

$$-\Delta u + \lambda b(x)u = u \log u^2 \quad \text{in } \mathbb{R}^N \tag{1.3}$$

has recently received much attention. For example, see [14–22] and references therein. Logarithmic nonlinear problems have a wide range of applications in fields such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, Bose–Einstein condensation and etc. Interested readers may refer to [23–25].

Different approaches have been developed to study the existence and multiplicity of solutions for nonlinear Schrödinger equations with logarithmic nonlinearities. Cazenave [14] worked in an Orlicz space endowed with a Luxemburg type norm in order to make the associated energy functional of Eq. (1.3) to be C^1. Squassina and Szulkin [20] studied the existence of multiple solutions by using non-smooth critical point theory (see also [15,16,18]). Tanaka and Zhang [21] applied the penalization technique to study multi-bump solutions of Eq. (1.3). For the idea of penalization, see also [17,26,27]. In [22], Wang and Zhang proved that the ground state solutions of the power-law scalar field equations $-\Delta u + \lambda u = |u|^{p-2}u$, as $p \downarrow 2$, converge to the ground state solution of the logarithmic-law equation $-\Delta u = \lambda u \log u^2$. Recently, several results are devoted to studying the sign-changing solutions. Chen and Tang [28] established the existence of least energy sign-changing solutions of some logarithmic Schrödinger equation in bounded domains of $\mathbb{R}^N$ using the constraint variational method. Shuai [19] obtained the existence of least energy sign-changing solutions for Eq. (1.3) under different types of potentials by using the directional derivative and constrained minimization method. Zhang and Wang investigated, in [29], the existence and concentration behaviors of sign-changing solutions for logarithmic scalar field equations in the semiclassical setting. Ji [30] established the existence and multiplicity of multi-bump type nodal solutions for Eq. (1.3). For more studies on logarithmic nonlinear equations, one may refer to [14–16,18,20,31,32] and their references.

The goal of this work is to show the existence of least energy sign-changing solutions of (1.2) and their asymptotic behavior as $\lambda \to +\infty$. To the best of our knowledge, there is no result on sign-changing solutions for logarithmic Schrödinger problems on locally finite graphs.

In the sequel of this paper, we make the assumption that there exists a constant $\mu_{\min} > 0$ such that the measure $\mu(x) \geq \mu_{\min} > 0$ for all $x \in V$. As for the potential $a = a(x)$, we assume that:

(A_1) $a(x) \geq 0$ and the potential well $\Omega = \{x \in V : a(x) = 0\}$ is a non-empty, connected and bounded domain in V;

(A_2) there exists $M > 0$ such that the volume of the set D_M is finite, namely,

$$Vol(D_M) = \sum_{x \in D_M} \mu(x) < \infty,$$

where $D_M = \{x \in V : a(x) < M\}$.

To explain our result, we first introduce some necessary notations. For any function $u : V \to \mathbb{R}$, the graph Laplacian of u is defined by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} \left(u(y) - u(x)\right). \tag{1.4}$$

The integral of u over V is defined by $\int_V u d\mu = \sum_{x\in V} \mu(x)u(x)$, and the gradient form of the two functions u, v on V is defined by

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy} (u(y) - u(x)) (v(y) - v(x)) . \tag{1.5}$$

Write $\Gamma(u) = \Gamma(u, u)$, and sometimes we use $\nabla u \nabla v$ to replace $\Gamma(u, v)$. The length of the gradient of u is defined by

$$|\nabla u|(x) = \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy} (u(y) - u(x))^2 \right)^{1/2} . \tag{1.6}$$

Denote by $C_c(V)$ the set of all functions with compact support, and let $H^1(V)$ be the completion of $C_c(V)$ under the norm

$$\|u\|_{H^1(V)} = \left(\int_V \left(|\nabla u|^2 + u^2\right) d\mu \right)^{1/2} .$$

Then, $H^1(V)$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_V (\Gamma(u, v) + uv)\, d\mu, \quad \forall u,\ v \in H^1(V).$$

We write $\|u\|_p = \left(\int_V |u|^p d\mu\right)^{1/p}$ for $p \in [1, +\infty)$ and $\|u\|_{L^\infty} = \sup_{x\in V} |u(x)|$.

For each $\lambda > 0$ we introduce a space

$$\mathcal{H}_\lambda = \left\{ u \in H^1(V) : \int_V \lambda a(x) u^2 d\mu < +\infty \right\}$$

with norm

$$\|u\|^2_{\mathcal{H}_\lambda} \doteq \int_V \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu,$$

which is induced by the inner product

$$\langle u, v\rangle_{\mathcal{H}_\lambda} = \int_V (\Gamma(u, v) + (\lambda a(x) + 1)uv)\, d\mu,\ \forall u,\ v \in \mathcal{H}_\lambda.$$

Clearly, $\mathcal{H}_\lambda$ is also a Hilbert space.

Note that Eq. (1.2) is formally associated with the energy functional $J_\lambda :\ H^1(V) \to \mathbb{R} \cup \{+\infty\}$ given by

$$J_\lambda(u) = \frac{1}{2} \int_V \left(|\nabla u|^2 + (\lambda a(x) + 1)u^2\right) d\mu - \frac{1}{2} \int_V u^2 \log u^2 d\mu. \tag{1.7}$$

Clearly, J_λ fails to be C^1 in $H^1(V)$. In fact, for some $G = (V, E)$ with suitable measure μ, there exists $u \in H^1(V)$ but $\int_V u^2 \log u^2 d\mu = -\infty$. (For example, see [33].)

When $a(x)$ satisfies (A_1) and (A_2), we consider the functional J_λ in (1.7) on the set

$$\mathcal{D}_\lambda = \left\{ u \in \mathcal{H}_\lambda : \int_V u^2 |\log u^2| d\mu < \infty \right\} .$$

That is,

$$J_\lambda(u) = \frac{1}{2} \|u\|^2_{\mathcal{H}_\lambda} - \frac{1}{2} \int_V u^2 \log u^2 d\mu, \quad \forall u \in \mathcal{D}_\lambda.$$

Define the Nehari manifold and sign-changing Nehari set respectively by

$$\mathcal{N}_\lambda = \left\{ u \in \mathcal{D}_\lambda \setminus \{0\} : J'_\lambda(u) \cdot u = 0 \right\},$$

$$\mathcal{M}_\lambda = \left\{ u \in \mathcal{D}_\lambda : u^\pm \neq 0 \text{ and } J'_\lambda(u) \cdot u^+ = J'_\lambda(u) \cdot u^- = 0 \right\},$$

where $u^+ = \max\{u,\ 0\}$ and $u^- = \min\{u,\ 0\}$. Clearly, $\mathcal{N}_\lambda$ contains all the nontrivial solutions of Eq. (1.2) and the set $\mathcal{M}_\lambda$ contains all the sign-changing solutions of Eq. (1.2). Set

$$c_\lambda = \inf_{u\in \mathcal{N}_\lambda} J_\lambda(u), \quad m_\lambda = \inf_{u\in \mathcal{M}_\lambda} J_\lambda(u).$$

Our main results are as follows.

Theorem 1.1. *Suppose that $G = (V, E)$ is a connected locally finite graph and the potential $a : V \to \mathbb{R}$ satisfies conditions (A_1) and (A_2). Then, there exists a constant $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, Eq. (1.2) admits a least energy sign-changing solution $u_\lambda \in \mathcal{D}_\lambda$ such that $J_\lambda(u_\lambda) = m_\lambda$. Moreover, $m_\lambda > 2c_\lambda$.*

We recall that $D \subset V$ is a bounded domain if the distance $d(x, y)$ between any $x, y \in D$ is uniformly bounded. The boundary of D is defined by

$$\partial D \doteq \{y \notin D : \text{there exists } x \in D \text{ such that } xy \in E\}$$

and the interior of D is denoted by D°. Obviously, we have $D^\circ = D$.

Set $\Omega = \{x \in V : a(x) = 0\}$. Let $H_0^1(\Omega)$ be the completion of $C_c(\Omega)$ under the norm

$$\|u\|_{H_0^1(\Omega)} = \left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu\right)^{1/2}.$$

Then, $H_0^1(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\Omega\cup\partial\Omega} \Gamma(u, v) d\mu + \int_\Omega uv d\mu, \quad \forall u,\ v \in H_0^1(\Omega).$$

Consider the following Dirichlet problem

$$\begin{cases} -\Delta u = u \log u^2 & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

The energy functional $J_\Omega : H_0^1(\Omega) \to \mathbb{R}$ associated with problem (1.8) is given by

$$J_\Omega(u) \doteq \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{1}{2}\int_\Omega u^2 \log u^2 d\mu, \quad \forall u \in H_0^1(\Omega).$$

Define

$$\mathcal{N}_\Omega = \left\{u \in H_0^1(\Omega) \setminus \{0\} : J'_\Omega(u) \cdot u = 0\right\},$$

$$\mathcal{M}_\Omega = \left\{u \in H_0^1(\Omega) : u^\pm \neq 0 \text{ and } J'_\Omega(u) \cdot u^+ = J'_\Omega(u) \cdot u^- = 0\right\}.$$

Set

$$c_\Omega = \inf_{u\in\mathcal{N}_\Omega} J_\lambda(u), \quad m_\Omega = \inf_{u\in\mathcal{M}_\Omega} J_\Omega(u).$$

Similar to Theorem 1.1, problem (1.8) also has a least energy sign-changing solution.

Theorem 1.2. *Let $G = (V, E)$ be a connected locally finite graph. Assume $\Omega = \{x \in V : a(x) = 0\}$ is a non-empty, connected and bounded domain in V. Then problem (1.8) admits a least energy sign-changing solution $u_0 \in H_0^1(\Omega)$ such that $J_\Omega(u_\Omega) = m_\Omega$. Moreover, $m_\Omega > 2c_\Omega$.*

Finally, we prove that the least energy sign-changing solution u_λ converges to a least energy sign-changing solution of problem (1.8).

Theorem 1.3. *Under the assumptions of Theorem 1.1, we conclude that for any sequence $\lambda_k \to +\infty$, up to a subsequence, the corresponding least energy sign-changing solution u_{λ_k} of Eq. (1.2) converges in $H^1(V)$ to a least energy sign-changing solution of problem (1.8).*

One of the main challenges in proving Theorem 1.1–1.3 is to deal with the logarithmic term in Eq. (1.2). In the Euclidean space, the logarithmic Sobolev inequality plays a significant role in studying logarithmic Schrödinger equation (see [19,20,27] etc.). While, on discrete graphs, the logarithmic Sobolev inequality is only available under a positive curvature condition, which requires the measure μ to be finite (see [34] for details). In our case, the measure μ has a uniform positive lower bound, which violates the positive curvature condition. To overcome this difficulty, we will develop new and delicate arguments which do not rely on the logarithmic Sobolev inequality.

Furthermore, the associated energy functional with Eq. (1.2) is not well-defined in the setting of discrete graphs (see [33]). Inspired by ideas in [19,22], we will restrict $u^2 \log u^2 \in L^1(V)$ which is suitable for finite energy solutions. However, new challenge arises since the techniques in [19,22] are not applicable here because the graph Laplacian operator is non-local. To be precise, in [19], the following decomposition

$$I(u) = I(u^+) + I(u^-), \quad \left\langle I'(u), u\right\rangle = \left\langle I'(u^+), u^+\right\rangle + \left\langle I'(u^-), u^-\right\rangle, \tag{1.9}$$

plays a key role in studying nodal solutions. Here I is the corresponding energy functional. But in our case, such a decomposition does not hold. Actually, by a direct computation, it follows that for each $u \in \mathcal{D}_\lambda \setminus \{0\}$,

$$J_\lambda(u) = J_\lambda(u^+) + J_\lambda(u^-) - \frac{1}{2}K_V(u),$$

$$J'_\lambda(u) \cdot u^\pm = J'_\lambda(u^\pm) \cdot u^\pm - \frac{1}{2}K_V(u),$$

where $K_V(u) = \sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[u^+(x)u^-(y)+u^-(x)u^+(y)\right] < 0$, see Section 2 for details. Clearly, $J_\lambda(u) \neq J_\lambda(u^+)+J_\lambda(u^-)$ and $\langle J'_\lambda(u), u\rangle \neq \langle J'_\lambda(u^+), u^+\rangle + \langle J'_\lambda(u^-), u^-\rangle$, which imply that (1.9) fails. Motivated by [35,36], we will develop new variational arguments involving nonlocal operator based on directional derivative to the logarithmic Schrödinger equation on locally finite graphs.

The paper is organized as follows. In Section 2, we introduce some notations, definitions and preliminary lemmas. In Section 3, we apply the Nehari manifold method to prove the existence of least energy sign-changing solution of Eq. (1.2) and the Dirichlet problem (1.8). In Section 4, we give the proof of Theorem 1.3.

2. Some preliminary results

2.1. Some definitions

To prove Theorem 1.1, we need the definition of the directional derivative.

Definition 2.1. Given $u \in \mathcal{D}_\lambda$ and $\phi \in C_c(V)$, the derivative of J_λ in the direction ϕ at u, denoted by $J'_\lambda(u)\cdot\phi$, is defined as $\lim_{t\to 0^+}\frac{1}{t}\left[J_\lambda(u+t\phi)-J_\lambda(u)\right]$.

It is easy to check that

$$J'_\lambda(u)\cdot\phi = \int_V \left(\Gamma(u,\phi)+(\lambda a(x)+1)\,u\phi\right)d\mu - \int_V u\phi\log u^2 d\mu.$$

In fact, it suffices to show the following

$$\begin{aligned}
&\lim_{t\to 0^+}\frac{1}{t}\left[\int_V \left(\Gamma(u+t\phi)-\Gamma(u)\right)d\mu\right]\\
&=\lim_{t\to 0^+}\frac{1}{t}\left[\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(((u+t\phi)(y)-(u+t\phi(x)))^2-(u(y)-u(x))^2\right)\right]\\
&=\lim_{t\to 0^+}\frac{1}{2t}\left[\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(t^2\left(\phi(y)-\phi(x)\right)^2+2t\left(u(y)-u(x)\right)\left(\phi(y)-\phi(x)\right)\right)\right]\\
&=\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)\left(\phi(y)-\phi(x)\right)\\
&=2\int_V \Gamma(u,\phi)d\mu.
\end{aligned}$$

Definition 2.2.

(1) For $u, v \in \mathcal{D}_\lambda$, we define

$$J'_\lambda(u)\cdot v := \int_V \left(\Gamma(u,v)+\lambda a(x)uv\right)d\mu - \int_V uv\log u^2 d\mu.$$

Clearly, $\int_V uv\log u^2 d\mu$ is well-defined for $u, v \in \mathcal{D}_\lambda$.

(2) We say that $u \in \mathcal{H}_\lambda$ is a critical point of J_λ if $u \in \mathcal{D}_\lambda$ and $J'_\lambda(u)\cdot v = 0$ for all $v \in \mathcal{D}_\lambda$. We also say that $d_\lambda \in \mathbb{R}$ is a critical value for J_λ if there exists a critical point $u \in \mathcal{H}_\lambda$ such that $J_\lambda(u) = d_\lambda$.

It is easily seen that, u is a weak solution to Eq. (1.2) if and only if u is a critical point of J_λ.

For the functional J_Ω of problem (1.8), note that, for any $0 < \varepsilon < 1$, there exists $C_\varepsilon > 0$ such that

$$|u^2\log u^2| \le C_\varepsilon(|u|^{2-\varepsilon}+|u|^{2+\varepsilon}).$$

Since $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $p \in [1,+\infty]$, by a standard argument, we have $J_\Omega \in C^1(H^1_0(\Omega),\mathbb{R})$ and

$$J'_\Omega(u)\cdot v = \int_{\Omega\cup\partial\Omega}\nabla u\nabla v d\mu - \int_\Omega uv\log u^2 d\mu, \forall u, v \in H^1_0(\Omega).$$

Clearly, u is a weak solution to problem (1.8) if and only if u is a critical point of J_Ω.

Lemma 2.3. *If $u \in \mathcal{D}_\lambda$ is a weak solution of Eq. (1.2), then u is a point-wise solution of Eq. (1.2).*

Proof. If $u \in \mathcal{D}_\lambda$ is a weak solution of (1.2), then for any $\varphi \in \mathcal{D}_\lambda$, there holds

$$\int_V \left(\Gamma(u,\varphi)+\lambda a(x)u\varphi\right)d\mu = \int_V u\varphi\log u^2 d\mu.$$

Using $C_c(V)$ is dense in $\mathcal{D}_\lambda$ and ω_{xy} is symmetric, for any $\varphi \in C_c(V)$, by integration by parts, we have

$$\begin{aligned}\int_V \Gamma(u,\varphi)d\mu =&\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)\left(\varphi(y)-\varphi(x)\right)\\ =&\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)\varphi(y)-\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)\varphi(x)\\ =&-\frac{1}{2}\sum_{y\in V}\sum_{x\sim y}\omega_{xy}\left(u(y)-u(x)\right)\varphi(x)-\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)\varphi(x)\\ =&-\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left(u(y)-u(x)\right)\varphi(x)\\ =&-\int_V \Delta u\varphi d\mu,\end{aligned}$$

which gives

$$\int_V \left(-\Delta u+\lambda a(x)u\right)\varphi d\mu=\int_V u\varphi\log u^2 d\mu,\quad \forall\varphi\in C_c(V). \tag{2.1}$$

For any fixed $y \in V$, take a test function $\varphi : V \to \mathbb{R}$ in (2.1) with

$$\varphi(x)=\begin{cases}1, & x=y,\\ 0, & x\neq y.\end{cases}$$

Clearly, $\varphi \in \mathcal{D}_\lambda$, and $-\Delta u(y)+\lambda a(y)u(y)-u(y)\log\left(u(y)\right)^2=0$. Since y is arbitrary, we conclude that u is a point-wise solution of (1.2). □

Similarly, we obtain

Lemma 2.4. *If $u \in H_0^1(\Omega)$ is a weak solution of problem (1.8), then u is a point-wise solution of problem (1.8).*

Next, we have the following observations:

$$\begin{aligned}&\int_V \Gamma(u^++u^-)d\mu\\ =&\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[(u^++u^-)(y)-(u^++u^-)(x)\right]^2\\ =&\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[\left(u^+(y)-u^+(x)\right)^2+\left(u^-(y)-u^-(x)\right)^2-2\left[u^+(x)u^-(y)+u^-(x)u^+(y)\right]\right]\\ =&\int_V \Gamma(u^+)d\mu+\int_V \Gamma(u^-)d\mu-K_V(u),\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\int_V \Gamma(u^++u^-,u^+)d\mu\\ =&\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[(u^++u^-)(y)-(u^++u^-)(x)\right]\left[u^+(y)-u^+(x)\right]\\ =&\frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[|u^+(y)|^2-\left[u^+(x)u^-(y)+u^-(x)u^+(y)\right]\right]\\ =&\int_V \Gamma(u^+)d\mu-\frac{1}{2}K_V(u).\end{aligned} \tag{2.3}$$

Similarly, we have

$$\int_V \Gamma(u^++u^-,u^-)d\mu=\int_V \Gamma(u^-)d\mu-\frac{1}{2}K_V(u). \tag{2.4}$$

Then, for each $u \in \mathcal{D}_\lambda$, we have

$$\begin{aligned}&J_\lambda(u)=J_\lambda(u^+)+J_\lambda(u^-)-\frac{1}{2}K_V(u),\\ &J'_\lambda(u)\cdot u^\pm=J'_\lambda(u^\pm)\cdot u^\pm-\frac{1}{2}K_V(u),\end{aligned}$$

and for each $u \in H_0^1(\Omega)$,

$$J_\Omega(u) = J_\Omega(u^+) + J_\Omega(u^-) - \frac{1}{2}K_\Omega(u),$$
$$J'_\Omega(u)\cdot u^\pm = J'_\Omega(u^\pm)\cdot u^\pm - \frac{1}{2}K_\Omega(u),$$

where $K_\Omega(u) := \sum_{x\in\Omega\cup\partial\Omega}\sum_{y\sim x}\omega_{xy}\left[u^+(x)u^-(y) + u^-(x)u^+(y)\right]$.

2.2. Sobolev embedding

In this subsection, we establish a Sobolev embedding result.

Lemma 2.5. *If $\mu(x) \geq \mu_{\min} > 0$ and $a(x)$ satisfies $(A_1) - (A_2)$, then there exist a constant $\lambda_0 > 0$ such that, for all $\lambda \geq \lambda_0$, the space $\mathcal{H}_\lambda$ is compactly embedded into $L^p(V)$ for all $2 \leq p \leq +\infty$.*

Proof. For all $\lambda > 0$, at any vertex $x_0 \in V$, by (A_1) we have

$$\begin{aligned}
\|u\|^2_{\mathcal{H}_\lambda} &= \int_V \left(|\nabla u|^2 + (\lambda a(x)+1)u^2\right) d\mu \\
&\geq \int_V u^2 d\mu \\
&= \sum_{x\in V}\mu(x)u^2(x) \\
&\geq \mu_{\min}u^2(x_0),
\end{aligned}$$

which implies that $|u(x_0)| \leq \sqrt{\frac{1}{\mu_{\min}}}\|u\|_{\mathcal{H}_\lambda}$. Thus $\mathcal{H}_\lambda \hookrightarrow L^\infty(V)$ continuously. Hence, using interpolation gives that $\mathcal{H}_\lambda \hookrightarrow L^p(V)$ continuously for all $2 \leq p \leq \infty$. Assuming $\{u_k\}$ is bounded in $\mathcal{H}_\lambda$, we have that, up to a subsequence, $u_k \rightharpoonup u$ in $\mathcal{H}_\lambda$. In particular, $\{u_k\} \subset \mathcal{H}_\lambda$ is also bounded in $L^2(V)$ and by the weak convergence in $L^2(V)$ it follows that, for any $\varphi \in L^2(V)$,

$$\lim_{k\to\infty}\int_V (u_k - u)\varphi d\mu = \lim_{k\to\infty}\sum_{x\in V}\mu(x)\left(u_k(x) - u(x)\right)\varphi(x) = 0. \tag{2.5}$$

Take any $x_0 \in V$ and let

$$\varphi_0(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases}$$

Obviously, $\varphi_0(x) \in L^2(V)$. By substituting φ_0 into (2.5), we can get that $\lim_{k\to\infty} u_k(x) = u(x)$ for any fixed $x \in V$.

We now prove that there exist a constant $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$, we have $u_k \to u$ in $L^p(V)$ for all $2 \leq p \leq \infty$. Since u_k is bounded in $\mathcal{H}_\lambda$ and $u \in \mathcal{H}_\lambda$, there exists some constant C_1 such that

$$\lambda\int_V a(x)(u_k - u)^2 d\mu \leq C_1.$$

We claim that, up to a subsequence,

$$\lim_{k\to+\infty}\int_V (u_k - u)^2 d\mu = 0.$$

In fact, since $a(x)$ satisfies (A_2), there exists some $M > 0$ such that

$$\begin{aligned}
\int_V (u_k - u)^2 d\mu &= \int_{D_M}(u_k - u)^2 d\mu + \int_{V\setminus D_M}(u_k - u)^2 d\mu \\
&\leq \int_{D_M}(u_k - u)^2 d\mu + \int_{V\setminus D_M}\frac{1}{\lambda M}\lambda a(x)(u_k - u)^2 d\mu \\
&\leq \int_{D_M}(u_k - u)^2 d\mu + \frac{C_1}{\lambda M}.
\end{aligned}$$

We can see that, for all $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that when $\lambda > \lambda_0$, we have $\frac{C_1}{\lambda M} < \varepsilon$. Moreover, up to a subsequence, we have

$$\lim_{k\to+\infty}\int_{D_M}(u_k - u)^2 d\mu = 0.$$

Hence the claim holds. Then, in view of $\|u_k - u\|_{L^\infty}^2 \leq \frac{1}{\mu_{\min}} \int_V |u_k - u|^2 d\mu$, we obtain, for any $2 < p < \infty$,

$$\int_V |u_k - u|^p d\mu \leq \left(\frac{1}{\mu_{\min}}\right)^{\frac{p-2}{2}} \left(\int_V |u_k - u|^2 d\mu\right)^{\frac{p}{2}}.$$

Therefore, up to a subsequence, $u_k \to u$ in $L^p(V)$ for all $2 \leq p \leq +\infty$. $\square$

3. Existence of least energy sign-changing solutions

This section is devoted to proving that Eq. (1.2), as well as (1.8), admits a least energy sign-changing solution by using the Nehari manifold method based on directional derivative.

The following result will be useful.

Lemma 3.1. *For all $u \in \mathcal{M}_\lambda$ and $s, t > 0$, there holds*

$$J_\lambda(u) \geq J_\lambda(su^+ + tu^-).$$

The "=" holds if and only if $s = t = 1$.

Proof. For any $u \in \mathcal{M}_\lambda$,

$$\begin{aligned}
J_\lambda(u) =& J_\lambda(u) - \frac{1}{2}J'_\lambda(u)\cdot u^+ - \frac{1}{2}J'_\lambda(u)\cdot u^- \\
=& J_\lambda(u^+) - \frac{1}{2}J'_\lambda(u^+)\cdot u^+ + J_\lambda(u^-) - \frac{1}{2}J'_\lambda(u^-)\cdot u^- \\
=& \left(\frac{1}{2}\|u^+\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}\int_V |u^+|^2 \log|u^+|^2 d\mu\right) - \left(\frac{1}{2}\|u^+\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}\int_V |u^+|^2 \log|u^+|^2 d\mu - \frac{1}{2}\|u^+\|_2^2\right) \\
&+ \left(\frac{1}{2}\|u^-\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}\int_V |u^-|^2 \log|u^-|^2 d\mu\right) - \left(\frac{1}{2}\|u^-\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}\int_V |u^-|^2 \log|u^-|^2 d\mu - \frac{1}{2}\|u^+\|_2^2\right) \\
=& \frac{1}{2}\|u^+\|_2^2 + \frac{1}{2}\|u^-\|_2^2.
\end{aligned}$$

For $s, t > 0$, we have

$$\begin{aligned}
&\int_V \Gamma(su^+ + tu^-) d\mu \\
=& \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[(su^+ + tu^-)(y) - (su^+ + tu^-)(x)\right]^2 \\
=& \frac{1}{2}\sum_{x\in V}\sum_{y\sim x}\omega_{xy}\left[\left(su^+(y) - su^+(x)\right)^2 + \left(tu^-(y) - tu^-(x)\right)^2 - 2st\left[u^+(x)u^-(y) + u^-(x)u^+(y)\right]\right] \\
=& \int_V \Gamma(su^+) d\mu + \int_V \Gamma(tu^-) - stK_V(u).
\end{aligned} \tag{3.1}$$

Hence, we have

$$\begin{aligned}
&J_\lambda(su^+ + tu^-) \\
=& J_\lambda(su^+) + J_\lambda(tu^-) - \frac{st}{2}K_V(u) \\
=& s^2 J_\lambda(u^+) - \frac{1}{2}s^2\log s^2\|u^+\|_2^2 + t^2 J_\lambda(u^-) - \frac{1}{2}t^2\log t^2\|u^-\|_2^2 - \frac{st}{2}K_V(u) \\
=& s^2\left[J_\lambda(u^+) - \frac{1}{2}J'_\lambda(u)\cdot u^+\right] - \frac{1}{2}s^2\log s^2\|u^+\|_2^2 + t^2\left[J_\lambda(u^-) - \frac{1}{2}J'_\lambda(u)\cdot u^-\right] \\
&- \frac{1}{2}t^2\log t^2\|u^-\|_2^2 - \frac{st}{2}K_V(u) \\
=& s^2\left[J_\lambda(u^+) - \frac{1}{2}J'_\lambda(u^+)\cdot u^+ + \frac{1}{4}K_V(u)\right] - \frac{1}{2}s^2\log s^2\|u^+\|_2^2 \\
&+ t^2\left[J_\lambda(u^-) - \frac{1}{2}J'_\lambda(u^-)\cdot u^- + \frac{1}{4}K_V(u)\right] - \frac{1}{2}t^2\log t^2\|u^-\|_2^2 - \frac{st}{2}K_V(u) \\
=& \frac{1}{2}(s^2 - s^2\log s^2)\|u^+\|_2^2 + \frac{1}{2}(t^2 - t^2\log t^2)\|u^-\|_2^2 + \frac{(s-t)^2}{4}K_V(u).
\end{aligned}$$

Therefore, defining $f(\tau) = \tau^2 - \tau^2 \log \tau^2 - 1$ for any $\tau \geq 0$, we have

$$\begin{aligned} & J_\lambda(su^+ + tu^-) - J_\lambda(u) \\ =& \frac{1}{2}(s^2 - s^2 \log s^2 - 1)\|u^+\|_2^2 + \frac{1}{2}(t^2 - t^2 \log t^2 - 1)\|u^-\|_2^2 + \frac{(s-t)^2}{4} K_V(u) \\ =& \frac{1}{2} f(s)\|u^+\|_2^2 + \frac{1}{2} f(t)\|u^-\|_2^2 + \frac{(s-t)^2}{4} K_V(u). \end{aligned}$$

Since $f(0) = -1, f(1) = 0$ and $f(\tau) < 0$ if $\tau \neq 1$, $\frac{(s-t)^2}{4} K_V(u) < 0$ for any $s \neq t$, the conclusions follow. □

Next we show $\mathcal{M}_\lambda \neq \emptyset$.

Lemma 3.2. *If $u \in \mathcal{D}_\lambda \setminus \{0\}$ with $u^\pm \neq 0$, then there exists a unique positive number pair (s_u, t_u) satisfying $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.*

Proof. For $s, t > 0$, we have

$$\int_V \Gamma(su^+ + tu^-, su^+) d\mu = \int_V \Gamma(su^+) d\mu - \frac{st}{2} K_V(u) \tag{3.2}$$

and

$$\int_V \Gamma(su^+ + tu^-, tu^-) d\mu = \int_V \Gamma(tu^-) d\mu - \frac{st}{2} K_V(u). \tag{3.3}$$

Let

$$\begin{aligned} g_1(s,t) \doteq& J_\lambda'(su^+ + tu^-) \cdot (su^+) \\ =& J_\lambda'(su^+) \cdot (su^+) - \frac{st}{2} K_V(u) \\ =& s^2 \|u^+\|_{\mathcal{H}_\lambda}^2 - s^2 \int_V |u^+|^2 \log |u^+|^2 d\mu - s^2 \log s^2 \|u^+\|_2^2 - s^2 \|u^+\|_2^2 - \frac{st}{2} K_V(u) \end{aligned}$$

and

$$\begin{aligned} g_2(s,t) \doteq& J_\lambda'(su^+ + tu^-) \cdot (tu^-) \\ =& J_\lambda'(tu^-) \cdot (tu^-) - \frac{st}{2} K_V(u) \\ =& t^2 \|u^-\|_{\mathcal{H}_\lambda}^2 - t^2 \int_V |u^-|^2 \log |u^-|^2 d\mu - t^2 \log t^2 \|u^-\|_2^2 - t^2 \|u^-\|_2^2 - \frac{st}{2} K_V(u). \end{aligned}$$

We can see that there exists $r_1 > 0$ small enough and $R_1 > 0$ large enough such that

$$\begin{aligned} & g_1(s,s) > 0,\ g_2(s,s) > 0 \text{ for all } s \in (0, r_1), \\ & g_1(s,s) < 0,\ g_2(s,s) < 0 \text{ for all } s \in (R_1, +\infty). \end{aligned}$$

Hence, there exist $0 < r < R$ such that

$$\begin{aligned} & g_1(r,t) > 0,\ g_1(R,t) < 0 \text{ for all } t \in [r, R], \\ & g_2(s,r) > 0,\ g_2(s,R) < 0 \text{ for all } s \in [r, R]. \end{aligned}$$

Applying Miranda's theorem [37], there exist some $s_u, t_u \in [r, R]$ such that $g_1(s_u, t_u) = g_2(s_u, t_u) = 0$, which implies that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$.

In what follows, we prove the uniqueness of the pair (s_u, t_u). If $u \in \mathcal{M}_\lambda$, then

$$0 = J_\lambda'(u) \cdot u^+ = J_\lambda'(u^+) \cdot u^+ - \frac{1}{2} K_V(u) \tag{3.4}$$

and

$$0 = J_\lambda'(u) \cdot u^- = J_\lambda'(u^-) \cdot u^- - \frac{1}{2} K_V(u). \tag{3.5}$$

We claim that $(s_u, t_u) = (1, 1)$ is the unique pair of positive numbers such that $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$. Indeed, if $(s_u, t_u) = (1, 1)$ satisfies $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$, without loss of generality, we assume that $0 < s_u \leq t_u$. Then

$$\begin{aligned} 0 =& J_\lambda'(s_u u^+ + t_u u^-) \cdot (s_u u^+) \\ =& J_\lambda'(s_u u^+) \cdot (s_u u^+) - \frac{s_u t_u}{2} K_V(u) \\ =& s_u^2 J_\lambda'(u^+) \cdot u^+ - s_u^2 \log s_u^2 \|u^+\|_2^2 - \frac{s_u t_u}{2} K_V(u) \\ \geq& s_u^2 J_\lambda'(u^+) \cdot u^+ - s_u^2 \log s_u^2 \|u^+\|_2^2 - \frac{s_u^2}{2} K_V(u) \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} 0 =& J_\lambda'(s_u u^+ + t_u u^-) \cdot (t_u u^-) \\ =& J_\lambda'(t_u u^-) \cdot (t_u u^-) - \frac{s_u t_u}{2} K_V(u) \\ =& t_u^2 J_\lambda'(u^-) \cdot u^- - t_u^2 \log t_u^2 \|u^-\|_2^2 - \frac{s_u t_u}{2} K_V(u) \\ \leq& t_u^2 J_\lambda'(u^-) \cdot u^- - t_u^2 \log t_u^2 \|u^-\|_2^2 - \frac{t_u^2}{2} K_V(u). \end{aligned} \tag{3.7}$$

Together with (3.4) and (3.6), we get

$$s_u^2 \log s_u^2 \int_V |u^+|^2 d\mu \geq 0,$$

Similarly, by (3.5) and (3.7), we can deduce that

$$t_u^2 \log t_u^2 \int_V |u^-|^2 d\mu \leq 0,$$

which implies that $s_u \geq 1$ and $t_u \leq 1$. In view of $0 < s_u \leq t_u$, it follows that $s_u = t_u = 1$.

If $u \notin \mathcal{M}_\lambda$, let (s_1, t_1) and (s_2, t_2) be the two different positive pairs such that $v_i := s_i u^+ + t_i u^- \in \mathcal{M}_\lambda, i = 1, 2$, which shows that

$$\frac{s_2}{s_1} v_1^+ + \frac{t_2}{t_1} v_1^- = v_2 \in \mathcal{M}_\lambda.$$

By similar analysis as above, we can obtain that

$$\frac{s_2}{s_1} = \frac{t_2}{t_1} = 1.$$

This implies that $(s_1, t_1) = (s_2, t_2)$ and the uniqueness is obtained. □

Lemma 3.3. *Let $u \in \mathcal{D}_\lambda$ with $u^\pm \neq 0$ such that $J_\lambda'(u) \cdot u^\pm \leq 0$. Then the unique pair (s_u, t_u) obtained in Lemma 3.2 satisfies $s_u, t_u \in (0, 1]$. In particular, the "=" holds if and only if $s_u = t_u = 1$.*

Proof. Without loss of generality, we assume that $0 < t_u \leq s_u$. Since $s_u u^+ + t_u u^- \in \mathcal{M}_\lambda$, then

$$\begin{aligned} 0 =& J_\lambda'(s_u u^+ + t_u u^-) \cdot (s_u u^+) \\ =& s_u^2 J_\lambda'(u^+) \cdot u^+ - s_u^2 \log s_u^2 \|u^+\|_2^2 - \frac{s_u t_u}{2} K_V(u). \end{aligned} \tag{3.8}$$

Note that $K_V(x, y) < 0$. Since $J_\lambda'(u) \cdot u^+ \leq 0$, from (3.8), we can deduce that

$$\begin{aligned} 0 \leq& s_u^2 \left(J_\lambda'(u^+) \cdot u^+ - \frac{1}{2} K_V(x, y) \right) - s_u^2 \log s_u^2 \|u^+\|_2^2 \\ =& s_u^2 J_\lambda'(u) \cdot u^+ - s_u^2 \log s_u^2 \|u^+\|_2^2 \\ \leq& - s_u^2 \log s_u^2 \|u^+\|_2^2, \end{aligned}$$

which implies that $0 < s_u \leq 1$. Therefore, $0 < t_u \leq s_u \leq 1$. □

Similarly, we have

Lemma 3.4. *If $u \in H_0^1(\Omega) \setminus \{0\}$ with $u^\pm \neq 0$, then there exists a unique positive number pair (s_u, t_u) satisfying $s_u u^+ + t_u u^- \in \mathcal{M}_\Omega$.*

Lemma 3.5. *Let $u \in H_0^1(\Omega)$ with $u^{\pm} \neq 0$ such that $J'_{\Omega}(u) \cdot u^{\pm} \leq 0$. Then the unique pair (s_u, t_u) obtained in Lemma 3.4 satisfies $s_u, t_u \in (0, 1]$. In particular, the "=" holds if and only if $s_u = t_u = 1$.*

Now we prove that the minimizer of J_λ on $\mathcal{M}_\lambda$ is achieved.

Lemma 3.6. *Supposed (A_1) and (A_2) hold. Then $m_\lambda > 0$ is achieved.*

Proof. Taking a minimizing sequence $\{u_k\} \subset \mathcal{M}_\lambda$ of J_λ yields

$$\begin{aligned}\lim_{k\to+\infty} J_\lambda(u_k) &= \lim_{k\to+\infty}\left[J_\lambda(u_k) - \frac{1}{2}J'_\lambda(u_k)\cdot u_k^+ - \frac{1}{2}J'_\lambda(u_k)\cdot u_k^-\right]\\ &= \lim_{k\to+\infty}\left[J_\lambda(u_k^+) - \frac{1}{2}J'_\lambda(u_k^+)\cdot u_k^+ + J_\lambda(u_k^-) - J'_\lambda(u_k^-)\cdot u_k^-\right]\\ &= \lim_{k\to+\infty}\left(\frac{1}{2}\|u_k^+\|_2^2 + \frac{1}{2}\|u_k^-\|_2^2\right) = m_\lambda.\end{aligned} \tag{3.9}$$

By Lemma 2.5, the Hölder's inequality and Young inequality, for any $\varepsilon \in (0, 1)$, there exist $C_\varepsilon, C'_\varepsilon, C''_\varepsilon > 0$ such that

$$\begin{aligned}\int_V |u_k^\pm|^2 \log|u_k^\pm|^2 d\mu \leq \int_V (|u_k^\pm|^2\log|u_k^\pm|^2)^+ d\mu &\leq C_\varepsilon \int_V |u_k^\pm|^{2+\varepsilon} d\mu\\ &\leq C_\varepsilon\left(\int_V |u_k^\pm|^2 d\mu\right)^{\frac{1}{2}}\left(\int_V |u_k^\pm|^{2(1+\varepsilon)} d\mu\right)^{\frac{1}{2}}\\ &\leq C'_\varepsilon \|u_k^\pm\|_2 \|u_k^\pm\|_{\mathcal{H}_\lambda}^{1+\varepsilon}\\ &\leq \frac{1}{2}\|u_k^\pm\|_{\mathcal{H}_\lambda}^2 + C''_\varepsilon \|u_k^\pm\|_2^{\frac{2}{1-\varepsilon}}.\end{aligned}$$

Since $\{u_k\} \subset \mathcal{M}_\lambda$, we deduce that

$$\begin{aligned}&\|u_k^\pm\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}K_V^k(x, y)\\ =& \int_V |u_k^\pm|^2 \log|u_k^\pm|^2 d\mu + \|u_k^\pm\|_2^2\\ \leq& \frac{1}{2}\|u_k^\pm\|_{\mathcal{H}_\lambda}^2 + C''_\varepsilon\|u_k^\pm\|_2^{\frac{2}{1-\varepsilon}} + \|u_k^\pm\|_2^2,\end{aligned} \tag{3.10}$$

where $K_V^k(u) = \sum_{x\in V}\sum_{y\sim x}\left[u_k^+(x)u_k^-(y) + u_k^-(x)u_k^+(y)\right]$. This together with (3.9) implies that $\{u_k^\pm\}$ is bounded in $\mathcal{H}_\lambda$ and $\{u_k\}$ is also bounded in $\mathcal{H}_\lambda$. Then, there exists $\lambda_0 > 0$ such that $\lambda \geq \lambda_0$, by Lemma 2.5, there exists $u_\lambda \in \mathcal{H}_\lambda$ such that

$$\begin{cases} u_k \rightharpoonup u_\lambda & \text{weakly in } \mathcal{H}_\lambda,\\ u_k \to u_\lambda & \text{point-wisely in } V,\\ u_k \to u_\lambda & \text{strongly in } L^p(V) \text{ for } p \in [2, +\infty].\end{cases}$$

Thus, together with the weak-lower semi-continuity of norm and Fatou's lemma, we get

$$\begin{aligned}&\int_V \left(\Gamma(u_\lambda^+) + (\lambda a(x) + 1)|u_\lambda^+|^2\right) d\mu - \int_V (|u_\lambda^+|^2\log|u_\lambda^+|^2)^- d\mu - \frac{1}{2}K_V^\lambda(u)\\ \leq& \liminf_{k\to+\infty}\left[\int_V\left(\Gamma(u_k^+) + (\lambda a(x) + 1)|u_k^+|^2\right) d\mu - \int_V (|u_k^+|^2\log|u_k^+|^2)^- d\mu - \frac{1}{2}K_V^k(u)\right]\\ =& \liminf_{k\to+\infty}\int_V \left(|u_k^+|^2 + (|u_k^+|^2\log|u_k^+|^2)^+\right) d\mu\\ =& \int_V |u_\lambda^+|^2 d\mu + \int_V (|u_\lambda^+|^2 \log|u_\lambda^+|^2)^+ d\mu,\end{aligned}$$

where $K_V^\lambda(u) = \sum_{x\in V}\sum_{y\sim x}\left[u_\lambda^+(x)u_\lambda^-(y) + u_\lambda^-(x)u_\lambda^+(y)\right]$. It follows that

$$J'_\lambda(u_\lambda)\cdot u_\lambda^+ = \int_V \left(\Gamma(u_\lambda^+) + \lambda a(x)|u_\lambda^+|^2\right) d\mu - \int_V |u_\lambda^+|^2\log|u_\lambda^+|^2 d\mu - \frac{1}{2}K_V^\lambda(u) \leq 0. \tag{3.11}$$

Similarly, it holds that

$$J'_\lambda(u_\lambda)\cdot u_\lambda^- = \int_V \left(\Gamma(u_\lambda^-) + \lambda a(x)|u_\lambda^-|^2\right) d\mu - \int_V |u_\lambda^-|^2\log|u_\lambda^-|^2 d\mu - \frac{1}{2}K_V^\lambda(u) \leq 0. \tag{3.12}$$

In view of Lemmas 3.2 and 3.3, there exist two constants $s, t \in (0, 1]$ such that $\widetilde{u} = su_\lambda^+ + tu_\lambda^- \in \mathcal{M}_\lambda$. Then

$$\begin{aligned}
m_\lambda \leq & J_\lambda(\widetilde{u}) = J_\lambda(\widetilde{u}) - \frac{1}{2}J_\lambda'(\widetilde{u}) \cdot (su_\lambda^+) - \frac{1}{2}J_\lambda'(\widetilde{u}) \cdot (tu_\lambda^-) \\
= & J_\lambda(su_\lambda^+) - \frac{1}{2}J_\lambda'(su_\lambda^+) \cdot (su_\lambda^+) + J_\lambda(tu_\lambda^-) - \frac{1}{2}J_\lambda'(tu_\lambda^-) \cdot (tu_\lambda^-) \\
= & \frac{s^2}{2}\|u_\lambda^+\|_2^2 + \frac{t^2}{2}\|u_\lambda^-\|_2^2 \\
\leq & \frac{1}{2}\|u_\lambda^+\|_2^2 + \frac{1}{2}\|u_\lambda^-\|_2^2 \\
\leq & \liminf_{k\to+\infty} \left[\frac{1}{2}\|u_k^+\|_2^2 + \frac{1}{2}\|u_k^-\|_2^2\right] \\
= & \liminf_{k\to+\infty} \left[J_\lambda(u_k^+) - \frac{1}{2}J_\lambda'(u_k^+) \cdot u_k^+ + J_\lambda(u_k^-) - \frac{1}{2}J_\lambda'(u_k^-) \cdot u_k^-\right] \\
= & \liminf_{k\to+\infty} \left[J_\lambda(u_k) - \frac{1}{2}J_\lambda'(u_k) \cdot u_k^+ - \frac{1}{2}J_\lambda'(u_k) \cdot u_k^-\right] \\
= & \liminf_{k\to+\infty} J_\lambda(u_k) = m_\lambda.
\end{aligned}$$

This implies that $s = t = 1$, i.e., $u_\lambda \in \mathcal{M}_\lambda$ satisfying $J_\lambda(u_\lambda) = m_\lambda$.

We claim that $m_\lambda > 0$. In fact, if $m_\lambda = 0$, we have

$$0 = J_\lambda(u_\lambda) - \frac{1}{2}J'(u_\lambda) \cdot u_\lambda^+ - \frac{1}{2}J'(u_\lambda) \cdot u_\lambda^- = \frac{1}{2}\|u_\lambda^+\|_2^2 + \frac{1}{2}\|u_\lambda^-\|_2^2.$$

Then, by similar arguments as in (3.10), it follows that $\|u_\lambda^\pm\|_{\mathcal{H}_\lambda} = 0$. However, by Lemma 2.5, for any $q > 2$, there exists $C_q > 0$ such that

$$\|u_\lambda^\pm\|_{\mathcal{H}_\lambda}^2 < \int_V |u_\lambda^\pm|^2 \log |u_\lambda^\pm|^2 d\mu \leq \int_V (|u_\lambda^\pm|^2 \log |u_\lambda^\pm|^2)^+ d\mu \leq C_q \int_V |u_\lambda^\pm|^q d\mu \leq C\|u_\lambda^\pm\|_{\mathcal{H}_\lambda}^q,$$

which implies

$$\|u_\lambda^\pm\|_{\mathcal{H}_\lambda} \geq \left(\frac{1}{C}\right)^{\frac{1}{q-2}} > 0,$$

which provides a contradiction, hence the claim holds. □

The following lemma completes the proof of Theorem 1.1.

Lemma 3.7. *If $u \in \mathcal{M}_\lambda$ with $J_\lambda(u) = m_\lambda$, then u is a sign-changing solution of Eq. (1.2). Moreover, $m_\lambda > 2c_\lambda$.*

Proof. We assume by contradiction that $u \in \mathcal{M}_\lambda$ with $J_\lambda(u) = m_\lambda$, but u is not a solution of Eq. (1.2). Then we can find a function $\phi \in C_c(V)$ such that

$$\int_V (\nabla u \nabla \phi + \lambda a(x) u\phi)\, d\mu - \int_V u\phi \log u^2 d\mu \leq -1,$$

which implies that, for some $\varepsilon > 0$ small enough,

$$J_\lambda'(su^+ + tu^- + \sigma\phi) \cdot \phi \leq -\frac{1}{2} \text{ for all } |s-1| + |t-1| + |\sigma| \leq \epsilon.$$

In what follows, we estimate $\sup_{s,t} J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right)$, where η is a cut-off function such that

$$\eta(s,t) = \begin{cases} 1 & \text{if } |s-1| \leq \frac{1}{2}\varepsilon \text{ and } |t-1| \leq \frac{1}{2}\varepsilon, \\ 0 & \text{if } |s-1| \geq \varepsilon \text{ or } |t-1| \geq \varepsilon. \end{cases}$$

In the case of $|s-1| \leq \varepsilon$ and $|t-1| \leq \varepsilon$, we have

$$\begin{aligned}
J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right) = & J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right) - J_\lambda(su^+ + tu^-) + J_\lambda(su^+ + tu^-) \\
= & J_\lambda(su^+ + tu^-) + \int_0^1 J_\lambda'\left(su^+ + tu^- + \sigma\varepsilon\eta(s,t)\phi\right) \cdot (\varepsilon\eta(s,t)\phi)\, d\sigma \\
= & J_\lambda(su^+ + tu^-) + \varepsilon\eta(s,t) \int_0^1 J_\lambda'\left(su^+ + tu^- + \sigma\varepsilon\eta(s,t)\phi\right) \cdot \phi d\sigma \\
\leq & J_\lambda(su^+ + tu^-) - \frac{1}{2}\varepsilon\eta(s,t).
\end{aligned}$$

For the other case, that is $|s-1| \geq \varepsilon$ or $|t-1| \geq \varepsilon$, $\eta(s,t)=0$, the above estimate is obvious. Now since $u \in \mathcal{M}_\lambda$, for $(s,t) \neq (1,1)$, by Lemma 3.1, we have $J_\lambda(su^+ + tu^-) < J_\lambda(u)$. Hence

$$J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right) \leq J_\lambda(su^+ + tu^-) < J_\lambda(u) \text{ for all } (s,t) \neq (1,1).$$

For $(s,t) = (1,1)$,

$$J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right) \leq J_\lambda(su^+ + tu^-) - \frac{1}{2}\varepsilon\eta(1,1) = J_\lambda(u) - \frac{1}{2}\varepsilon.$$

In any case, we have $J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right) < J_\lambda(u) = m_\lambda$. In particular, for $0 < \varepsilon < 1-\varepsilon$,

$$\sup_{\varepsilon \leq s,t \leq 2-\varepsilon} J_\lambda\left(su^+ + tu^- + \varepsilon\eta(s,t)\phi\right) = \widetilde{m}_\lambda < m_\lambda.$$

Set $v = su^+ + tu^- + \varepsilon\eta(s,t)\phi$ and define

$$H(s,t) = (F_1(s,t), F_2(s,t)) \doteq \left(J'_\lambda(v)\cdot v^+, J'_\lambda(v)\cdot v^-\right).$$

By the definition of η, when $s=\varepsilon,\ t \in (\epsilon, 2-\epsilon)$, we have $\eta(s,t)=0$ and $s<t$. Hence

$$\begin{aligned}
F_1(\varepsilon,t) &\doteq J'_\lambda(su^+ + tu^-)\cdot(su^+)\Big|_{s=\varepsilon}\\
&= \left[J'_\lambda(su^+)\cdot(su^+) - \frac{st}{2}K_V(u)\right]_{s=\varepsilon}\\
&= \left[s^2 J'_\lambda(u^+)\cdot u^+ - \frac{st}{2}K_V(u) - s^2\log s^2\|u^+\|_2^2\right]_{s=\varepsilon}\\
&> \left[s^2\left(J'_\lambda(u^+) - \frac{1}{2}K_V(u)\right) - s^2\log s^2\|u^+\|_2^2\right]_{s=\varepsilon}\\
&= -s^2\log s^2\|u^+\|_2^2\Big|_{s=\varepsilon}\\
&= -\varepsilon^2\log\varepsilon^2\|u^+\|_2^2\\
&> 0.
\end{aligned}$$

When $s = 2-\varepsilon,\ t \in (\epsilon, 2-\epsilon)$, we have $\eta(s,t)=0$ and $s>t$. Therefore,

$$\begin{aligned}
F_1(2-\varepsilon,t) &\doteq J'_\lambda(su^+ + tu^-)\cdot(su^+)\Big|_{s=2-\varepsilon}\\
&= \left[J'_\lambda(su^+)\cdot(su^+) - \frac{st}{2}K_V(u)\right]_{s=2-\varepsilon}\\
&= \left[s^2 J'_\lambda(u^+)\cdot u^+ - \frac{st}{2}K_V(u) - s^2\log s^2\|u^+\|_2^2\right]_{s=2-\varepsilon}\\
&< \left[s^2\left(J'_\lambda(u^+) - \frac{1}{2}K_V(u)\right) - s^2\log s^2\|u^+\|_2^2\right]_{s=2-\varepsilon}\\
&= -s^2\log s^2\|u^+\|_2^2\Big|_{s=2-\varepsilon}\\
&= -(2-\varepsilon)^2\log(2-\varepsilon)^2\|u^+\|_2^2\\
&< 0.
\end{aligned}$$

That is

$$F_1(\varepsilon,t) > 0,\ \ F_1(2-\varepsilon,t) < 0 \text{ for all } t \in (\varepsilon, 2-\varepsilon).$$

Similarly, we have

$$F_2(s,\varepsilon) > 0,\ \ F_2(s,2-\varepsilon) < 0 \text{ for all } s \in (\varepsilon, 2-\varepsilon).$$

Thus, applying Miranda's theorem [37], there exists $(s_0,t_0) \in (\varepsilon, 2-\varepsilon)\times(\varepsilon, 2-\varepsilon)$ such that $\widetilde{u} = s_0u^+ + t_0u^- + \varepsilon\eta(s_0,t_0)\phi \in \mathcal{M}_\lambda$ and $J_\lambda(\widetilde{u}) < m_\lambda$. This give a contradiction to the definition of m_λ.

Next, we prove that $m_\lambda > 2c_\lambda$. Assume that $u \in \mathcal{M}_\lambda$ such that $J_\lambda(u) = m_\lambda$. Then $u^\pm \neq 0$. Similar to the proof of Lemmas 3.2 and 3.3, we can deduce that there exists a unique $s_{u^+} \in (0,1]$ such that $s_{u^+}u^+ \in \mathcal{N}_\lambda$, and a unique $t_{u^-} \in (0,1]$ such that $t_{u^-}u^- \in \mathcal{N}_\lambda$. Similar to the proofs of Lemma 3.6 and Lemma 3.7, we can deduce that $c_\lambda > 0$ can be achieved. Furthermore, if $u \in \mathcal{N}_\lambda$ with $J_\lambda(u) = c_\lambda$, then u is a least energy solution.

By the definition of J_λ and $K_V(x,y) < 0$, we have

$$\begin{aligned}
J_\lambda(s_{u^+}u^+ + t_{u^-}u^-) &= J_\lambda(s_{u^+}u^+) + J_\lambda(t_{u^-}u^-) - \frac{s_{u^+}t_{u^-}}{2}K_V(u)\\
&> J_\lambda(s_{u^+}u^+) + J_\lambda(t_{u^-}u^-).
\end{aligned}$$

By Lemma 3.1, we deduce that

$$m_\lambda = J_\lambda(u^+ + u^-) \geq J_\lambda(s_{u^+}u^+ + t_{u^-}u^-) > J_\lambda(s_{u^+}u^+) + J_\lambda(t_{u^-}u^-) \geq 2c_\lambda.$$

This completes the proof. □

Proof of Theorem 1.2. Similar to the proof of Theorem 1.1, we can also obtain the existence of a least energy sign-changing solution u_0 of problem (1.8), which achieves the minimum m_Ω of the functional J_Ω in $\mathcal{M}_\Omega$ and the least energy solution of problem (1.8), which achieves the minimum c_Ω of the functional J_Ω in $\mathcal{N}_\Omega$. Moreover, $m_\Omega > 2c_\Omega$. □

4. Convergence of least energy sign-changing solutions

In this section, we shall study the asymptotic behavior of the least energy sign-changing $u_\lambda \in H_\lambda$ of Eq. (1.2) as $\lambda \to +\infty$. First we show that the family of solutions $\{u_\lambda\}$ is uniformly bounded above and below away from zero.

Lemma 4.1. *There exists $\sigma > 0$ (independent of λ) such that $\|u\|_{\mathcal{H}_\lambda} \geq \|u\|_{H^1(V)} \geq \sigma$ for all $u \in \mathcal{M}_\lambda$.*

Proof. Note that for all $\varepsilon > 0$, if $s \geq e^{-\frac{1}{2}}$, then

$$e^{\frac{\varepsilon}{2}} s^{2+\varepsilon} \geq s^2. \tag{4.1}$$

Since $u \in \mathcal{M}_\lambda$, by Lemma 2.5 and (4.1), we have

$$\begin{aligned}
0 =& J_\lambda'(u)\cdot u^+ = J_\lambda'(u^+)\cdot u^+ - \frac{1}{2}K_V(u)\\
\geq& \int_V \left(\Gamma(u^+) + (\lambda a(x)+1)|u^+|^2\right) d\mu - \int_V |u^+|^2 d\mu - \int_V |u^+|^2 \log|u^+|^2 d\mu\\
=& \|u^+\|^2_{\mathcal{H}_\lambda} - \int_{|u^+|<e^{-\frac{1}{2}}} \left(|u^+|^2 + |u^+|^2\log|u^+|^2\right) d\mu - \int_{|u^+|\geq e^{-\frac{1}{2}}} |u^+|^2 d\mu\\
&- \int_{e^{-\frac{1}{2}}\leq|u^+|\leq 1} |u^+|^2\log|u^+|^2 d\mu - \int_{|u^+|>1} |u^+|^2 \log|u^+|^2 d\mu\\
\geq& \|u^+\|^2_{\mathcal{H}_\lambda} - e^{\frac{\varepsilon}{2}} \int_{|u^+|\geq e^{-\frac{1}{2}}} |u^+|^{2+\varepsilon} d\mu - C_\varepsilon \int_{|u^+|>1} |u^+|^{2+\varepsilon} d\mu\\
\geq& \|u^+\|^2_{\mathcal{H}_\lambda} - C'_\varepsilon \int_V |u^+|^{2+\varepsilon} d\mu\\
\geq& \|u^+\|^2_{H^1(V)} - C''_\varepsilon \|u^+\|^{2+\varepsilon}_{H^1(V)}.
\end{aligned}$$

Then

$$\|u^+\|_{\mathcal{H}_\lambda} \geq \|u^+\|_{H^1(V)} \geq (C''_\varepsilon)^{-\frac{1}{\varepsilon}} > 0.$$

Similarly, we get

$$\|u^-\|_{\mathcal{H}_\lambda} \geq \|u^-\|_{H^1(V)} \geq (C''_\varepsilon)^{-\frac{1}{\varepsilon}} > 0.$$

Hence,

$$\|u\|^2_{\mathcal{H}_\lambda} \geq \|u\|^2_{H^1(V)} = \|u^+\|^2_{H^1(V)} + \|u^-\|^2_{H^1(V)} - K_V(u) > \|u^+\|^2_{H^1(V)} + \|u^-\|^2_{H^1(V)} \geq 2(C''_\varepsilon)^{-\frac{2}{\varepsilon}}.$$

Thus we can choose $\sigma = \sqrt{2}(C''_\varepsilon)^{-\frac{1}{\varepsilon}}$ such that $\|u\|_{\mathcal{H}_\lambda} \geq \|u\|_{H^1(V)} \geq \sigma$. □

Lemma 4.2. *There exists $c_0 > 0$ (independent of λ) such that if sequence $\{u_k\} \subset \mathcal{M}_\lambda$ of J_λ with $\lim_{k\to\infty} J_\lambda(u_k) = m_\lambda$, then $\|u_k\|_{\mathcal{H}_\lambda} \leq c_0$.*

Proof. Since $\mathcal{M}_\Omega \subset \mathcal{M}_\lambda$, it is easily seen that $m_\lambda \leq m_\Omega$ for any $\lambda > 0$. Since $\{u_k\} \subset \mathcal{M}_\lambda$ and $\lim_{k\to\infty} J_\lambda(u_k) = m_\lambda$, we have

$$\begin{aligned}
\lim_{k\to+\infty} J_\lambda(u_k) &= \lim_{k\to+\infty}\left[J_\lambda(u_k) - \frac{1}{2}J_\lambda'(u_k)\cdot u_k^+ - \frac{1}{2}J_\lambda'(u_k)\cdot u_k^-\right]\\
&= \lim_{k\to+\infty}\left[J_\lambda(u_k^+) - \frac{1}{2}J_\lambda'(u_k^+)\cdot u_k^+ + J_\lambda(u_k^-) - \frac{1}{2}J_\lambda'(u_k^-)\cdot u_k^-\right]\\
&= \lim_{k\to+\infty}\left(\frac{1}{2}\|u_k^+\|_2^2 + \frac{1}{2}\|u_k^-\|_2^2\right) = m_\lambda \leq m_\Omega.
\end{aligned} \tag{4.2}$$

By Lemma 2.5, the Hölder's inequality and Young inequality, for any $\varepsilon \in (0, 1)$, there exist $C_\varepsilon, C'_\varepsilon, C''_\varepsilon > 0$ such that

$$\begin{aligned}\int_V |u_k^{\pm}|^2 \log |u_k^{\pm}|^2 d\mu \le \int_V (|u_k^{\pm}|^2 \log |u_k^{\pm}|^2)^+ d\mu \le C_\varepsilon \int_V |u_k^{\pm}|^{2+\varepsilon} d\mu \\ \le C_\varepsilon \left(\int_V |u_k^{\pm}|^2 d\mu\right)^{\frac{1}{2}} \left(\int_V |u_k^{\pm}|^{2(1+\varepsilon)} d\mu\right)^{\frac{1}{2}} \\ \le C'_\varepsilon \|u_k^{\pm}\|_2 \|u_k^{\pm}\|_{\mathcal{H}_\lambda}^{1+\varepsilon} \\ \le \frac{1}{2}\|u_k^{\pm}\|_{\mathcal{H}_\lambda}^2 + C''_\varepsilon \|u_k^{\pm}\|_2^{\frac{2}{1-\varepsilon}}.\end{aligned}$$

Since $\{u_k\} \subset \mathcal{M}_\lambda$, we deduce that

$$\begin{aligned}\|u_k^{\pm}\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}K_V^k(x, y) = \int_V |u_k^{\pm}|^2 \log |u_k^{\pm}|^2 d\mu + \|u_k^{\pm}\|_2^2 \\ \le \frac{1}{2}\|u_k^{\pm}\|_{\mathcal{H}_\lambda}^2 + C''_\varepsilon \|u_k^{\pm}\|_2^{\frac{2}{1-\varepsilon}} + \|u_k^{\pm}\|_2^2.\end{aligned}$$

This together with (4.2) we get

$$\begin{aligned}&\lim_{k\to+\infty} \left(\|u_k^{\pm}\|_{\mathcal{H}_\lambda}^2 - \frac{1}{2}K_V^k(u)\right) \\ \le &\lim_{k\to+\infty} \left(2C''_\varepsilon \|u_k^{\pm}\|_2^{\frac{2}{1-\varepsilon}} + 2\|u_k^{\pm}\|_2^2\right) \\ \le &C'''_\varepsilon \left(m_\Omega^{\frac{1}{1-\varepsilon}} + m_\Omega\right).\end{aligned}$$

From Lemma 3.6 we know that $m_\lambda > 0$ and then $m_\Omega > 0$. Therefore it suffices to choose $c_0 = C'''_\varepsilon \left(m_\Omega^{\frac{1}{1-\varepsilon}} + m_\Omega\right)$. □

Secondly, we have the following relation about the ground state energy m_λ and m_Ω.

Lemma 4.3. *$m_\lambda \to m_\Omega$ as $\lambda \to +\infty$.*

Proof. By $m_\lambda \le m_\Omega$ for any $\lambda > 0$, passing to subsequence if necessary, we may take a sequence $\lambda_k \to +\infty$ such that

$$\lim_{k\to\infty} m_{\lambda_k} = \eta \le m_\Omega, \tag{4.3}$$

where $m_{\lambda_k} = \inf_{u_k \in \mathcal{M}_{\lambda_k}} J_{\lambda_k}(u_k)$ and u_{λ_k} is a least energy sign-changing solution of Eq. (1.2). Then, combining Lemma 4.1 and (1.8), it is easy to get $\eta > 0$. By Lemma 4.2, we have that $\{u_{\lambda_k}\}$ is uniformly bounded in $\mathcal{H}_{\lambda_k}$. Consequently, $\{u_{\lambda_k}\}$ is also bounded in $H^1(V)$ and thus, up to a subsequence, there exists some $u_0 \in H^1(V)$ such that

$$\begin{cases} u_{\lambda_k} \rightharpoonup u_0 & \text{weakly in } H^1(V), \\ u_{\lambda_k} \to u_0 & \text{point-wisely in } V, \\ u_{\lambda_k} \to u_0 & \text{strongly in } L^p(V) \text{ for } p \in [2, +\infty]. \end{cases} \tag{4.4}$$

We claim that $u_0 |_{\Omega^c} = 0$. In fact, if there exists a vertex $x_0 \in \Omega^c$ such that $u_0(x_0) \neq 0$. Since $u_{\lambda_k} \in \mathcal{M}_{\lambda_k}$, we have

$$\begin{aligned}J_{\lambda_k}(u_{\lambda_k}) &= \frac{1}{2}\|u_{\lambda_k}\|_{\mathcal{H}_{\lambda_k}}^2 - \frac{1}{2}\int_V u_{\lambda_k}^2 \log u_{\lambda_k}^2 d\mu \\ &\ge \frac{\lambda_k}{2}\int_V a(x) u_{\lambda_k}^2 d\mu - \frac{1}{2}\int_V (u_{\lambda_k}^2 \log u_{\lambda_k}^2)^+ d\mu \\ &\ge \frac{\lambda_k}{2}\int_V a(x) u_{\lambda_k}^2 d\mu - \frac{C_\varepsilon}{2}\int_V |u_{\lambda_k}|^{2+\varepsilon} d\mu \\ &\ge \frac{\lambda_k}{2}\sum_{x\in V} \mu(x) a(x) u_{\lambda_k}^2(x) - C'_\varepsilon \|u_{\lambda_k}\|_{H^1(V)}^{2+\varepsilon} \\ &\ge \frac{\lambda_k}{2}\mu_{\min} a(x_0) u_{\lambda_k}^2(x_0) - C''_\varepsilon.\end{aligned}$$

Since $a(x_0) > 0$, $u_{\lambda_k}(x_0) \to u_0(x_0) \neq 0$ and $\lambda_k \to +\infty$, we get

$$\lim_{k\to+\infty} J_{\lambda_k}(u_{\lambda_k}) = +\infty,$$

This is in contradiction with (4.3). Hence the claim holds.

Since $u_0 \mid_{\Omega^c} = 0$, by the weak lower semi-continuity of the norm $\|\cdot\|_{H^1(V)}$ and Fatou's lemma, taking $u_{\lambda_k}^+$ as test function in Eq. (1.2), we get

$$
\begin{aligned}
&\int_{\Omega\cup\partial\Omega} \Gamma(u_0^+)d\mu + \int_\Omega |u_0^+|^2 d\mu - \int_{\{\Omega:|u_0^+|\le 1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu - \frac{1}{2}K_\Omega^0(u) \\
&\le \int_V \left(\Gamma(u_0^+) + |u_0^+|^2\right) d\mu - \int_{\{V:|u_0^+|\le 1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu - \frac{1}{2}K_V^0(u) \\
&\le \liminf_{k\to+\infty}\left[\int_V \left(\Gamma(u_{\lambda_k}^+) + |u_{\lambda_k}^+|^2\right) d\mu - \int_{\{V:|u_{\lambda_k}^+|\le 1\}} |u_{\lambda_k}^+|^2 \log|u_{\lambda_k}^+|^2 d\mu - \frac{1}{2}K_V^{\lambda_k}(u)\right] \\
&\le \liminf_{k\to+\infty}\left[\int_V \left(\Gamma(u_{\lambda_k}^+) + (\lambda_k a(x)+1)\,|u_{\lambda_k}^+|^2\right) d\mu - \int_{\{V:|u_{\lambda_k}^+|\le 1\}} |u_{\lambda_k}^+|^2 \log|u_{\lambda_k}^+|^2 d\mu - \frac{1}{2}K_V^{\lambda_k}(u)\right] \\
&= \liminf_{k\to+\infty}\left[\int_V |u_{\lambda_k}^+|^2 d\mu + \int_{\{V:|u_{\lambda_k}^+|>1\}} |u_{\lambda_k}^+|^2 \log|u_{\lambda_k}^+|^2 d\mu\right] \\
&= \int_V |u_0^+|^2 d\mu + \int_{\{V:|u_0^+|>1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu \\
&= \int_\Omega |u_0^+|^2 d\mu + \int_{\{\Omega:|u_0^+|>1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu,
\end{aligned}
$$

where

$$
\begin{aligned}
K_\Omega^0(u) &= \sum_{x\in\Omega\cup\partial\Omega}\sum_{y\sim x}\left[u_0^+(x)u_0^-(y) + u_0^-(x)u_0^+(y)\right], \\
K_V^0(u) &= \sum_{x\in V}\sum_{y\sim x}\left[u_0^+(x)u_0^-(y) + u_0^-(x)u_0^+(y)\right], \\
K_V^{\lambda_k}(u) &= \sum_{x\in V}\sum_{y\sim x}\left[u_{\lambda_k}^+(x)u_{\lambda_k}^-(y) + u_{\lambda_k}^-(x)u_{\lambda_k}^+(y)\right].
\end{aligned}
$$

Then

$$
J_\Omega'(u_0)\cdot u_0^+ = \int_{\Omega\cup\partial\Omega} \Gamma(u_0^+)d\mu - \int_\Omega |u_0^+|^2 \log|u_0^+|^2 d\mu - \frac{1}{2}K_\Omega^0(u) \le 0. \tag{4.5}
$$

Similarly, it holds that

$$
J_\Omega'(u_0)\cdot u_0^- = \int_{\Omega\cup\partial\Omega} \Gamma(u_0^-)d\mu - \int_\Omega |u_0^-|^2 \log|u_0^-|^2 d\mu - \frac{1}{2}K_\Omega^0(u) \le 0. \tag{4.6}
$$

In view of Lemmas 3.4 and 3.5, there exist two constants $s, t \in (0, 1]$ such that $\widetilde{u}_0 = su_0^+ + tu_0^- \in \mathcal{M}_\Omega$. Then

$$
\begin{aligned}
m_\Omega &\le J_\Omega(\widetilde{u}_0) = J_\Omega(\widetilde{u}_0) - \frac{1}{2}J_\Omega'(\widetilde{u}_0)\cdot(su_0^+) - \frac{1}{2}J_\Omega'(\widetilde{u}_0)\cdot(tu_0^-) \\
&= J_\Omega(su_0^+) - \frac{1}{2}J_\Omega'(su_0^+)\cdot(su_0^+) + J_\Omega(tu_0^-) - \frac{1}{2}J_\Omega'(tu_0^-)\cdot(tu_0^-) \\
&= \frac{s^2}{2}\|u_0^+\|_{L^2(\Omega)}^2 + \frac{t^2}{2}\|u_0^-\|_{L^2(\Omega)}^2 \\
&\le \frac{1}{2}\|u_0^+\|_2^2 + \frac{1}{2}\|u_0^-\|_2^2 \\
&\le \liminf_{k\to\infty}\left[\frac{1}{2}\|u_{\lambda_k}^+\|_2^2 + \frac{1}{2}\|u_{\lambda_k}^-\|_2^2\right] \\
&= \liminf_{k\to\infty}\left[J_{\lambda_k}(u_{\lambda_k}^+) - \frac{1}{2}J_{\lambda_k}'(u_{\lambda_k}^+)\cdot u_{\lambda_k}^+ + J_{\lambda_k}(u_{\lambda_k}^-) - \frac{1}{2}J_{\lambda_k}'(u_{\lambda_k}^-)\cdot u_{\lambda_k}^-\right] \\
&= \liminf_{k\to\infty}\left[J_{\lambda_k}(u_{\lambda_k}) - \frac{1}{2}J_{\lambda_k}'(u_{\lambda_k})\cdot u_{\lambda_k}^+ - \frac{1}{2}J_{\lambda_k}'(u_{\lambda_k})\cdot u_{\lambda_k}^-\right] \\
&= \liminf_{k\to+\infty} J_{\lambda_k}(u_{\lambda_k}) = \eta \le m_\Omega.
\end{aligned}
$$

Hence,

$$\lim_{\lambda\to+\infty} m_\lambda = m_\Omega.$$

This completes the proof. □

Next, we prove Theorem 1.3.

Proof of Theorem 1.3. Assume that $u_{\lambda_k} \in \mathcal{M}_{\lambda_k}$ satisfies $J_{\lambda_k}(u_{\lambda_k}) = m_{\lambda_k}$. We shall prove that u_{λ_k} converges in $H^1(V)$ to a least energy sign-changing solution u_0 of Eq. (1.8) along a subsequence.

Lemma 4.2 gives that $u_{\lambda_k} \in \mathcal{H}_{\lambda_k}$ is uniformly bounded. Consequently, we have that $\{u_{\lambda_k}\}$ is also bounded in $H^1(V)$. Therefore, we can assume that for any $p \in [2,\infty)$, $u_{\lambda_k} \to u_0$ in $L^p(V)$ and $u_{\lambda_k} \rightharpoonup u_0$ in $H^1(V)$. Moreover, in view of $u_0 \in \mathcal{N}_\Omega$ and we get from Lemma 4.1 that $u_0 \not\equiv 0$. As proved in Lemma 4.3, we can prove that $u_0 \mid_{\Omega^c}= 0$. Then it suffices to show that, as $k \to +\infty$, we have $\lambda_k \int_V a(x)|u^{\pm}_{\lambda_k}|^2 d\mu \to 0$ and $\int_V \Gamma(u^{\pm}_{\lambda_k})d\mu \to \int_V \Gamma(u^{\pm}_0)d\mu$. If not, we may assume that

$$\lim_{k\to+\infty} \lambda_k \int_V a(x)|u^{\pm}_{\lambda_k}|^2 d\mu = \delta > 0.$$

Since $u_0 \mid_{\Omega^c}= 0$, by weak lower semi-continuity of the norm $\|\cdot\|_{H^1(V)}$ and Fatou's lemma, taking $u^+_{\lambda_k}$ as test function in Eq. (1.2), we get

$$\begin{aligned}
&\int_{\Omega\cup\partial\Omega} \Gamma(u_0^+)d\mu + \int_\Omega |u_0^+|^2 d\mu - \int_{\{\Omega:|u_0^+|\le 1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu - \frac{1}{2}K^0_\Omega(u)\\
&< \int_V \left(\Gamma(u_0^+) + |u_0^+|^2\right) d\mu + \delta - \int_{\{V:|u_0^+|\le 1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu - \frac{1}{2}K^0_V(u)\\
&\le \liminf_{k\to+\infty}\left[\int_V \left(\Gamma(u^+_{\lambda_k}) + (\lambda_k a(x)+1)\,|u^+_{\lambda_k}|^2\right) d\mu - \int_{\{V:|u^+_{\lambda_k}|\le 1\}} |u^+_{\lambda_k}|^2 \log|u^+_{\lambda_k}|^2 d\mu - \frac{1}{2}K^{\lambda_k}_V(u)\right]\\
&= \liminf_{k\to+\infty}\left[\int_V |u^+_{\lambda_k}|^2 d\mu + \int_{\{V:|u^+_{\lambda_k}|>1\}} |u^+_{\lambda_k}|^2 \log|u^+_{\lambda_k}|^2 d\mu\right]\\
&= \int_V |u_0^+|^2 d\mu + \int_{\{V:|u_0^+|>1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu\\
&= \int_\Omega |u_0^+|^2 d\mu + \int_{\{\Omega:|u_0^+|>1\}} |u_0^+|^2 \log|u_0^+|^2 d\mu,
\end{aligned}$$

which implies that

$$J'_\Omega(u_0)\cdot u_0^+ = \int_{\Omega\cup\partial\Omega} \Gamma(u_0^+)d\mu - \int_\Omega |u_0^+|^2 \log|u_0^+|^2 d\mu - \frac{1}{2}K^0_\Omega(u) < 0. \tag{4.7}$$

Similarly, it holds that

$$J'_\Omega(u_0)\cdot u_0^- = \int_{\Omega\cup\partial\Omega} \Gamma(u_0^-)d\mu - \int_\Omega |u_0^-|^2 \log|u_0^-|^2 d\mu - \frac{1}{2}K^0_\Omega(u) < 0. \tag{4.8}$$

By similar arguments as above, if

$$\lim_{k\to+\infty} \int_V \Gamma(u^{\pm}_{\lambda_k})d\mu > \int_V \Gamma(u^{\pm}_0)d\mu,$$

we also have (4.7) and (4.8).

In view of Lemmas 3.4 and 3.5, there exist two constants $s, t \in (0, 1)$ such that $\widetilde{u}_0 = su_0^+ + tu_0^- \in \mathcal{M}_\Omega$. Consequently, we have

$$\begin{aligned}
m_\Omega \le& J_\Omega(\widetilde{u}_0) = J_\Omega(\widetilde{u}_0) - \frac{1}{2}J'_\Omega(\widetilde{u}_0)\cdot(su_0^+) - \frac{1}{2}J'_\Omega(\widetilde{u}_0)\cdot(tu_0^-)\\
=& J_\Omega(su_0^+) - \frac{1}{2}J'_\Omega(su_0^+)\cdot(su_0^+) + J_\Omega(tu_0^-) - \frac{1}{2}J'_\Omega(tu_0^-)\cdot(tu_0^-)\\
=& \frac{s^2}{2}\|u_0^+\|^2_{L^2(\Omega)} + \frac{t^2}{2}\|u_0^-\|^2_{L^2(\Omega)}\\
<& \frac{1}{2}\|u_0^+\|_2^2 + \frac{1}{2}\|u_0^-\|_2^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \liminf_{k\to+\infty}\left[\frac{1}{2}\|u_{\lambda_k}^+\|_2^2+\frac{1}{2}\|u_{\lambda_k}^-\|_2^2\right]\\
&=\liminf_{k\to+\infty}\left[J_{\lambda_k}(u_{\lambda_k}^+)-\frac{1}{2}J_{\lambda_k}'(u_{\lambda_k}^+)\cdot u_{\lambda_k}^+ + J_{\lambda_k}(u_{\lambda_k}^-)-\frac{1}{2}J_{\lambda_k}'(u_{\lambda_k}^-)\cdot u_{\lambda_k}^-\right]\\
&=\liminf_{k\to+\infty}\left[J_{\lambda_k}(u_{\lambda_k})-\frac{1}{2}J_{\lambda_k}'(u_{\lambda_k})\cdot u_{\lambda_k}^+ - \frac{1}{2}J_{\lambda_k}'(u_{\lambda_k})\cdot u_{\lambda_k}^-\right]\\
&=\liminf_{k\to+\infty} J_{\lambda_k}(u_{\lambda_k})\\
&=\liminf_{k\to+\infty} m_{\lambda_k}=m_{\Omega},
\end{aligned}
$$

which leads to a contradiction. Hence, we obtain that $u_{\lambda_k} \to u_0$ in $H^1(V)$ and u_0 is a least energy sign-changing solution of problem (1.8). □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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