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Multiple and Nodal Solutions for Parametric Dirichlet Equations Driven by the Double Phase Differential Operator

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Abstract

We consider a nonlinear parametric Dirichlet problem driven by the double phase differential operator. Using variational tools combined with critical groups, we show that for all small values of the parameter, the problem has at least three nontrivial bounded solutions which are ordered and we provide the sign information for all of them. Two solutions are of constant sign and the third one is nodal. Finally, we determine the asymptotic behavior of the nodal solution as the parameter converges to zero.

Keywords Double phase differential operator · Extremal constant sign solutions · Critical groups · Generalized Orlicz spaces

Mathematics Subject Classification Primary: 35J30 · 35J60; Secondary: 58E05

1 Introduction

Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 3)$ be a bounded domain with a Lipschitz boundary $\partial\Omega$. In this paper we study the following parametric Dirichlet problem (nonlinear eigenvalue problem)

$$\left\{\begin{array}{l} -\Delta_p^a u - \Delta_q u = \lambda f(z,u) \text{ in } \Omega, \\ u|_{\partial\Omega} = 0,\ 1 < q < p < N,\ \lambda > 0. \end{array}\right\} \tag{P_λ}$$

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This paper is dedicated with esteem to Professor Steven G. Krantz on the occasion of his 70th anniversary.

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In this problem, for $a \in L^\infty(\Omega) \setminus \{0\}$, $a(z) \geq 0$ for a.a. $z \in \Omega$, Δ_p^a denotes the weighted p-Laplace differential operator defined by

$$\Delta_p^a = \operatorname{div}\left(a(z)|Du|^{p-2}Du\right).$$

If $a \equiv 1$, then we have the standard p-Laplace differential operator. Problem (P_λ) is driven by the sum of two such operators with different exponents. So, the differential operator $u \mapsto -\Delta_p^a u - \Delta_q u$ driving problem (P_λ) is not homogeneous. In the reaction, the hypotheses on $f(z,x)$ are minimal and essentially involve restrictions on $f(z,\cdot)$ near zero. Our aim is to prove a multiplicity theorem for (P_λ) producing nodal (sign changing) solutions and finding their asymptotic behavior as $\lambda \to 0^+$. Recently, Leonardi and Papageorgiou [11] examined a similar problem. They studied a Robin (p,q)-equation with an indefinite potential term. Their equation is driven by the (p,q)-Laplace operator

$$-\Delta_p u - \Delta_q u.$$

This operator is associated with the energy functional

$$\int_\Omega \left(|Du|^p + |Du|^q\right) dz.$$

In this functional, the density function is

$$\hat{\eta}(t) = t^p + t^q,$$

which exhibits balanced growth, namely we have

$$t^p \leq \hat{\eta}(t) \leq c_0\left(1 + t^p\right) \text{ for all } t \geq 0, \text{ for some } c_0 > 0.$$

This property leads to a global regularity theory (regularity up to the boundary), which is included in the work of Lieberman [12].

In problem (P_λ) we do not assume that the weight $a(\cdot)$ is bounded away from zero, namely we do not require that $0 < \operatorname{ess\,inf}_\Omega a$. Then the operator of problem (P_λ) is associated with the so-called double phase integral functional

$$\int_\Omega [a(z)|Du|^p + |Du|^q] dz.$$

The density of this functional is

$$\eta(z,t) = a(z)t^p + t^q \text{ for all } z \in \Omega, \text{ all } t \geq 0.$$

Since $a(\cdot)$ may vanish, the integrand $\eta(z,t)$ exhibits unbalanced growth, namely

$$t^q \leq \eta(z,t) \leq c_1\left(1 + t^p\right) \text{ for a.a. } z \in \Omega, \text{ all } t \geq 0, \text{ for some } c_1 > 0.$$

Such operators are used in the description of diffusion-type processes in a space, where the behavior changes in different subdomains. For example, we can model composite materials with energy density of q-growth on $\{a = 0\}$ and of p-growth on $\{a > 0\}$. Such functionals were first studied by Marcellini [15, 16] and by Zhikov [25, 26], in the context of problems of the calculus of variations and of nonlinear elasticity theory including the Lavientiev gap phenomenon. Recently, the interest for such problems was revived and there have been efforts to develop a regularity theory. So far, there have been only local regularity results for minimizers of integral functionals of this kind. A global (up to the boundary) regularity theory remains elusive. The progress in this direction can be traced in the works of Marcellini [17], Mingione-Rădulescu [18] and Ragusa-Tachikawa [24]. The absence of such a global theory removes from consideration many effective tools which are readily available when dealing with balanced growth problems. This makes the study of double phase problems more difficult. A typical example of such a useful result is the equivalence between local Hölder and Sobolev minimizers. This was first observed by Brezis-Nirenberg [1] for semilinear problems driven by the Laplacian. Their result was later extended to p-Laplacian equations by Garcia Azorero-Manfredi-Peral Alonso [4] and to more general anisotropic operators by Papageorgiou-Rădulescu-Zhang [22]. This equivalence result proved to be very effective in obtaining multiplicity results for different kinds of nonlinear elliptic boundary value problems. Therefore the methods and techniques employed by Leonardi-Papageorgiou [11] can not be used here and we need to come up with a new approach.

We mention that recently there have been existence and multiplicity results for double phase equations. We mention the works of Gasinski-Papageorgiou [6], Gasinski-Winkert [7], Liu-Dai [13], Liu-Papageorgiou [14], Papageorgiou-Pudelko-Rădulescu [19], Papageorgiou-Vetro-Vetro [23]. Of the aforementioned works only Gasinski-Papageorgiou [6] and Liu-Papageorgiou [14] produce nodal solutions but under stronger conditions on the reaction and using different methods and techniques.

2 Mathematical Background-Hypotheses

As a consequence of the unbalanced growth of the differential operator, we have to abandon the convenient functional framework of standard Sobolev spaces and use generalized Orlicz-Sobolev spaces. A comprehensive presentation of the theory of these spaces can be found in the book of Harjulehto-Hästo [9].

In what follows $C^{0,1}(\bar{\Omega}) := \{u : \bar{\Omega} \to \mathbb{R} \text{ is Lipschitz}\}$. Our hypotheses on the weight function $a(\cdot)$ and the exponents p, q are the following:

$$\mathrm{H}_0 : a \in C^{0,1}(\bar{\Omega})\backslash\{0\}, a(z) \geq 0 \text{ for any } z \in \bar{\Omega},\ 1 < q < p < N,\ 2 \leq p,\ \frac{p}{q} < 1 + \frac{1}{N}.$$

Remark 2.1 The assumption that $a \in C^{0,1}(\bar{\Omega})$ guarantees that the Poincaré inequality holds in the corresponding generalized Orlicz-Sobolev space (see Harjulehto-Hästo [9, pp.100,138]). The last condition $\frac{p}{q} < 1 + \frac{1}{N}$ is common in Dirichlet double phase problems and says that the two exponents p and q can not be far apart. It implies that $p < q^* = \frac{Nq}{N-q}$, which in turn leads to some useful compact embeddings of some relevant spaces.

Let $L^0(\Omega)$ be the space of all measurable functions $u : \Omega \to \mathbb{R}$. As usual, we identity two such functions which differ only on a Lebesque-null subset of Ω. Also $\eta(z,t)$ denotes the double phase density defined by

$$\eta(z,t) = a(z)t^p + t^q \quad \text{for all } z \in \Omega, \text{ all } t \geq 0.$$

Then the generalized Orlicz-Lebesgue space $L^\eta(\Omega)$ is defined by

$$L^\eta(\Omega) = \left\{ u \in L^0(\Omega) : \rho_\eta(u) = \int_\Omega \eta(z, |u|)\, dz < \infty \right\}.$$

The function $\rho_\eta(\cdot)$ is known as the "modular function" corresponding to η. On $L^\eta(\Omega)$ we introduce the so-called "Luxemburg norm" $\|\cdot\|_\eta$ defined by

$$\|u\|_\eta = \inf \left\{ \lambda > 0 : \rho_\eta\left(\frac{u}{\lambda}\right) \leq 1 \right\} \text{ for all } u \in L^\eta(\Omega).$$

Equipped with this norm, the space $L^\eta(\Omega)$ becomes a Banach space which is separable and reflexive (in fact uniformly convex). Using $L^\eta(\Omega)$, we can define the corresponding generalized Orlicz-Sobolev space $W^{1,\eta}(\Omega)$ by

$$W^{1,\eta}(\Omega) = \{u \in L^\eta(\Omega) : |Du| \in L^\eta(\Omega)\}.$$

Here by Du we denote the weak gradient of u. The norm $\|\cdot\|_{1,\eta}$ of this space is given

$$\|u\|_{1,\eta} = \|u\|_\eta + \|Du\|_\eta \quad \text{for all } u \in W^{1,\eta}(\Omega).$$

Note that $\|Du\|_\eta = \||Du|\|_\eta$. Also, we define

$$W_0^{1,\eta}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{1,\eta}}.$$

Both spaces $W^{1,\eta}(\Omega)$, $W_0^{1,\eta}(\Omega)$ are separable, reflexive (in fact uniformly convex) Banach spaces. On account of hypotheses H_0(in particular since $a \in C^{0,1}(\bar{\Omega})$), the Poincaré inequality holds on $W_0^{1,\eta}(\Omega)$, namely we can find $\hat{c} = \hat{c}(\Omega) > 0$ such that

$$\|u\|_\eta \leq \hat{c}\|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Therefore on $W_0^{1,\eta}(\Omega)$ we consider the equivalent norm $\|\cdot\|$ defined by

$$\|u\| = \|Du\|_\eta \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

There is a close relation between the norm $\|\cdot\|$ and the modular function $\rho_\eta(\cdot)$ defined on $W_0^{1,\eta}(\Omega)$.

Proposition 2.2 *The following results hold:*

(a) $\|u\| = \lambda \Leftrightarrow \rho_\eta\left(\frac{Du}{\lambda}\right) = 1$.
(b) $\|u\| < 1(resp. = 1, > 1) \Leftrightarrow \rho_\eta(Du) < 1(resp. = 1, > 1)$.
(c) $\|u\| < 1 \Rightarrow \|u\|^p \le \rho_\eta(Du) \le \|u\|^q$.
(d) $\|u\| > 1 \Rightarrow \|u\|^q \le \rho_\eta(Du) \le \|u\|^p$.
(e) $\|u\| \to 0(resp. \to +\infty) \Leftrightarrow \rho_\eta(Du) \to 0(resp. \to +\infty)$.

Also the following embeddings are helpful and generalize to the present setting the Sobolev embedding theorem.

Proposition 2.3 *The following results hold:*

(a) $L^\eta(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W_0^{1,\eta}(\Omega) \hookrightarrow W_0^{1,r}(\Omega)$ *continuously for all* $1 \le r \le q$.
(b) $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ *continuously if* $1 \le r \le q^*$ *and compactly if* $1 \le r < q^*$.
(c) $L^p(\Omega) \hookrightarrow L^\eta(\Omega)$ *continuously.*

Let $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ be the nonlinear operator defined by

$$\langle V(u), h\rangle = \int_\Omega (a(z)|Du|^{p-2}Du + |Du|^{q-2}Du, Dh)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,\eta}(\Omega).$$

This operator has the following properties(see Liu-Dai [13, Proposition 3.1]).

Proposition 2.4 *The operator* $V : W_0^{1,\eta}(\Omega) \to W_0^{1,\eta}(\Omega)^*$ *is bounded(that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type* $(S)_+$*, that is, "if* $u_n \xrightarrow{w} u$ *in* $W_0^{1,\eta}(\Omega)$ *and* $\limsup_{n\to\infty}\langle V(u_n), u_n - u\rangle \le 0$*, then* $u_n \to u$ *in* $W_0^{1,\eta}(\Omega)$*."*

We introduce also the following modular function

$$\rho_a(Du) = \int_\Omega a(z)|Du|^p\, dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

This function is continuous, convex, thus weakly lower semicontinuous. If $u \in L^0(\Omega)$, then we set

$$u^\pm(z) = \max\{\pm u(z), 0\} \quad \text{for all } z \in \Omega.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$. Also, if $u \in W_0^{1,\eta}(\Omega)$, then $u^\pm \in W_0^{1,\eta}(\Omega)$. If $u \in L^0(\Omega)$, we write $0 \prec u$, if for all $K \subseteq \Omega$ compact, we have $0 < c_K \le u(z)$ for a.a. $z \in K$. Clearly such a function satisfies $0 < u(z)$ for a.a. $z \in \Omega$. If $u, v \in L^0(\Omega)$ and $u(z) \le v(z)$ for a.a. $x \in \Omega$, then

$$[u, v] = \{h \in W_0^{1,\eta}(\Omega) : u(z) \le h(z) \le v(z) \text{ for a.a. } z \in \Omega\}.$$

To overcome the absence of a global regularity theory, we will use critical groups(Morse theory). Let X be a Banach space, $\varphi \in C^1(X)$, $c \in \mathbb{R}$. We set

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}(\text{the critical set of } \varphi),$$
$$\varphi^c = \{u \in X : \varphi(u) \le c\}.$$

Consider a topological pair (Y_1, Y_2) such that $Y_2 \subseteq Y_1 \subseteq X$. For $k \in \mathbb{N}_0$, by $H_k(Y_1, Y_2)$ we denote the k-th relative singular homology group with integer coefficients. Suppose $u \in K_\varphi$ is isolated and $c = \varphi(u)$. Then the critical groups of φ at u are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap \mathcal{U}, \varphi^c \cap \mathcal{U} \backslash \{u\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with $\mathcal{U}$ being an open neighborhood of u such that $K_\varphi \cap \varphi^c \cap \mathcal{U} = \{u\}$. The excision property of singular homology implies that this definition is independent of the isolating neighborhood $\mathcal{U}$. Suppose that $\varphi \in C^1(X)$ satisfies the C-condition (see [21, p.366] and that $-\infty < \inf \varphi(K_\varphi)$. Then the critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \quad \text{for all } k \in \mathbb{N}_0.$$

The second deformation theorem (see [21, p.366]) implies that this definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$. Suppose that K_φ is finite. We introduce the following series in $t \in \mathbb{R}$.

$$M(t, u) = \sum_{k \in \mathbb{N}_0} \text{rank} C_k(\varphi, u) t^k \quad \text{for all } u \in K_\varphi,$$
$$P(t, \infty) = \sum_{k \in \mathbb{N}_0} \text{rank} C_k(\varphi, \infty) t^k.$$

The "Morse relation" says that

$$\sum_{u \in K_\varphi} M(t, u) = P(t, \infty) + (1 + t) Q(t), \; t \in \mathbb{R}, \tag{1}$$

with $Q(t) = \sum_{k \in \mathbb{N}_0} \beta_k t^k$ being a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

To exploit the properties of critical groups, we will need the following notion. Suppose X is a finite dimensional Banach space and $g : X \to \mathbb{R}$. We say that $g(\cdot)$ is locally Lipschitz, if for every $K \subseteq X$ compact, $g|_K$ is Lipschitz with constant $\theta_K > 0$, that is,

$$|g(u) - g(v)| \le \theta_K \|u - v\|_X \quad \text{for all } u, v \in K.$$

If $\hat{g} : \Omega \times X \to \mathbb{R}$, then we say that $\hat{g}(\cdot, \cdot)$ is an L^∞-locally Lipschitz integrand, if for all $x \in \mathbb{R}$, $z \mapsto \hat{g}(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \mapsto \hat{g}(z, x)$ is locally

Lipschitz with constant $\theta_K \in L^\infty(\Omega)$ for all $K \subseteq \mathbb{R}$ compact. Such a function is jointly measurable (see Hu-Papageorgiou [10, p.59]).

The hypotheses on the reaction $f(z,x)$ are the following:

H_1 : $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is on L^∞-locally Lipschitz integrand such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,x)x \geq 0$ for all $x \in \mathbb{R}$ and

(i) $|f(z,x)| \leq \hat{a}(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $\hat{a} \in L^\infty(\Omega)$, $p < r < q^*$;

(ii) if $F(z,x) = \int_0^x f(z,s)\,ds$, then there exist $\delta > 0$ and $\tau \in (1,q)$ such that

$$c_2|x|^\tau \leq f(z,x)x \leq \tau F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta, \text{ some } c_2 > 0,$$

$$\limsup_{x \to 0} \frac{F(z,x)}{|x|^\tau} \leq c^* \quad \text{for a.a. } z \in \Omega, \text{ some } c^* > 0.$$

Remark 2.5 The hypotheses on $f(z,x)$ are minimal and imply the presence of a "concave" term near zero.

To produce nodal solutions, we will first show that the problem has extremal constant sign solutions, that is, a smallest positive solution u_λ^* and a biggest negative solution v_λ^*. Then we will look at the order interval $[v_\lambda^*, u_\lambda^*]$ and try to obtain a nontrival solution of (P_λ) distinct from u_λ^* and v_λ^*. The extremality of u_λ^* and v_λ^* will imply that such a solution is nodal. To do this, we need to use critical groups since the lack of regularity properties on the solutions does not permit the use of more direct methods.

3 An Auxiliary Problem

To implement the strategy outlined above, we need to produce extremal constant sign solutions. To do this first we consider the following auxiliary double phase problem, the solutions of which will be helpful in producing the extremal constant sign solutions.

The auxiliary double phase problem is the following

$$\begin{Bmatrix} -\Delta_p^a u - \Delta_q u = \lambda c_2 |u|^{\tau-2} u \text{ in } \Omega, \\ u|_{\partial\Omega} = 0, \ 1 < q < p < N, \ \lambda > 0. \end{Bmatrix} \tag{Q_λ}$$

Proposition 3.1 *If hypotheses H_0 hold and $\lambda > 0$, then problem (Q_λ) has a unique positive solution*

$$\bar{u}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), \ 0 \prec \bar{u}_\lambda,$$

and since the equation is odd, $\bar{v}_\lambda = -\bar{u}_\lambda$ is the unique negative solution of (Q_λ).

Proof Consider the C^1-functional $\beta_\lambda^+ : W_0^{1,\eta}(\Omega) \to \mathbb{R}$ defined by

$$\beta_\lambda^+(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \frac{\lambda c_2}{\tau}\|u^+\|_\tau^\tau.$$

Suppose that $\|u\| \geq 1$. Then

$$\beta_\lambda^+(u) \geq \frac{1}{p}\|u\|^q - \lambda c_3 \|u\|^\tau \text{ for some } c_3 > 0 \text{(see Propositions 2.2 and 2.3)}.$$

Since $q > \tau$, we see that $\beta_\lambda^+(\cdot)$ is coercive. Also using Proposition 2.3, we see that $\beta_\lambda^+(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\bar{u}_\lambda \in W_0^{1,\eta}(\Omega)$ such that

$$\beta_\lambda^+(\bar{u}_\lambda) = \inf\{\beta_\lambda^+(u) : u \in W_0^{1,\eta}(\Omega)\}. \tag{2}$$

Let $u \in W_0^{1,\eta}(\Omega)\backslash\{0\}$ with $u(z) \geq 0$ for a.a. $z \in \Omega$. For $t \in (0, 1)$, we have

$$\beta_\lambda^+(tu) = \frac{t^p}{p}\rho_a(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{\lambda c_2 t^\tau}{\tau}\|u\|_\tau^\tau.$$

Since $\tau < q < p$, choosing $t \in (0, 1)$ small, we see that

$$\begin{aligned}
&\beta_\lambda^+(tu) < 0, \\
\Rightarrow &\beta_\lambda^+(\bar{u}_\lambda) < 0 = \beta_\lambda^+(0) \text{(see (2))}, \\
\Rightarrow &\bar{u}_\lambda \neq 0.
\end{aligned}$$

From (2), we have

$$\begin{aligned}
&\langle(\beta_\lambda^+)'(\bar{u}_\lambda), h\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega), \\
\Rightarrow &\langle V(\bar{u}_\lambda), h\rangle = \lambda c_2 \int_\Omega (\bar{u}_\lambda^+)^{\tau-1} h\, dz \quad \text{for all } h \in W_0^{1,\eta}(\Omega),
\end{aligned} \tag{3}$$

In (3) we use the test function $h = -\bar{u}_\lambda^- \in W_0^{1,\eta}(\Omega)$ and obtain

$$\begin{aligned}
&\varphi_\eta(D\bar{u}_\lambda^-) = 0, \\
\Rightarrow &\bar{u}_\lambda \geq 0,\ \bar{u}_\lambda \neq 0 \text{ (see Proposition 2.2)}.
\end{aligned}$$

Then $\bar{u}_\lambda$ is a positive solution of (Q_λ). Thereom 3.1 of Gasinski-Winkert [7] implies that $\bar{u}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. Finally, Proposition 2.4 of Papageorgiou-Vetro-Vetro [23] says that $0 \prec \bar{u}_\lambda$.

Next we show the uniqueness of this positive solution. For this purpose, we introduce the integral functional $j : L^1(\Omega) \mapsto \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\rho_a\left(Du^{\frac{1}{q}}\right) + \frac{1}{q}\|Du^{\frac{1}{q}}\|_q^q & \text{if } u \geq 0, u^{\frac{1}{q}} \in W_0^{1,\eta}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathrm{dom}\, j = \{u \in L^1(\Omega) : j(u) < \infty\}$(the effective domain of $j(\cdot)$). Consider the intergrand $\hat{\eta}(z,t)$ defined by

$$\hat{\eta}(z,t) = \frac{1}{p}a(z)t^p + \frac{1}{q}t^q \quad \text{for all } z \in \Omega,\ \text{all } t \geq 0.$$

Evidently $\hat{\eta}(\cdot,\cdot)$ is continuous and for all $z \in \Omega$, $\hat{\eta}(z,\cdot)$ is increasing and $t \mapsto \hat{\eta}\left(z, t^{\frac{1}{q}}\right)$ is convex. Let $u_1, u_2 \in \mathrm{dom}\, j$ and set

$$u = (tu_1 + (1-t)u_2)^{\frac{1}{q}} \quad \text{with} \quad t \in [0,1].$$

From Diaz-Saa [3](proof of Lemma 1), we have

$$\begin{aligned}
&|Du| \leq \left(t\left|Du_1^{\frac{1}{q}}\right|^q + (1-t)\left|Du_2^{\frac{1}{q}}\right|^q\right)^{\frac{1}{q}} \\
\Rightarrow &\hat{\eta}(z,|Du|) \leq \hat{\eta}\left(z, \left(t\left|Du_1^{\frac{1}{q}}\right|^q + (1-t)\left|Du_2^{\frac{1}{q}}\right|^q\right)^{\frac{1}{q}}\right) \text{(since } \hat{\eta}(z,\cdot) \text{ is increasing)} \\
&\leq t\hat{\eta}\left(z, |Du_1|^{\frac{1}{q}}\right) + (1-t)\hat{\eta}\left(z, |Du_2|^{\frac{1}{q}}\right) \text{(since } t \mapsto \hat{\eta}\left(z, t^{\frac{1}{q}}\right) \text{ is convex).}
\end{aligned}$$

Then $j(\cdot)$ is convex. Suppose $\bar{\omega}_\lambda$ is another positive solution of problem (Q_λ). Again we have

$$\bar{\omega}_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega), \quad 0 \prec \bar{\omega}_\lambda.$$

Let $\epsilon > 0$ and define $\bar{u}_\lambda^\epsilon = \bar{u}_\lambda + \epsilon$, $\bar{w}_\lambda^\epsilon = \bar{w}_\lambda + \epsilon$. Let $L^\infty(\Omega)_+$ denote the positive (order) cone of $L^\infty(\Omega)$(that is, $L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega) : u(z) \geq 0$ for a.a. $z \in \Omega\}$). This cone has a nonempty interior given by $\mathrm{int}L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega)_+ : 0 < \mathrm{essinf}_\Omega u\}$. Evidently $\bar{u}_\lambda^\epsilon, \bar{w}_\lambda^\epsilon \in \mathrm{int}L^\infty(\Omega)_+$. Then using Proposition 4.1.22 of Papageorgiou-Rădulescu-Repovs [21, p.274], we have

$$\frac{\bar{u}_\lambda^\epsilon}{\bar{w}_\lambda^\epsilon} \in L^\infty(\Omega),\ \frac{\bar{w}_\lambda^\epsilon}{\bar{u}_\lambda^\epsilon} \in L^\infty(\Omega). \tag{4}$$

Let $h = (\bar{u}_\lambda^\epsilon)^q - (\bar{w}_\lambda^\epsilon)^q \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. On account of (4), we see that for $t \in (0,1)$ small,

$$(\bar{u}_\lambda^\epsilon)^q + th \in \mathrm{dom}\, j, \quad (\bar{w}_\lambda^\epsilon)^q + th \in \mathrm{dom}\, j.$$

We compute the directional derivatives of $j(\cdot)$ at $(\bar{u}_\lambda^\epsilon)^q$ and at $(\bar{w}_\lambda^\epsilon)^q$ in the direction h. From the convexity of $j(\cdot)$ and using the nonlinear Green's identity (see [21, p.34]), we have

$$
\begin{aligned}
j'((\bar{u}_\lambda^\epsilon)^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^a \bar{u}_\lambda - \Delta_q \bar{u}_\lambda}{(\bar{u}_\lambda^\epsilon)^{q-1}} h\,dz \\
&= \frac{\lambda c_2}{q}\int_\Omega \frac{\bar{u}_\lambda^{\tau-1}}{(\bar{u}_\lambda^\epsilon)^{q-1}} h\,dz, \\
j'((\bar{w}_\lambda^\epsilon)^q)(h) &= \frac{1}{q}\int_\Omega \frac{-\Delta_p^a \bar{w}_\lambda - \Delta_q \bar{w}_\lambda}{(\bar{w}_\lambda^\epsilon)^{q-1}} h\,dz \\
&= \frac{\lambda c_2}{q}\int_\Omega \frac{\bar{w}_\lambda^{\tau-1}}{(\bar{w}_\lambda^\epsilon)^{q-1}} h\,dz.
\end{aligned}
$$

The convexity of $j(\cdot)$ implies the monotonicity of the directional derivative. Therefore we have

$$
0 \le \int_\Omega \left(\frac{\bar{u}_\lambda^{\tau-1}}{(\bar{u}_\lambda^\epsilon)^{q-1}} - \frac{\bar{w}_\lambda^{\tau-1}}{(\bar{w}_\lambda^\epsilon)^{q-1}} \right) ((\bar{u}_\lambda^\epsilon)^q - (\bar{w}_\lambda^\epsilon)^q)\,dz \tag{5}
$$

Note that as $\epsilon \to 0^+$, we have

$$
\begin{aligned}
&\left(\frac{\bar{u}_\lambda^{\tau-1}}{(\bar{u}_\lambda^\epsilon)^{q-1}} - \frac{\bar{w}_\lambda^{\tau-1}}{(\bar{w}_\lambda^\epsilon)^{q-1}} \right) ((\bar{u}_\lambda^\epsilon)^q - (\bar{w}_\lambda^\epsilon)^q) \\
\longrightarrow &\left(\frac{1}{\bar{u}_\lambda^{q-\tau}} - \frac{1}{\bar{w}_\lambda^{q-\tau}} \right) (\bar{u}_\lambda^q - \bar{w}_\lambda^q) \quad \text{for a.a. } z \in \Omega.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\left| \left(\frac{\bar{u}_\lambda^{\tau-1}}{(\bar{u}_\lambda^\epsilon)^{q-\tau}} - \frac{\bar{w}_\lambda^{\tau-1}}{(\bar{w}_\lambda^\epsilon)^{q-\tau}} \right) ((\bar{u}_\lambda^\epsilon)^q - (\bar{w}_\lambda^\epsilon)^q) \right| \\
&\le c_4[\|\bar{u}_\lambda\|_\infty^\tau + \|\bar{w}_\lambda\|_\infty^\tau + 1] \quad \text{for some } c_4 > 0,\ a.a.\ z \in \Omega \,(\text{see } (4)).
\end{aligned}
$$

Therefore applying the Lebesgue dominated convergence theorem on (5) as $\epsilon \to 0^+$, we obtain

$$
\begin{aligned}
&0 \le \int_\Omega \left[\frac{1}{\bar{u}_\lambda^{q-\tau}} - \frac{1}{\bar{w}_\lambda^{q-\tau}} \right] (\bar{u}_\lambda^q - \bar{w}_\lambda^q)\,dz \le 0, \\
&\Rightarrow \bar{u}_\lambda = \bar{w}_\lambda \left(x \mapsto \frac{1}{x^{q-\tau}} \text{ is strictly decreasing on } \mathring{\mathbb{R}}_+ = (0,\infty) \right).
\end{aligned}
$$

This proves the uniqueness of the positive solution of (Q_λ). Since (Q_λ) is odd, $\bar{v}_\lambda = -\bar{u}_\lambda$ is the unique negative solution of (Q_λ). □

4 Extremal Constant Sign Solutions

In this section, we look for solutions of (P_λ) which have constant sign and we will produce a smallest positive solution and a biggest negative solution (extremal constant sign solutions).

we introduce the following sets

$$\begin{aligned} S_\lambda^+ &: \text{ set of positive solutions of } (P_\lambda), \\ S_\lambda^- &: \text{ set of negative solutions of } (P_\lambda). \end{aligned}$$

We have $S_\lambda^+, S_\lambda^- \subseteq W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. First we show that for $\lambda > 0$ small, these sets are nonempty.

Proposition 4.1 *If hypotheses H_0, H_1 hold, then there exists $\lambda^* > 0$ such that $S_\lambda^\pm \neq \emptyset$ for all $\lambda \in (0, \lambda^*)$ and $0 \prec u$ for all $u \in S_\lambda^+$, $0 \prec -v$ for all $v \in S_\lambda^-$.*

Proof First we show the nonemptiness of S_λ^+. To this end, let $\varphi_\lambda^+ : W_0^{1,\eta}(\Omega) \mapsto \mathbb{R}$ be the C^1-functional defined by

$$\varphi_\lambda^+(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \lambda F(z, u^+)\, dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Hypotheses H_1 imply that

$$F(z, x) \leq c_5(|x|^\tau + |x|^r) \text{ for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{some } c_5 > 0. \tag{6}$$

Then for $u \in W_0^{1,\eta}(\Omega)$ with $\|u\| \leq 1$, we have

$$\begin{aligned} \varphi_\lambda^+(u) &\geq \frac{1}{p}\rho_\eta(Du) - \lambda c_5(\|u\|_\tau^\tau + \|u\|_r^r)(\text{see (6) and recall } q < p) \\ &\geq \frac{1}{p}\|u\|^p - \lambda c_5(\|u\|^\tau + \|u\|^r) \text{ for some } c_6 > 0(\text{see Propositions 2.2 and 2.3}). \end{aligned} \tag{7}$$

Let $\theta \in \left(0, \frac{1}{p-\tau}\right)$ and consider $\lambda \in (0, 1)$ and $u \in W_0^{1,\eta}(\Omega)$ with $\|u\| = \lambda^\theta < 1$. From (7), we have

$$\varphi_\lambda^+(u) \geq \left(\frac{1}{p} - c_6\left(\lambda^{1-(p-\tau)\theta} + \lambda^{1+(r-p)\theta}\right)\right)\lambda^{\theta p}. \tag{8}$$

From the choice of θ, we have $1 - \theta(p - \tau) > 0$. Also $1 + \theta(r - p) > 1$. Hence $\gamma(\lambda) = \lambda^{1-(p-\tau)\theta} + \lambda^{1+(r-p)\theta} \to 0$ as $\lambda \to 0^+$. So we can find $\lambda^* \in (0, 1)$ such that $c_6\gamma(\lambda) < \frac{1}{p}$ for all $\lambda \in (0, \lambda^*)$. From (8), we have

$$\varphi_\lambda^+(u) \geq \left(\frac{1}{p} - c_6\gamma(\lambda)\right)\lambda^{\theta p} > 0 \text{ for all } \|u\| = \lambda^\theta,\ \text{all } \lambda \in (0, \lambda^*).$$

Let $\bar{B}_\lambda = \{u \in W_0^{1,\eta}(\Omega) : \|u\| \leq \lambda^\theta\}$. The reflexivity of $W_0^{1,\eta}(\Omega)$ and the Eberlein-Smulian theorem imply that $\bar{B}_\lambda$ is sequentially weakly compact. Also $\varphi_\lambda^+(\cdot)$ is sequentially weakly lower-semicontinuous. So, by the Weierestrass-Tonelli theorem, we can find $u_\lambda \in \bar{B}_\lambda$ such that

$$\varphi_\lambda^+(u_\lambda) = \inf\left\{\varphi_\lambda^+(u) : u \in \bar{B}_\lambda\right\}. \tag{9}$$

Let $u \in C_0^1(\bar{\Omega})_+$, $u \neq 0$ and choose $t \in (0,1)$ small so that

$$0 \leq tu(z) \leq \delta \quad \text{for all } z \in \bar{\Omega},$$

where $\delta > 0$ is as postulated by hypothesis H_1(ii). We have

$$\varphi_\lambda^+(tu) \leq \frac{t^q}{q}\rho_\eta(Du) - \frac{\lambda c_2 t^\tau}{\tau}\|u\|_\tau^\tau (\text{since } q < p, t \in (0,1) \text{ and see } \mathrm{H}_1(\mathrm{ii})).$$

Since $\tau < q$, chooseing $t \in (0,1)$ even smaller if necessary, we have

$$\begin{aligned} &\varphi_\lambda^+(tu) < 0, \\ \Rightarrow &\varphi_\lambda^+(u_\lambda) < 0 = \varphi_\lambda^+(0) (\text{see } (9)), \\ \Rightarrow &u_\lambda \neq 0. \end{aligned}$$

From (9), we have

$$\begin{aligned} &\langle(\varphi_\lambda^+)'(u_\lambda), h\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega), \\ \Rightarrow &\langle V(u_\lambda), h\rangle = \int_\Omega \lambda f(z, u_\lambda^+) h\, dz \quad \text{for all } h \in W_0^{1,\eta}(\Omega). \end{aligned}$$

Let $h = -u_\lambda^- \in W_0^{1,\eta}(\Omega)$. We obtain

$$\begin{aligned} &\rho_\eta(Du_\lambda^-) = 0, \\ \Rightarrow &u_\lambda \geq 0,\ u_\lambda \neq 0 \,(\text{see Proposition 2.2}). \end{aligned}$$

So, $u_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ is a positive solution of (P_λ) and we infer that $S_\lambda^+ \neq \emptyset$ for $\lambda \in (0, \lambda^*)$. If $u \in S_\lambda^+$, then $u \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$. Let $\rho = \|u\|_\infty$. On account of hypothesis H_1(ii), we can find $\hat{F}_\rho > 0$ such that $f(z,x) + \hat{F}_\rho x^{p-1} \geq 0$ for a.a. $z \in \Omega$. Hence

$$\begin{aligned} &-\Delta_p^a u - \Delta_q u + \hat{F}_\rho u^{p-1} \geq 0 \quad \text{in } \Omega, \\ \Rightarrow &0 \prec u (\text{see Papageorgiou-Vetro-Vetro [23, Proposition2.4]}). \end{aligned}$$

For the negative solutions, we work with the C^1-functional

$$\varphi_\lambda^-(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \int_\Omega \lambda F(z, -u^-)\,dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Reasoning as above we show that $S_\lambda^- \neq \emptyset$ for all $\lambda \in (0, \lambda^*)$(by taking $\lambda^* > 0$ even smaller if necessary) and if $v \in S_\lambda^-$, then $0 \prec -v$. □

From Papageorgiou-Rădulescu-Repovs [20] (proof of Proposition 7), we have that S_λ^+ is downward directed (that is, if $u_1,\ u_2 \in S_\lambda^+$, there is $u \in S_\lambda^+$ with $u \leq u_1$, $u \leq u_2$), S_λ^- is upward directed (that is, if $v_1, v_2 \in S_\lambda^-$, there is $v \in S_\lambda^-$ with $v_1 \leq v$, $v_2 \leq v$).

Using these properties of the solution sets, we can generate extremal constant sign solutions, that is, we can prove that there is $u_\lambda^* \in S_\lambda^+$ such that $u_\lambda^* \leq u$ for all $u \in S_\lambda^+$ and there is $v_\lambda^* \in S_\lambda^-$ such that $v \leq v_\lambda^*$ for all $v \in S_\lambda^-$.

Proposition 4.2 *if hypotheses H_0, H_1 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has extremal constant sign solutions*

$$u_\lambda^* \in S_\lambda^+ \quad \textit{and} \quad v_\lambda^* \in S_\lambda^-.$$

Proof Theorem 5.109, p.308 of Hu-Papageorgiou [10] says that we can find a decreasing sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq S_\lambda^+$ such that

$$\inf S_\lambda^+ = \inf_{n\in\mathbb{N}} u_n.$$

We have

$$\langle V(u_n), h\rangle = \int_\Omega \lambda f(z, u_n)h\,dz \quad \text{for all } h \in W_0^{1,\eta}(\Omega),\ \text{all } n \in \mathbb{N}, \tag{10}$$

$$0 \leq u_n \leq u_1 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

In (10) we choose the test function $h = u_n \in W_0^{1,\eta}(\Omega)$. Using (11) and hypothesis H_1(i), we obtain

$$\rho_\eta(Du_n) \leq \lambda c_7 \quad \text{for some } c_7 > 0,\ \text{all } n \in \mathbb{N},$$

$$\Rightarrow \{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,\eta}(\Omega) \text{ is bounded (see Proposition 2.2).} \tag{12}$$

From Colasuonno-Squassina [2] (Section 3.2), for $m \in \left(\frac{N}{q}, \infty\right)$, we have

$$\|u_n\|_\infty \leq \lambda c_8 \|f(\cdot, u_n(\cdot))\|_m^{\frac{1}{q-1}} \quad \text{for some } c_8 > 0,\ \text{all } n \in \mathbb{N}. \tag{13}$$

From (12), we see that at least for a subsequence, we have

$$u_n \xrightarrow{w} u_\lambda^* \text{ in } W_0^{1,\eta}(\Omega),\ u_n \to u_\lambda^* \text{ in } L^r(\Omega) \text{(see Proposition 2.3).} \tag{14}$$

In (10) we use $h = u_n - u_\lambda^* \in W_0^{1,\eta}(\Omega)$, pass to the limit as $n \to \infty$ and use (14). We obtain

$$\begin{aligned}
&\lim_{n\to\infty} \langle V(u_n), u_n - u_\lambda^* \rangle = 0, \\
\Rightarrow & u_n \to u_\lambda^* \quad \text{in } W_0^{1,\eta}(\Omega) (\text{see Proposition 2.4}).
\end{aligned} \tag{15}$$

Suppose that $u_\lambda^* = 0$. From (13) and (15), we see that

$$\|u_n\|_\infty \to 0.$$

Therefore we can find $n_0 \in \mathbb{N}$ such that

$$\begin{aligned}
&0 \le u_n(z) \le \delta \quad \text{for a.a. } z \in \Omega, \ \text{all } n \ge n_0, \\
\Rightarrow & c_2 u_n(z)^{\tau-1} \le f(z, u_n(z)) \quad \text{for a.a. } z \in \Omega, \ \text{all } n \ge n_0 (\text{see hypothesis } \mathrm{H}_1(\text{ii})).
\end{aligned} \tag{16}$$

Fix $n \ge n_0$ and consider the Caratheodary function $k_+(z, x)$ defined by

$$k_+(z, x) = \begin{cases} c_2 (x^+)^{\tau-1} & \text{if } x \le u_n(z) \\ c_2 u_n(z)^{\tau-1} & \text{if } u_n(z) < x \end{cases} \tag{17}$$

We set $K_+(z, x) = \int_0^x k_+(z, s)\, ds$ and consider the C^1-functional $\varphi_\lambda^+ : W_0^{1,\eta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_\lambda^+(u) = \frac{1}{p} \rho_a(Du) + \frac{1}{q} \|Du\|_q^q - \int_\Omega \lambda K_+(z, u)\, dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

From (17), we see that $\varphi_\lambda^+(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda \in W_0^{1,\eta}(\Omega)$ such that

$$\varphi_\lambda^+(\tilde{u}_\lambda) = \inf \left\{ \varphi_\lambda^+(u) : u \in W_0^{1,\eta}(\Omega) \right\}. \tag{18}$$

Let $v \in C_0^1(\bar{\Omega})_+$, $v \neq 0$ and $t \in (0,1)$. We have

$$
\begin{aligned}
\varphi_\lambda^+(tv) &\le \frac{t^q}{q}\rho_\eta(Dv) - \int_\Omega \lambda K_+(z,tv)\,dz (\text{since } q < p)\\
&= \frac{t^q}{q}\rho_\eta(Dv) - \frac{\lambda c_2 t^\tau}{\tau}\int_{\{tv\le v_n\}} v^\tau\,dz\\
&\quad - \frac{\lambda c_2}{\tau}\int_\Omega u_n^\tau\,dz - \lambda c_2\int_{\{u_n<tv\}} u_n^{\tau-1}(tv-u_n)\,dz (\text{see } (17))\\
&\le \frac{t^q}{q}\rho_\eta(Dv) - \frac{\lambda c_2 t^\tau}{\tau}\int_{\{tv\le u_n\}} v^\tau\,dz\\
&= \frac{t^q}{q}\rho_\eta(Dv) - \frac{\lambda c_2 t^\tau}{\tau}\int_\Omega v^\tau\,dz + \frac{\lambda c_2}{\tau}\int_{\{u_n<tv\}} v^\tau\,dz\\
&= \left[\frac{t^{q-\tau}}{q}\rho_\eta(Dv) - \frac{\lambda c_2}{\tau}\|v\|_\tau^\tau + \frac{\lambda c_2}{\tau}\int_{\{u_n<tv\}} v^\tau\,dz\right]t^\tau
\end{aligned}
$$

If by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, then $|\{u_n < tv\}|_N \to 0$ as $t \to 0^+$ (recall that $0 \prec u_n$). So, for $t \in (0,1)$ small, we have

$$
\begin{aligned}
&\varphi_\lambda^+(tv) < 0,\\
\Rightarrow &\varphi_\lambda^+(\tilde{u}_\lambda) < 0 = \varphi_\lambda^+(0) (\text{see } (18)),\\
\Rightarrow &\tilde{u}_\lambda \neq 0.
\end{aligned}
$$

From (18), we have

$$
\begin{aligned}
&\langle(\varphi_\lambda^+)'(\tilde{u}_\lambda), h\rangle = 0 \quad \text{for all } h \in W_0^{1,\eta}(\Omega),\\
\Rightarrow &\langle V(\tilde{u}_\lambda), h\rangle = \int_\Omega \lambda k_+(z,\tilde{u}_\lambda)h\,dz \quad \text{for all } h \in W_0^{1,\eta}(\Omega).
\end{aligned} \tag{19}
$$

In (19) first we choose $h = -\tilde{u}_\lambda^- \in W_0^{1,\eta}(\Omega)$ and obtain

$$
\begin{aligned}
&\rho_\eta(D\tilde{u}_\lambda^-) = 0 (\text{see } (17)),\\
\Rightarrow &\tilde{u}_\lambda \ge 0,\ \tilde{u}_\lambda \neq 0.
\end{aligned}
$$

Next in (19) we use the test function $h = (\tilde{u}_\lambda - u_n)^+ \in W_0^{1,\eta}(\Omega)$. We obtain

$$
\begin{aligned}
&\langle V(\tilde{u}_\lambda), (\tilde{u}_\lambda - u)^+\rangle\\
&= \int_\Omega \lambda c_2 u_n^{\tau-1}(\tilde{u}_\lambda - u_n)^+\,dz\\
&\le \int_\Omega \lambda f(z,u_n)(\tilde{u}_\lambda - u_n)^+\,dz \quad \text{since } n \ge n_0 (\text{see } (16))
\end{aligned}
$$

$$= \langle V(u_n), (\tilde{u}_\lambda - u_n)^+\rangle (\text{since } u_n \in S_\lambda^+),$$
$$\Rightarrow \tilde{u}_\lambda \le u_n \quad (\text{see Proposition (2.4)}).$$

So, we have proved that

$$\tilde{u}_\lambda \in [0, u_n],\ \ \tilde{u}_\lambda \neq 0. \tag{20}$$

From (17, (19), (20)) and Proposition 3.1, we infer that $\tilde{u}_\lambda = \bar{u}_\lambda$. We have

$$\bar{u}_\lambda \le u_n \quad \text{for all } n \ge n_0,$$

contradicting the assumption that $u_\lambda^* = 0$ and so that $u_n \to 0$ in $L^\infty(\Omega)$. Therefore $u_\lambda^* \neq 0$. If in (10) we pass to the limit as $n \to \infty$ and use (15), then

$$\langle V(u_\lambda^*), h\rangle = \lambda \int_\Omega f(z, u_\lambda^*) h\, dz \quad \text{for all } h \in W_0^{1,\eta}(\Omega),$$
$$\Rightarrow u_\lambda^* \in S_\lambda^+,\ \ u_\lambda^* = \inf S_\lambda^+.$$

Similarly for S_λ^- which is upward directed and so we can find $\{v_n\}_{n\in\mathbb{N}} \subseteq S_\lambda^-$ increasing such that $\sup S_\lambda^- = \sup_{n\in\mathbb{N}} v_n$. □

5 Nodal Solutions

In this section, using the extremal constant sign solutions and the theory of critical groups (see [21]), we will produce nodal solutions and determine their asymptotic behavior as $\lambda \to 0^+$.

We introduce the energy functional for problem (P_λ), $\varphi_\lambda : W_0^{1,\eta}(\Omega) \mapsto \mathbb{R}$ defined by

$$\varphi_\lambda(u) = \frac{1}{p}\rho_a(Du) + \frac{1}{q}\|Du\|_q^q - \lambda \int_\Omega F(z, u)\, dz \quad \text{for all } u \in W_0^{1,\eta}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W_0^{1,\eta}(\Omega))$.

Proposition 5.1 *If hypotheses H_0, H_1 hold, $\lambda > 0$ and $0 \in K_{\varphi_\lambda}$ is isolated, then $C_k(\varphi_\lambda, 0) = 0$ for all $k \in \mathbb{N}_0$.*

Proof Hypotheses H_1 imply that

$$c_2|x|^\tau - c_9|x|^r \le F(z, x) \quad \text{for a.a. } z \in \Omega,\ \text{all } x \in \mathbb{R},\ \text{some } c_9 > 0. \tag{21}$$

If $u \in W_0^{1,\eta}(\Omega)\backslash\{0\}$ and $t \in (0, 1)$, then

$$\varphi_\lambda(tu) \le \frac{t^q}{q}\rho_\eta(Du) + c_9 t^r \|u\|_r^r - c_2 t^\tau \|u\|_\tau^\tau (\text{recall } q < p,\ \text{see (21)}).$$

Since $\tau < q < r$, we see that there exists $t^* \in (0, 1)$ such that

$$\varphi_\lambda(tu) < 0 \quad \text{for all } t \in (0, t^*).$$

Consider $u \in W_0^{1,\eta}(\Omega)$ with $0 < \|u\| \leq 1$ which satisfies also $\varphi_\lambda(u) = 0$. We have

$$\begin{aligned}
\frac{d}{dt}\varphi_\lambda(tu)|_{t=1} &= \langle \varphi'_\lambda(u), u\rangle (\text{by the chain rule}) \\
&= \rho_\eta(Du) - \lambda \int_\Omega f(z, u)u\, dz \\
&= \left(1 - \frac{\tau}{p}\right) \rho_a(Du) + \left(1 - \frac{\tau}{q}\right) \|Du\|_q^q \\
&\quad + \lambda \int_\Omega [\tau F(z, u) - f(z, u)u]\, dz (\text{since } \varphi_\lambda(u) = 0) \\
&\geq \left(1 - \frac{\tau}{p}\right) \rho_a(Du) + \left(1 - \frac{\tau}{q}\right) \|Du\|_q^q \\
&\quad + \lambda \int_{\{|u|>\delta\}} [\tau F(z, u) - f(z, u)u]\, dz (\text{see } \mathrm{H}_1(\text{ii})) \\
&\geq \left(1 - \frac{\tau}{q}\right) \rho_\eta(Du) - \lambda c_{10}\|u\|^r \quad \text{for some } c_{10} > 0 \\
&\quad (\text{since } q < p,\ \ F \geq 0 \text{ and using } \mathrm{H}_1(\text{i})) \\
&\geq \left(1 - \frac{\tau}{q}\right) \|u\|^p - \lambda c_{10}\|u\|^r (\text{recall } \|u\| \leq 1 \text{ and see Proposition 2.2})
\end{aligned} \tag{22}$$

But $p < r$. So, from (22), we see that for $\rho \in (0, 1)$ small, we have

$$\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=1} > 0 \quad \text{for all } u \in W_0^{1,\eta}(\Omega) \text{ with } 0 < \|u\| \leq \rho,\ \varphi_\lambda(u) = 0. \tag{23}$$

Consider $u \in W_0^{1,\eta}(\Omega)$ as in (23). We will show that

$$\varphi_\lambda(tu) \leq 0 \quad \text{for all } t \in [0, 1]. \tag{24}$$

If (24) is not true, then we can find $t_0 \in (0, 1)$ such that

$$\varphi_\lambda(t_0 u) > 0.$$

Recall that $\varphi_\lambda(u) = 0$(see (23)). So, we can define

$$\begin{aligned}
&\hat{t} = \min\{t \in (t_0, 1] : \varphi_\lambda(tu) = 0\} > t_0 > 0, \\
&\Rightarrow \varphi_\lambda(tu) > 0 \quad \text{for all } t \in [t_0, \hat{t}).
\end{aligned} \tag{25}$$

Let $y = \hat{t}u$. Then

$$0 < \|y\| = \hat{t}\|u\| < \|u\| \leq \rho \quad \text{and} \quad \varphi_\lambda(y) = \varphi_\lambda(\hat{t}u) = 0.$$

Therefore from (23), we have

$$\frac{d}{dt}\varphi_\lambda(ty)\Big|_{t=1} > 0. \tag{26}$$

Also we have

$$\begin{aligned} &\varphi_\lambda(y) = \varphi_\lambda(\hat{t}u) = 0 < \varphi_\lambda(tu) \quad \text{for all } t \in [t_0, \hat{t})(\text{see } (25)) \\ \Rightarrow &\frac{d}{dt}\varphi_\lambda(ty)\Big|_{t=1} = \hat{t}\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=\hat{t}} = \hat{t}\lim_{t\to\hat{t}^-}\frac{\varphi_\lambda(tu)}{t-\hat{t}} \leq 0. \end{aligned} \tag{27}$$

We compare (26) and (27) and we have a contradiction. So (24) is true.

Since by hypothesis $0 \in K_{\varphi_\lambda}$ is isolated, we can always choose $\rho \in (0, 1)$ small so that $K_{\varphi_\lambda} \cap \bar{B}_\rho = \{0\}$ $\Big(\text{recall } \bar{B}_\rho = \Big\{u \in W_0^{1,\eta}(\Omega) : \|u\| \leq \rho\Big\}\Big)$. We consider the deformation

$$H(t, u) = (1 - t)u \quad \text{for all } t \in [0, 1], \text{ all } u \in \varphi_\lambda^0 \cap \bar{B}_\rho.$$

From (24), we see that this is a well-defined deformation of $\varphi_\lambda^0 \cap \bar{B}_\rho$. H shows shat $\varphi_\lambda^0 \cap \bar{B}_\rho$ is contractible. Let $u \in \bar{B}_\rho$ with $\varphi_\lambda(u) > 0$. We will show that there exists unique $t(u) \in (0, 1)$ such that

$$\varphi_\lambda(t(u)u) = 0.$$

From the first part of the proof and Bolzano's theorem, we know that such a $t(u) \in (0, 1)$ exists. So, we have to show that it is unique. Arguing by contradiction, suppose we can find $t_1 = t_1(u) < t_2 = t_2(u) < 1$ such that

$$\varphi_\lambda(t_1 u) = \varphi_\lambda(t_2 u) = 0.$$

From (24), we have

$$\varphi_\lambda(tt_2u) \leq 0 \quad \text{for all } t \in [0, 1].$$

Hence $\tilde{t} = \frac{t_1}{t_2} \in (0, 1)$ is a maximizer of the function $t \mapsto \varphi_\lambda(tt_2u)$ on $[0, 1]$. It follows that

$$0 = \frac{d}{dt}\varphi_\lambda(tt_2u)\Big|_{t=\frac{t_1}{t_2}} = \frac{d}{dt}\varphi_\lambda(tt_1u)\Big|_{t=1}. \tag{28}$$

Since $\varphi_\lambda(t_1 u) = 0$ and $0 < t_1\|u\| \le \|u\| \le \rho$, we see that (28) contradicts (26). This proves the uniqueness of $t(u) \in (0,1)$ such that $\varphi_\lambda(t(u)u) = 0$. Then we have

$$\begin{aligned} &\varphi_\lambda(tu) < 0 \quad \text{if } 0 < t < t(u) \\ &\varphi_\lambda(tu) > 0 \quad \text{if } t(u) < t \le 1. \end{aligned}$$

We introduce the map $\xi : \bar{B}_\rho \backslash \{0\} \mapsto [0,1]$ defined by

$$\xi(u) = \begin{cases} 1 & \text{if } u \in \bar{B}_\rho \backslash \{0\},\ \varphi_\lambda(u) \le 0 \\ t(u) & \text{if } u \in \bar{B}_\rho \backslash \{0\},\ \varphi_\lambda(u) > 0. \end{cases} \tag{29}$$

We claim that $\xi(\cdot)$ is continuous. We need to show continuity on the interface of the two branches. So, consider $u \in \bar{B}_\rho \backslash \{0\}$ with $\varphi_\lambda(u) = 0$ and a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq \bar{B}_\rho \backslash \{0\}$ such that

$$u_n \to u,\ \varphi_\lambda(u_n) > 0 \quad \text{for all } n \in \mathbb{N} (\text{see } (29)).$$

Arguing by contradiction, suppose that

$$\begin{aligned} &t(u_n) \le \tilde{t}_0 < 1 \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow &\varphi_\lambda(tu_n) > 0 \quad \text{for all } t \in (\tilde{t}_0, 1],\ \text{all } n \in \mathbb{N}, \\ \Rightarrow &\varphi_\lambda(tu) \ge 0 \quad \text{for all } t \in (\tilde{t}_0, 1], \\ \Rightarrow &\varphi_\lambda(tu) = 0 \quad \text{for all } t \in (\tilde{t}_0, 1] (\text{see } (24)), \\ \Rightarrow &\frac{d}{dt}\varphi_\lambda(tu)\Big|_{t=1} = 0, \end{aligned}$$

which contradicts (26). Therefore $\xi(\cdot)$ is continuous. Now consider the map $\xi^* : \bar{B}_\rho \backslash \{0\} \mapsto (\varphi_\lambda^0 \cap \bar{B}_\rho) \backslash \{0\}$ defined by

$$\xi^*(u) = \begin{cases} u & \text{if } u \in \bar{B}_\rho \backslash \{0\},\ \varphi_\lambda(u) \le 0 \\ \xi(u)u & \text{if } u \in \bar{B}_\rho \backslash \{0\},\ \varphi_\lambda(u) > 0. \end{cases}$$

The continuity of $\xi(\cdot)$ implies the continuity of $\xi^*(\cdot)$. Also, we have

$$\begin{aligned} &\xi^*\Big|_{(\varphi_\lambda^0 \cap \bar{B}_\rho) \backslash \{0\}} = \mathrm{id}_{(\varphi_\lambda^0 \cap \bar{B}_\rho) \backslash \{0\}} \\ \Rightarrow &(\varphi_\lambda^0 \cap \bar{B}_\rho) \backslash \{0\} \text{ is a retract of } \bar{B}_\rho \backslash \{0\}. \end{aligned}$$

The set $\bar{B}_\rho \backslash \{0\}$ is contractible. A retract of a contractible space is contractible. Therefore $(\varphi_\lambda^0 \cap \bar{B}_\rho) \backslash \{0\}$ is contractible. Earlier in the proof we showed that $\varphi_\lambda^0 \cap \bar{B}_\rho$ is contractible. Then from Papageorgiou-Rădulescu-Repovs [21, p.469], we have

$$\begin{aligned} &H_k(\varphi_\lambda^0 \cap \bar{B}_\rho, (\varphi_\lambda^0 \cap \bar{B}_\rho) \backslash \{0\}) = 0 \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow &C_k(\varphi_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

The proof is now complete. □

Let u_λ^*, v_λ^* be the two extremal constant sign solutions of problem (P_λ) produced in Proposition 4.2. We introduce the Caratheodary function $g_\lambda(z,x)$ defined by

$$g_\lambda(z,x)=\begin{cases}\lambda f(z,v_\lambda^*(z)) & \text{if } x<v_\lambda^*(z)\\ \lambda f(z,x) & \text{if } v_\lambda^*(z)\le x\le u_\lambda^*(z)\\ \lambda f(z,u_\lambda^*) & \text{if } u_\lambda^*<x.\end{cases} \tag{30}$$

Also we consider the positive and negative trucations of $g_\lambda(z,\cdot)$, namely the Caratheodary functions

$$g_\lambda^{\pm}(z,x)=g_\lambda(z,\pm x^{\pm}). \tag{31}$$

We set $G_\lambda(z,x)=\int_0^x g_\lambda(z,s)\,ds$, $G_\lambda^{\pm}(z,x)=\int_0^x g_\lambda^{\pm}(z,s)\,ds$ and consider the C^1-functionals σ_λ, $\sigma_\lambda^{\pm}: W_0^{1,\eta}(\Omega)\mapsto\mathbb{R}$ defined by

$$\sigma_\lambda(u)=\frac{1}{p}\rho_a(Du)+\frac{1}{q}\|Du\|_q^q-\int_\Omega G_\lambda(z,u)\,dz,$$
$$\sigma_\lambda^{\pm}(u)=\frac{1}{p}\rho_a(Du)+\frac{1}{q}\|Du\|_q^q-\int_\Omega G_\lambda^{\pm}(z,u)\,dz$$

for all $u\in W_0^{1,\eta}(\Omega)$.

Using (30), (31) and the extremality of u_λ^*, v_λ^*, we prove easily the following proposition.

Proposition 5.2 *If hypotheses H_0, H_1 hold and $\lambda\in(0,\lambda^*)$, then $K_{\sigma_\lambda}\subseteq[v_\lambda^*,u_\lambda^*]\cap L^\infty(\Omega)$, $K_{\sigma_\lambda^+}=\{0,u_\lambda^*\}$, $K_{\sigma_\lambda^-}=\{0,v_\lambda^*\}$.*

From this proposition, we see that we may assume that K_{δ_λ} is finite or otherwise we already have an infinity of bounded nodal solutions and so we are done. From (30), we see that

$$\sigma_\lambda'|_{[v_\lambda^*,u_\lambda^*]}=\varphi_\lambda'|_{[v_\lambda^*,u_\lambda^*]},$$
$$\Rightarrow K_{\varphi_\lambda}\cap[v_\lambda^*,u_\lambda^*]\text{ is finite},$$
$$\Rightarrow 0\in K_{\varphi_\lambda}\text{ is isolated.}$$

Proposition 5.3 *If hypotheses H_0, H_1 hold and $\lambda\in(0,\lambda^*)$, then $C_k(\sigma_\lambda,0)=0$ for all $k\in\mathbb{N}_0$.*

Proof we have

$$
\begin{aligned}
&|\sigma_\lambda(u) - \varphi_\lambda(u)| \\
&\le \int_\Omega |G_\lambda(z,u) - \lambda F(z,u)|\,dz \\
&= \lambda \int_{\{u<v_\lambda^*\}} |F(z,v_\lambda^*) + (u-v_\lambda^*) f(z,v_\lambda^*) - F(z,u)|\,dz \\
&\quad + \lambda \int_{\{u_\lambda^*<u\}} |F(z,u_\lambda^*) + (u-u_\lambda^*) f(z,u_\lambda^*) - F(z,u)|\,dz \\
&\le \lambda \int_{\{u<v_\lambda^*\}} (2F(z,u) + 2|f(z,v_\lambda^*)||u|)\,dz \\
&\quad + \lambda \int_{\{u_\lambda^*<u\}} (2F(z,u) + 2f(z,u_\lambda^*)u)\,dz (\text{from the sign condition}) \\
&\le 2\lambda \int_\Omega F(z,u)\,dz + \lambda c_{11}\|u\| \text{ for some } c_{11} > 0 \\
&\le 2\lambda c_{12}[\|u\|^\tau + \|u\|^r] + \lambda c_{11}\|u\| \text{ for some } c_{12} > 0 (\text{see hypotheses } \mathrm{H}_1) \\
&\le \lambda c_{13}\|u\| \text{ for some } c_{13} > 0 \text{ and for } \|u\| \le 1.
\end{aligned} \tag{32}
$$

Next let $h \in W_0^{1,\eta}(\Omega)$. We have

$$
\begin{aligned}
&|\langle \sigma_\lambda'(u) - \varphi_\lambda'(u), h\rangle| \\
&\le \int_\Omega |g_\lambda(z,u) - \lambda f(z,u)||h|\,dz \\
&= \int_{\{u<v_\lambda^*\}} \lambda|f(z,v_\lambda^*) - f(z,u)||h|\,dz + \int_{\{u_\lambda^*<u\}} \lambda|f(z,u) - f(z,u_\lambda^*)||h|\,dz \\
&\le \lambda c_{14}\left[\int_{\{u<v_\lambda^*\}} (|u|^{\tau-1} + |u|^{r-1})|h|\,dz + \int_{\{u_\lambda^*<u\}} (u^{\tau-1} + |u|^{r-1})|h|\,dz\right] \text{for some } c_{14} > 0 \\
&\quad (\text{note that } |v_\lambda^*| \le |u| \text{ on } \{u < v_\lambda^*\}) \\
&\le \lambda c_{14} \int_\Omega (|u|^{\tau-1} + |u|^{r-1})|h|\,dz.
\end{aligned}
$$

Since $h \in W_0^{1,\eta}(\Omega)$ and $W_0^{1,\eta}(\Omega) \hookrightarrow L^\tau(\Omega)$ and in $L^r(\Omega)$ continuously(in fact compactly, see Proposition 2.3), using Hölder's inequality, we have

$$
\begin{aligned}
|\langle \sigma_\lambda'(u) - \varphi_\lambda'(u), h\rangle| &\le \lambda c_{15}[\|u\|_\tau^{\tau-1} + \|u\|_r^{r-1}]\|h\| \text{ for some } c_{15} > 0, \\
&\le \lambda c_{16}\|u\| \cdot \|h\| \text{ for some } c_{16} > 0 \text{ (since } \|u\| \le 1,\ 1 < \tau < r).
\end{aligned}
$$

Then

$$
\|\sigma_\lambda'(u) - \varphi_\lambda'(u)\|_* \le \lambda c_{16}\|u\|. \tag{33}
$$

From (32) and (33), we see that given $\epsilon > 0$, for $\delta \in (0, 1)$ small, we have

$$\|\sigma_\lambda - \varphi_\lambda\|_{C^1(\bar{B}_\delta)} \leq \epsilon \text{ with } \bar{B}_\delta = \{u \in W_0^{1,\eta}(\Omega) : \|u\| \leq \delta\}.$$

The functional $\sigma_\lambda(\cdot)$ is coercive(see (30)). So, from Proposition 5.1.15 of [21, p.369], we know that $\sigma_\lambda(\cdot)$ satisfies the C-condition. So, we can use the C^1-continuity property of critical groups(see Gasinski-Papageorgiou [5], Theorem 5.129, p.836) and infer that

$$\begin{aligned} &C_k(\sigma_\lambda, 0) = C_k(\varphi_\lambda, 0) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow &C_k(\sigma_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0 \text{ (see Proposition 5.1).} \end{aligned}$$

This completes the proof. □

Now we are ready to produce a nodal solution y_λ and determine its asymptotic behavior as $\lambda \to 0^+$.

Proposition 5.4 *If hypotheses H_0, H_1 hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) has a nodal solution $y_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ and $y_\lambda \to 0$ in $W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ as $\lambda \to 0^+$.*

Proof From (30) and (31), it is clear that $\sigma_\lambda^+(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $\tilde{u}_\lambda^* \in W_0^{1,\eta}(\Omega)$ such that

$$\sigma_\lambda^+(\tilde{u}_\lambda^*) = \inf\{\sigma_\lambda^+(u) : u \in W_0^{1,\eta}(\Omega)\}. \tag{34}$$

Let $u \in C_0^1(\bar{\Omega}_+)$, $u \neq 0$ and choose $t \in (0, 1)$ small so that $0 \leq tu(z) \leq \delta$ for all $z \in \bar{\Omega}$, with $\delta > 0$ as postulated in hypothesis H_1(ii). We have

$$\begin{aligned} \sigma_\lambda^+(tu) &\leq \frac{t^p}{p}\rho_a(Du) + \frac{t^q}{q}\|Du\|_q^q - \frac{c_2 t^\tau}{\tau}\int_{\{tu \leq u_\lambda^*\}} u^\tau \, dz \text{(see hypothesis } H_1\text{(ii))} \\ &\leq \frac{t^q}{q}\rho_\eta(Du) - \frac{c_2 t^\tau}{\tau}\int_\Omega u^\tau \, dz + \frac{c_2 t^\tau}{\tau}\int_{\{u_\lambda^* < tu\}} u^\tau \, dz \\ &= \left[\frac{t^{q-\tau}}{q}\rho_\eta(Du) - \frac{c_2}{\tau}\|u\|_\tau^\tau + \frac{c_2}{\tau}\int_{\{u_\lambda^* < tu\}} u^\tau \, dz\right] t^\tau \end{aligned} \tag{35}$$

Note that $|\{u_\lambda^* < tu\}|_N \to 0$ as $t \to 0^+$ (recall that $0 \prec u_\lambda^*$). Then from (35), we see that for $t \in (0, 1)$ small, we have

$$\begin{aligned} &\sigma_\lambda^+(tu) < 0, \\ \Rightarrow &\sigma_\lambda^+(\tilde{u}_\lambda^*) < 0 = \sigma_\lambda^+(0) \text{(see (34))} \\ \Rightarrow &\tilde{u}_\lambda^* \neq 0. \end{aligned}$$

From (34), we have $\tilde{u}_\lambda^* \in K_{\sigma_\lambda^+}$. Then by Proposition 5.2, we have $\tilde{u}_\lambda^* = u_\lambda^*$. So, we have

$$C_k(\sigma_\lambda^+, u_\lambda^*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{36}$$

Claim: $C_k(\sigma_\lambda, u_\lambda^*) = C_K(\sigma_\lambda^+, u_\lambda^*)$ for all $k \in \mathbb{N}_0$. Note that $g_\lambda(z, u_\lambda^*) = g_\lambda^+(z, u_\lambda^*)$ and $G_\lambda(z, u_\lambda^*) = G_\lambda^+(z, u_\lambda^*)$ (see (30) and (31)). We have

$$|\sigma_\lambda(u) - \sigma_\lambda^+(u)| \leq \int_\Omega |G_\lambda(z, u) - G_\lambda(z, u_\lambda^*)|\, dz + \int_\Omega |G_\lambda^+(z, u_\lambda^*) - G_\lambda^+(z, u)|\, dz. \tag{37}$$

We estimate each term in the right hand side of (37). We have

$$\begin{aligned}
&\int_\Omega |G_\lambda(z, u) - G_\lambda(z, u_\lambda^*)|\, dz \\
&\leq \int_{\{u<v_\lambda^*\}} \big(\lambda|(u - v_\lambda^*) f(z, v_\lambda^*) + F(z, v_\lambda^*) - F(z, u_\lambda^*)|\big)\, dz \\
&\quad + \int_{\{v_\lambda^* \leq u \leq u_\lambda^*\}} \lambda |F(z, u) - F(z, u_\lambda^*)|\, dz \\
&\quad + \int_{\{u_\lambda^* < u\}} \lambda (u - u_\lambda^*) f(z, u_\lambda^*)\, dz.
\end{aligned} \tag{38}$$

Note that on $\{u < v_\lambda^*\}$ we have

$$\begin{aligned}
&|(u - v_\lambda^*) f(z, v_\lambda^*)| \\
&\leq c_{17} |u - v_\lambda^*| \quad \text{for some } c_{16} > 0 \text{ (see H}_1\text{(i))} \\
&\leq c_{17} (|u - u_\lambda^*| + (u_\lambda^* - v_\lambda^*)) \\
&\leq 2c_{17} |u - u_\lambda^*| (\text{since } u_\lambda^* - v_\lambda^* \leq u_\lambda^* - u \text{ on } \{u < v_\lambda^*\}).
\end{aligned}$$

Since by hypothesis $f(z, \cdot)$ is an L^∞-locally Lipschitz integrand, on $\{u < v_\lambda^*\}$ we have

$$\begin{aligned}
|F(z, v_\lambda^*) - F(z, u_\lambda^*)| &\leq c_{18} (u_\lambda^* - v_\lambda^*) \quad \text{for some } c_{18} > 0 \\
&\leq c_{18} (u_\lambda^* - u) \quad (\text{since } - v_\lambda^* \leq -u)
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
&\int_{\{u<v_\lambda^*\}} \big(\lambda|(u - v_\lambda^*) f(z, v_\lambda^*) + F(z, v_\lambda^*) - F(z, u_\lambda^*)|\big)\, dz \\
&\leq \lambda c_{19} \|u - u_\lambda^*\| \quad \text{for some } c_{19} > 0.
\end{aligned}$$

Also using once again the hypothesis that $f(z, \cdot)$ is an L^∞-locally Lipschitz integrand, we have

$$\int_{\{v_\lambda^* \leq u \leq u_\lambda^*\}} \lambda |F(z, u) - F(z, u_\lambda^*)|\, dz \leq \lambda c_{20} \|u - u_\lambda^*\| \quad \text{for some } c_{20} > 0.$$

Finally we have

$$\int_{\{u_\lambda^*<u\}} \lambda(u-u_\lambda^*)f(z,u_\lambda^*)\,dz \le \lambda c_{21}\|u-u_\lambda^*\| \quad \text{for some } c_{21}>0.$$

Returning to (38), we see that

$$\int_\Omega |G_\lambda(z,u)-G_\lambda(z,u_\lambda^*)|\,dz \le \lambda c_{22}\|u-u_\lambda^*\| \quad \text{for some } c_{22}>0. \tag{39}$$

In a similar fashion, we show that

$$\int_\Omega |G_\lambda^+(z,u_\lambda^*)-G_\lambda^+(z,u)|\,dz \le \lambda c_{23}\|u-u_\lambda^*\| \quad \text{for some } c_{23}>0. \tag{40}$$

Returning to (37) and using (39) and (40), we obtain

$$|\sigma_\lambda(u)-\sigma_\lambda^+(u)| \le \lambda c_{24}\|u-u_\lambda^*\| \quad \text{for some } c_{24}>0. \tag{41}$$

Now we perform a similar estimation for the derivatives. Let $h \in W_0^{1,\eta}(\Omega)$. As before we have

$$\begin{aligned}&|\langle \sigma_\lambda'(u)-(\sigma_\lambda^+)'(u),h\rangle|\\ &\le \int_\Omega |g_\lambda(z,u)-g_\lambda(z,u_\lambda^*)||h|\,dz + \int_\Omega |g_\lambda^+(z,u_\lambda^*)-g_\lambda^*(z,u)||h|\,dz\end{aligned} \tag{42}$$

We estimate the two terms in the right hand side of (42). We have

$$\begin{aligned}&\int_\Omega |g_\lambda(z,u)-g_\lambda(z,u_\lambda^*)||h|\,dz\\ &= \int_{\{u<v_\lambda^*\}} \lambda|f(z,v_\lambda^*)-f(z,u_\lambda^*)||h|\,dz + \int_{\{v_\lambda^*\le u\le u_\lambda^*\}} \lambda|f(z,u)-f(z,u_\lambda^*)||h|\,dz\\ &\le \lambda c_{25}\int_\Omega |u-u_\lambda^*||h|\,dz (\text{again we use that } u_\lambda^*-v_\lambda^* \le u_\lambda^*-u \text{ on } \{u<v_\lambda^*\}).\end{aligned}$$

On account of hypotheses H_0, if $s>1$ is close to 1, we have

$$u-u_\lambda^* \in L^{\frac{p}{s}}(\Omega) \quad \text{and} \quad h \in L^{(\frac{p}{s})'}(\Omega).$$

Since $\left(\frac{p}{s}\right)' = \frac{p}{p-s} < q^* = \frac{Nq}{N-q}$ (recall that $p \ge 2$ and $\frac{1}{q} < \frac{1}{p} + \frac{1}{Np}$ see hypotheses H_0). Therefore by Hölder's inequality, and Proposition 2.3, we have

$$\int_\Omega |g_\lambda(z,u)-g_\lambda(z,u_\lambda^*)||h|\,dz \le \lambda c_{26}\|u-u_\lambda^*\|\|h\| \quad \text{for some } c_{26}>0. \tag{43}$$

Similarly, we show that

$$\int_\Omega |g_\lambda^+(z,u_\lambda^*) - g_\lambda^+(z,u)|\,dz \le \lambda c_{27}\|u-u_\lambda^*\|\|h\| \quad \text{for some } c_{27}>0. \tag{44}$$

Using (43) and (44) in (42) and taking supremum over $h \in W_0^{1,\eta}(\Omega)$, $\|h\| \le 1$, we obtain

$$\|\sigma_\lambda'(u) - (\sigma_\lambda^+)'(u)\|_* \le \lambda c_{28}\|u-u_\lambda^*\| \quad \text{for some } c_{28}>0. \tag{45}$$

From (41) and (45), we infer that given $\epsilon > 0$, we can find $\delta \in (0,1)$ small such that

$$\|\sigma_\lambda - \sigma_\lambda^+\|_{C^1(\bar{B}_\delta(u_\lambda^*))} \le \epsilon$$

where $\bar{B}_\delta(u_\lambda^*) = \left\{u \in W_0^{1,\eta}(\Omega) : \|u-u_\lambda^*\| \le \delta\right\}$. Then the C^1-continunity property of critical groups(see [5, p.836]) implies that

$$C_k(\sigma_\lambda, u_\lambda^*) = C_k(\sigma_\lambda^+, u_\lambda^*) \quad \text{for all } k \in \mathbb{N}_0.$$

This proves the claim. From the claim and (36), we have

$$C_k(\sigma_\lambda, u_\lambda^*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{46}$$

Working with σ_λ^-, first we show as above that v_λ^* is a global minimizer of $\sigma_\lambda^-(\cdot)$ and so

$$C_k(\sigma_\lambda^-, v_\lambda^*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0.$$

Reasoning as in the claim, we show that

$$\begin{aligned} &C_k(\sigma_\lambda, v_\lambda^*) = C_k(\sigma_\lambda^-, v_\lambda^*) \quad \text{for all } k \in \mathbb{N}_0 \\ \Rightarrow\, &C_k(\sigma_\lambda, v_\lambda^*) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \end{aligned} \tag{47}$$

From Proposition 5.3, we know that

$$C_k(\sigma_\lambda, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{48}$$

The function $\sigma_\lambda(\cdot)$ is coercive and so from Proposition 6.2.24 of [21, p.491], we have

$$C_k(\sigma_\lambda, \infty) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{49}$$

Suppose $K_{\sigma_\lambda} = \{0, u_\lambda^*, v_\lambda^*\}$. Then from (46), (47), (48), (49) and the Morse relation with $t = -1$(see (1)), we have $2(-1)^0 = (-1)^0$, a contradiction. So, there exists

$y_\lambda \in K_{\sigma_\lambda} \subseteq [v_\lambda^*, u_\lambda^*] \cap L^\infty(\Omega)$(see Proposition 5.2), $y_\lambda \notin \{0, u_\lambda^*, v_\lambda^*\}$. On account of the extremality of u_λ^*, v_λ^*, this solution is nodal. We have

$$\langle V(y_\lambda), h\rangle = \int_\Omega \lambda f(z, y_\lambda) h\, dz \quad \text{for all } h \in W_0^{1,\eta}(\Omega).$$

Choosing $h = y_\lambda \in W_0^{1,\eta}(\Omega)$, we obtain

$$\rho_\eta(Dy_\lambda) \leq \lambda c_{29} \quad \text{for some } c_{29} > 0, \tag{50}$$

$$\Rightarrow y_\lambda \to 0 \quad \text{in } W_0^{1,\eta}(\Omega) \text{ as } \lambda \to 0^+. \tag{51}$$

Evidently $\{u_\lambda^*\}_{\lambda\in(0,1]}$ is decreasing and $\{v_\lambda^*\}_{\lambda\in(0,1]}$ is increasing as $\lambda \to 0^+$. Also, using the Moser iteration technique as in Guedda-Véron [8](see also [2]), for $m > \frac{N}{q}$, we have

$$\|y_\lambda\|_\infty \leq \lambda c_{30} \|f(\cdot, y_\lambda(\cdot))\|_m^{\frac{1}{q-1}} \quad \text{for some } c_{30} > 0, \text{ all } \lambda \in (0,1],$$
$$\Rightarrow y_\lambda \to 0 \quad \text{in } L^\infty(\Omega) \text{ as } \lambda \to 0^+.$$

Recall that $W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ is a Banach space with norm $|\cdot| = \max\{\|\cdot\|, \|\cdot\|_\infty\}$. Therefore we conclude that

$$y_\lambda \to 0 \quad \text{in } W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega) \quad \text{as } \lambda \to 0^+.$$

The proof is now complete. ◻

Summarizing our findings, we can state the following multiplicity theorem for problem (P_λ).

Theorem 5.5 *If hypotheses H_0, H_1 hold, then for all $\lambda > 0$ small problem (P_λ) has at least three nontrivial solutions $u_\lambda,\ v_\lambda,\ y_\lambda \in W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ such that*

$$0 \prec u_\lambda,\ 0 \prec -v_\lambda,\ y_\lambda \in [v_\lambda, u_\lambda] \text{ is nodal}$$

and $y_\lambda \to 0$ in $W_0^{1,\eta}(\Omega) \cap L^\infty(\Omega)$ as $\lambda \to 0^+$.

Remark 5.6 In this multiplicity theorem we provide sign information for all the solutions produced and the solutions are ordered.

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