because $|b| < \sqrt{b^2 + a^2}$. For a given a let $f(x, s) = a/(a^2 + s^2 + x^2)$. This function is continuous on the real (x, s)-plane. Observe that $f(x, s) \le a/(a^2 + x^2)$, and $\int_0^\infty a\,dx/(a^2 + x^2)$ is convergent. Now

$$\int_0^b \frac{a\,ds}{a^2 + s^2 + x^2} = \frac{a}{\sqrt{a^2 + x^2}} \tan^{-1} \frac{b}{\sqrt{a^2 + x^2}},$$

so

$$\begin{aligned}
\int_0^\infty \frac{a}{\sqrt{a^2 + x^2}} \tan^{-1} \frac{b}{\sqrt{a^2 + x^2}}\,dx &= \int_0^\infty \int_0^b \frac{a}{a^2 + s^2 + x^2}\,ds\,dx \\
&= \int_0^b \int_0^\infty \frac{a}{a^2 + s^2 + x^2}\,dx\,ds = \int_0^b \frac{a}{\sqrt{a^2 + x^2}} \left[\tan^{-1} \frac{x}{\sqrt{a^2 + s^2}}\right]_{x=0}^{x=\infty} ds \\
&= \frac{a\pi}{2} \int_0^b \frac{ds}{\sqrt{a^2 + x^2}} = \frac{a\pi}{2} \left[\log\left(s + \sqrt{a^2 + s^2}\right)\right]_{s=0}^{s=b} \\
&= \frac{a\pi}{2} \left[\log\left(b + \sqrt{a^2 + b^2}\right) - \log a\right].
\end{aligned}$$

Also solved by B. M. Abrégo, Z. Ahmed (India), S. Amghibech (Canada), T. Andebrhan (Eritrea), D. Arrigo, A. G. Astudio (Spain), W. S. Au, M. R. Avidon, R. Bagby, M. Bataille (France), D. Beckwith, M. Benito (Spain), K. Bernstein, K. N. Boyadzhiev, E. Braune (Austria), J. Borwein (Canada), P. Bracken, M. A. Carlton, N. Caro (Brazil), R. Chapman (U. K.), W. Chu (Italy), K. Dale (Norway), B. E. Davis, D. Donini (Italy), Y. Dumont (France), O. Furdui, M. L. Glasser, M. Goldenberg & M. Kaplan, O. P. Lossers (Netherlands), J.-P. Grivaux (France), J. A. Grzesik, C. B. Harris, E. A. Herman, G. L. Isaacs, S. Y. Jeon (Korea), K. T. Koo, G. Lamb, K. D. Lathrop, J. Marengo, T. McCoy, K. McInturff, K. D. McLenithan, A. Nijenhuis, N. Ortner (Austria), C. R. Pranesachar (India), H. Ricardo, O. G. Ruehr, I. Sofair, A. Stadler (Switzerland), R. Stong, N. Thornber, D. B. Tyler, E. I. Verriest, M. Vowe (Switzerland), P. Walker, Y. Zamani, L. Zhou, GCHQ Problem Solving Group (U. K.), NSA Problem Solving Group, NUIM Problem Solving Group (Ireland), and the proposer.

Periodic Solution of a Differential Equation

11104 [2004, 724]. *Proposed by Vicenţiu Rădulescu, University of Craiova, Craiova, Romania.* Let $0 < a < 1$, and let f be a twice-differentiable function from $\mathbb{R}$ to $\mathbb{R}$ satisfying the differential equation $f'' = -f(1 - f^2)$ and the initial condition $f(0) = a$. Suppose further that f is periodic, with principle period $T(a)$, and that f^2 achieves its maximum at the origin.
(**a**) Prove that $T(a) > 2\pi$.
(**b**) Let $t_0(a) = \sup\{t > 0\colon\ f(s) > 0 \text{ for } 0 < s < t\}$. Show that f is decreasing on $(0, t_0(a))$, and express $T(a)$ in terms of $t_0(a)$.
(**c**) Prove that $a^2 + (2\pi/T(a))^2 > 1$.
(**d**) Show that t_0 is an increasing function on $(0, 1)$, and compute the limit of $t_0(a)$ as a approaches 0 or 1 from within the interval.
(**e**) Prove that for $T > 2\pi$ there exists a unique periodic function g with period T from $\mathbb{R}$ to $\mathbb{R}$ such that $g'' = -g(1 - g^2)$ and g^2 achieves its maximum at the origin.

Solution by the GCHQ Problem Solving Group, Cheltenham, UK. Letting $v = f'$, we have

$$\frac{1}{2}\frac{d}{df}v^2 = v\frac{dv}{df} = f'' = -f(1 - f^2).$$

Integrating both sides with respect to f yields

$$\frac{1}{2}v^2 = \frac{1}{4}f^4 - \frac{1}{2}f^2 + k,$$

where k is some constant. Since by hypothesis f^2 achieves its maximum at the origin, we must have $v(0) = f'(0) = 0$, so $k = a^2/2 - a^4/4$. Therefore

$$v^2 = \frac{1}{2}(f^4 - a^4) - (f^2 - a^2) = \frac{1}{2}(a^2 - f^2)(2 - a^2 - f^2). \tag{1}$$

(**b**) Since $v' = -f(1 - f^2)$, we have $v' < 0$ on $(0, t_0(a))$. However, $v(0) = 0$. Hence $f' = v < 0$ on $(0, t_0(a))$, which implies that f is decreasing on $(0, t_0(a))$. By symmetry of (1), the region $(0, t_0(a))$ represents one quarter of the cycle. The four quarters are characterised by the following combinations for f and v: $f > 0, v < 0$; $f < 0, v < 0$; $f < 0, v > 0$; $f > 0, v > 0$. It follows that $T(a) = 4t_0(a)$.

(**a**), (**c**) In the interval $(0, t_0(a))$, we integrate (1) by separating it and using the substitution $f = a\sin\theta$:

$$t_0(a) = \sqrt{2}\int_0^a \frac{df}{\sqrt{(a^2 - f^2)(2 - a^2 - f^2)}} = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{2 - a^2 - a^2\sin^2\theta}}. \tag{2}$$

On $(0, \frac{\pi}{2})$, we have $0 < \sin\theta < 1$, from which it follows that $2 - a^2 > 2 - a^2 - a^2\sin^2\theta > 2(1 - a^2)$. Hence

$$\frac{1}{2\sqrt{1 - a^2}} > \frac{1}{\sqrt{2 - a^2 - a^2\sin^2\theta}} > \frac{1}{\sqrt{2 - a^2}}.$$

Integrating these inequalities, we obtain

$$\frac{\pi}{2}\frac{1}{\sqrt{1 - a^2}} > t_0(a) > \frac{\pi}{2}\sqrt{\frac{2}{2 - a^2}} > \frac{\pi}{2}.$$

Using (b), the right-hand inequality implies that $T(a) > 2\pi$, and the left-hand inequality can be rearranged to give

$$a^2 + \left(\frac{2\pi}{T(a)}\right)^2 > 1.$$

(**d**) From (2) it follows that

$$t_0'(a) = \sqrt{2}a\int_0^{\pi/2} \frac{1 + \sin^2\theta}{(2 - a^2 - a^2\sin^2\theta)^{3/2}}\,d\theta > 0.$$

Hence $t_0(a)$ is increasing. Moreover,

$$\lim_{a\to 0^+} t_0(a) = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{2}} = \frac{\pi}{2},$$

and

$$\lim_{a\to 1^-} t_0(a) = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - \sin^2\theta}} = \sqrt{2}\int_0^{\pi/2} \frac{d\theta}{\cos\theta} = +\infty,$$

as required.

(**e**) The hypothesis that g be positive at zero should be included in the claim; otherwise, $-g$ is also a solution, which violates uniqueness. Suppose that $g(0) > 0$, all else being the same. If $g(0) = 1$, then since $g'(0) = 0$ and $g'' = -g(1 - g^2)$, we conclude that $g \equiv 1$, contradicting the assumption concerning the fundamental period. If $g(0) > 1$, then $(g^2)'(0) = 2g(0)g'(0) = 0$ so $g'(0) = 0$ and $(g^2)''(0) = 2g(0)g''(0) = -2g^2(0)(1 - g^2(0) > 0$, which contradicts the assumption that g^2 achieves a maximum at zero. The remaining case, in which $0 < g(0) < 1$, has already been treated, with (b) and (d) implying the desired result.

Also solved by S. Amghibech (Canada), P. Bracken, R. Chapman (U. K.), J. H. Lindsey II, O. P. Lossers (Netherlands), K. McInturff, R. Stong, J. Vinuesa (Spain), Szeged Problem Solving Group Fejéntaláltuka (Hungary), and the proposer.

Cevians and Centroids

11105 [2004, 724]. *Proposed by Juan-Bosco Romero Marquez, Valladolid, Spain, Rick Luttmann, Sonoma (California) State University, Miguel de Guzman, Madrid, Spain, and G. Donald Chakerian, University of California at Davis.* The (V, P)-*Cevian* from a vertex V of a triangle through a point P inside that triangle is the line segment from V through P terminating on the side of the triangle opposite V.

Let T be a triangle in the plane, and call its vertices A_1, A_2, and A_3. Let D and E be points inside T. Construct and label Cevians through D and E as follows: for $i \in \{1, 2, 3\}$, with computation of indices modulo 3, let B_i be the intersection of the (A_{i+1}, D)-Cevian with the (A_{i-1}, E)-Cevian, and let C_i be the intersection of the (A_{i-1}, D)-Cevian with the (A_{i+1}, E)-Cevian. The line segments B_1C_1, B_2C_2, and B_3C_3 are then constructed.

(**a**) Prove that B_1C_1, B_2C_2, and B_3C_3 are concurrent.

(**b**) Prove that when D is the centroid and E is the Gergonne point, the point of concurrency in (**a**) is the incenter. (The *Gergonne point* of a triangle is the point at which the Cevians from the vertices to the points of tangency of the incircle meet.)

(**c**) Prove that when D is the centroid and E is any other point, the point of concurrency in (**a**) is the center of the unique ellipse inscribed in the triangle having the feet of the Cevians through E as its points of tangency.

Solution by Li Zhou, Polk Community College, Winter Haven, FL. (**a**) Take $A_1A_2A_3$ as coordinate triangle of the barycentric coordinate system, with $A_1 = (1 : 0 : 0)$, $A_2 = (0 : 1 : 0)$, and $A_3 = (0 : 0 : 1)$. Let $D = (r : s : t)$ and $E = (u : v : w)$. Determinants $\det(B_1, A_2, C_3)$ and $\det(B_1, A_3, C_2)$ are zero, so we get $B_1 = (ru : rv : tu)$. Similarly, $B_2 = (rv : sv : sw)$, $B_3 = (tu : sw : tw)$, $C_1 = (ru : su : rw)$, $C_2 = (su : sv : tv)$, and $C_3 = (rw : tv : tw)$. Let $F = (ru(sw + tv) : sv(rw + tu) : tw(rv + su))$. We check that $\det(B_i, C_i, F) = 0$ for $i = 1, 2, 3$, and it follows that B_1C_1, B_2C_2, and B_3C_3 concur at F.

(**b**) It is known (see, for example, P. Yiu's paper "The Uses of Homogeneous Barycentric Coordinates in Plane Euclidean Geometry," *Int. J. Math. Educ. Sci. Technol.* **31** (2000), 569–578) that $D = (1 : 1 : 1)$ and $E = ((s - a_2)(s - a_3) : (s - a_3)(s - a_1) : (s - a_1)(s - a_2))$, where $a_1 = A_2A_3$, $a_2 = A_3A_1$, $a_3 = A_1A_2$, and $s = (a_2 + a_2 + a_3)/2$. Hence $F = (a_1 : a_2 : a_3)$, which is the incenter (ibid.).

(**c**) Apply an orthogonal projection so that the ellipse becomes the incircle. Then the claim follows from (**b**) and the properties of orthogonal projections.

Editorial comment. Several solvers noted that part (**a**) follows from the dual of Pappus's theorem.

Also solved by S. Amghibech (Canada), M. Bataille (France), R. Chapman (U. K.), L. R. King, P. Lampe